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SOME REMARKS ON LIFTING INVERTIBLE ELEMENTS FROM QUOTIENT C^*-ALGEBRAS

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In some of his papers ([7], [8]) M. A. Rieffel introduced and used various types of stable ranks for C^*-algebras. These numbers have shown their use in the "non-stable" K-theoretical problem of classifying finitely generated projective modules up to isomorphism. In order to investigate the "cancellation property" a main tool that one uses is the theory of stable ranks. Since in the commutative case stable ranks can be recovered from the dimension of the spectrum one may consider the general case as a part of "non-commutative" dimension theory.

In this paper we shall apply a cancellation result to obtain a satisfactory answer to the following problem concerning extensions: "Given an ideal (by this we shall mean closed two-sided ideal) J in a unital C^*-algebra A, an integer n, and an invertible element x from $M_n(A/J)$, under what conditions can one lift x to a "good" element from $M_n(A)$?"

Our result is given in terms of stable ranks. We shall also obtain a little improvement for one of M. A. Rieffel's relations about the behaviour of stable ranks in extensions.

First we shall fix some notations and remind M. A. Rieffel's definitions and results.

For a unital C^*-algebra A and an integer n one considers $\mathrm{Lg}_n(A)$ to be the set of all n-tuples $(a_1, \ldots, a_n)$ from A^n which generate A as a left ideal (i.e. there exist $b_1, \ldots, b_n$ in A such that $b_1a_1 + \ldots + b_na_n = 1$). By $\mathrm{GL}_n(A)$ one denotes the group of invertible elements from $M_n(A)$ and by $\mathrm{GL}_n^0(A)$ the connected component of the identity. One considers $\mathrm{Lg}_n(A)$ as a set of column matrices on which multiplication on the left gives an obvious action of the group $\mathrm{GL}_n(A)$. By definitions 1.4, 4.7 and 10.1 from [7] one takes

$$\mathrm{tsr}(A) = \min\{n \in \mathbf{N} \mid \mathrm{Lg}_n(A) \text{ is dense in } A^n\},$$

$$\mathrm{csr}(A) = \min\{n \in \mathbf{N} \mid \mathrm{GL}_m^0(A) \text{ acts transitively on } \mathrm{Lg}_m(A) \text{ for every } m \geqslant n\},$$

$$\mathrm{gsr}(A) = \min\{n \in \mathbf{N} \mid \mathrm{GL}_m(A) \text{ acts transitively on } \mathrm{Lg}_m(A) \text{ for every } m \geqslant n\}.$$

If one of the above sets is empty the corresponding number is taken to be ∞. In the non-unital case one works in the algebra $\tilde{A}$ obtained from A by adjoining a unit.

Theorems 4.10 and 10.2 from [7] give

$$\mathrm{gsr}(A) \leqslant \mathrm{csr}(A) \leqslant \mathrm{tsr}(A) + 1.$$

The following cancellation property (Theorem 10.6 of [7]) will be useful:

THEOREM 1. *Let A be a unital C^*-algebra and let n be an integer, $n \geqslant \mathrm{gsr}(A)$. Suppose X is a finitely generated projective A-module such that*

$$X \oplus A \simeq A^n.$$

Then $X \simeq A^{n-1}$.

Only this fact is needed for the main result

THEOREM 2. *Let J be an ideal in the unital C^*-algebra A. Suppose that there are given $n \geqslant \mathrm{gsr}(J) - 1$, $n \in \mathbf{N}$ and $x \in M_n(A/J)$, unitary. Then we have the following properties:*

(i) $\partial[x] = 0$ *if and only if there exists a unitary X from $M_n(A)$ such that $\hat{X} = x$.*

(ii) $\partial[x] \leqslant 0$ *if and only if there exists an isometry X in $M_n(A)$ such that $\hat{X} = x$.*

(iii) $\partial[x] \geqslant 0$ *if and only if there exists a coisometry X in $M_n(A)$ such that $\hat{X} = x$.*

We use the notation $X \to \hat{X}$ for the quotient mapping: $M_n(A) \to M_n(A/J)$ and ∂ stands for the "index" homomorphism: $\mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$; the inequalities that appear in (ii) and (iii) should be understood in the ordered group $\mathrm{K}_0(\tilde{J})$.

Proof. Since (iii) follows from (ii) by passing to adjoints, it is not necessary to prove this part. Also the "if" parts are known from K-theory ([10], [1]). So it remains to prove the "only if" parts of (i) and (ii) and we shall do it simultaneously. For this purpose we repeat the construction of the "index" homomorphism. One takes

$$w = \begin{pmatrix} x & 0 \\ 0 & x^* \end{pmatrix} \in M_{2n}(A/J).$$

Since $w \in \mathrm{GL}_{2n}^0(A/J)$, it can be lifted to a matrix $W \in \mathrm{GL}_{2n}^0(A)$, that is $\hat{W} = w$. Going on, one takes

$$P = W \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} W^*.$$

P is a projection in $M_{2n}(\tilde{J})$ because

$$\hat{P} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

By definition $\partial[x] = [P]_{K_0(\tilde{J})} - [I_n]_{K_0(\tilde{J})} \in K_0(J)$.

The hypothesis of (ii) (respectively (i)) is that

$$[P]_{K_0(\tilde{J})} - [I_n]_{K_0(\tilde{J})} \leqslant 0$$

(resp.

$$[P]_{K_0(\tilde{J})} - [I_n]_{K_0(\tilde{J})} = 0)$$

in $K_0(\tilde{J})$.

But this means that there exist an integer m and a projection Q in $M_n(\tilde{J})$ such that

$$[P]_{K_0(\tilde{J})} + [Q]_{K_0(\tilde{J})} = [I_n]_{K_0(\tilde{J})}$$

(resp.

$$[P]_{K_0(\tilde{J})} = [I_n]_{K_0(\tilde{J})}).$$

Unifying the proof, in case (i) we take $Q = 0$.

Then the following finitely generated projective $\tilde{J}$-modules will be stably isomorphic: $P(\tilde{J}^{2n}) \oplus Q(\tilde{J}^m)$ and $\tilde{J}^n$. This is the place where we apply Theorem 1 (since $n \geqslant \mathrm{gsr}(J) - 1$) and conclude that these modules are actually isomorphic. Consequently $P(\tilde{J}^{2n})$ is isomorphic to a direct summand of $\tilde{J}^n$ (respectively in case (i) this summand is the entire $\tilde{J}^n$). Turning into the C^*-algebraic language this means that the projection P is equivalent (in the sense of Murray-von Neumann) to a subprojection of $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ (respectively to the entire projection). So there exists a partial isometry Z in $M_{2n}(\tilde{J})$ such that

$$Z^*Z = P = W\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}W^*$$

and

$$ZZ^* \leqslant \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \quad \left(\text{respectively } ZZ^* = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Let us take $Z_1 = ZW$. The above relations give

$$Z_1^*Z_1 = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Z_1 Z_1^* \leqslant \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \left(\text{respectively } Z_1 Z_1^* = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\right).$$

But in this situation Z_1 is forced to have the following form:

$$Z_1 = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } S_1 \text{ in } M_n(\tilde{J}) \text{ and } S_1^* S_1 = I_n,$$

$$S_1 S_1^* \leqslant I_n \quad (\text{respectively } S_1 S_1^* = I_n).$$

This proves that S_1 is an isometry (respectively a unitary) from $M_n(A)$. Let us compare x with $\hat{S}_1$. We have

$$\begin{pmatrix} \hat{S}_1 x^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \hat{S}_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^* & 0 \\ 0 & x \end{pmatrix} = \hat{Z}_1 w^* = \widehat{Z_1 W^*} = \hat{Z} \in M_{2n}(\mathbf{C}),$$

so $\hat{S}_1 x^*$ is a matrix with scalar entries (considered in $M_n(A/J)$). This matrix is an isometry, so it is actually a unitary and can be lifted in an obvious manner in $M_n(A)$ (take the matrix with same scalar entries). Take then a unitary S_2 in $M_n(A)$ such that $\hat{S}_2 = \hat{S}_1 x^*$. Finally take $X = S_2^* S_1$. X is an isometry (respectively a unitary) and $\hat{X} = \hat{S}_2^* \hat{S}_1 = (\hat{S}_1 x^*)^* \hat{S}_1 = x \hat{S}_1^* \hat{S}_1 = x$, which ends the proof.

REMARK 1. This theorem is useful when one can give estimates for the gsr. One example is that of the ideals which can be represented as $B \otimes K$ with B a unital C^*-algebra. In such a situation $\text{gsr}(B \otimes K) \leqslant \text{csr}(B \otimes K) \leqslant 2$ (see [3] or [9]), and we get the following

COROLLARY 1. *For an exact sequence of C^*-algebras*

$$0 \to B \otimes K \to A \to D \to 0$$

with B, A and D unital, given a unitary u in $M_n(D)$, one has:

(i) *u can be lifted to a unitary in $M_n(A)$ if and only if $\partial[u] = 0$.*

(ii) *u can be lifted to an isometry in $M_n(A)$ if and only if $\partial[u] \leqslant 0$.*

(iii) *u can be lifted to a coisometry in $M_n(A)$ if and only if $\partial[u] \geqslant 0$.*

In this case, of course, $\partial : \mathrm{K}_1(D) \to \mathrm{K}_0(B)$.

In fact in "good" situations (when $\text{Ann}(A, B \otimes K) = 0$, where $\text{Ann}(A, J) = \{x \in A \mid xy = 0 \ \forall y \in J\}$) each unitary from $M_n(D)$ can be lifted to a partial isometry in $M_n(A)$ (see [6]).

In order to obtain this well-known result from Theorem 2, we shall prove the following general statement:

PROPOSITION 1. *Let J be a non-unital C^*-algebra which satisfies the following conditions:*

a) $K_0(J)$ *is generated by projections from* $\bigcup_{m\in \mathbf{N}} M_m(J)$ *(that is* $K_0(J) = K_{00}(J)$ *using notations from* [1]*).*

b) *For every projection P in* $\bigcup_{m\in \mathbf{N}} M_m(J)$ *one has* $[P]_{K_0(M(J))} = 0$ *($M(J)$ stands for the algebra of multipliers of J).*

c) *There exists an integer n such that the map* $GL_n(Q(J)) \to K_1(Q(J))$ *is surjective and* $n \geqslant \mathrm{gsr}(J) - 1$ *($Q(J) = M(J)/J$).*

Suppose J is an ideal in the unital C^*-algebra A with $\mathrm{Ann}(A, J) = 0$. *Then for every unitary u in $M_n(A/J)$ there exists a partial isometry U in $M_n(A)$ such that $\hat{U} = u$. If $\partial[u] = 0$, U can be taken unitary and if $\partial[u] \leqslant 0$ (resp. $\partial[u] \geqslant 0$) U can be taken isometry (resp. coisometry), but this last statement uses only the assumption* $n \geqslant \mathrm{gsr}(J) - 1$.

Proof. Before proving the proposition we remark that if $J = B \otimes K$ with B unital, conditions a), b) and c) are fulfilled (see [1]), for every $n \geqslant 1$.

It is well-known that the condition upon the annhilators gives two injective unital homomorphisms φ and ψ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \to & J & \hookrightarrow & A & \longrightarrow & A/J & \to & 0 \\ & & \| & & \downarrow \varphi & & \downarrow \psi & & \\ 0 & \to & J & \hookrightarrow & M(J) & \xrightarrow{\pi} & Q(J) & \to & 0. \end{array} \tag{1}$$

Of course one has a similar situation for matrix algebras.

By the naturality of the exact sequence of K-theory, one has a commutative diagram of groups

$$\begin{array}{ccccc} K_1(A/J) & \xrightarrow{\partial} & K_0(J) & \longrightarrow & K_0(A) \\ \downarrow \psi_* & & \| & & \downarrow \varphi_* \\ K_1(Q(J)) & \xrightarrow{\partial'} & K_0(J) & \to & K_0(M(J)). \end{array}$$

Conditions a) and b) show that $\partial' : K_1(Q(J)) \to K_0(J)$ is surjective. Take $x = \psi(u) \in M_n(Q(J))$ and put (by condition a)) $\partial'[x] = [P] - [Q]$ with P and Q projections in some matrix algebras over J. Since ∂' is surjective, by c) one can find y in $GL_n(Q(J))$ unitary such that $\partial'[y] = -[Q] \leqslant 0$ in $K_0(J)$. Applying Theorem 2 we get an isometry Y in $M_n(M(J))$ such that $\pi(Y) = y$. Consider $z = y^*x$, then $\partial'[z] = \partial'[x] + \partial'[y^*] = \partial'[x] - \partial'[y] = [P] \geqslant 0$. Again we apply Theorem 2 and get a coisometry Z in $M_n(M(J))$ such that $\pi(Z) = z$. Of course $X = YZ$ is a partial isometry in $M_n(M(J))$ and $\pi(X) = x$. Take W in $M_n(A)$ such that $\hat{W} = u$; by the commutativity of (1) we have

$$\pi(X - \varphi(W)) = \pi(X) - \pi \circ \varphi(W) = \pi(X) - \psi(\hat{W}) = \pi(X) - x = 0.$$

So $X - \varphi(W)$ belongs to $\varphi(M_n(J)) = M_n(J)$. Hence X is in $\varphi(M_n(A))$ and, since φ is injective, X comes from a partial isometry U in $M_n(A)$, that is $\varphi(U) = X$. Finally $\psi(u) = x = \pi(X) = \pi(\varphi(U)) = \psi(\hat{U})$ and we conclude that $u = \hat{U}$.

REMARK 2. Since one has "gsr $\leqslant$ tsr + 1", a "topological" version for Theorem 2 can be stated with the condition "$n \geqslant \mathrm{tsr}(J)$".

In the final part of the paper we shall use Theorem 2 to study the behaviour of stable ranks in extensions.

COROLLARY 2. *Let J be an ideal in a C^*-algebra A such that* $\mathrm{tsr}(J) = \mathrm{tsr}(A/J) = 1$. *Then* $\mathrm{tsr}(A) = 1$ *if and only if the "index" homomorphism* $\partial : \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ *is null.*

Proof. (For an alternative proof see [4].)

Suppose first that $\mathrm{tsr}(A) = 1$. One can consider only the unital case (see [7]). Take x in $\mathrm{GL}_n(A/J)$ and consider X in $M_n(A)$ such that $\hat{X} = x$. Since (Theorem 3.3 of [7]) $\mathrm{tsr}(M_n(A)) = 1$, for every $\varepsilon > 0$ there exists Y in $\mathrm{GL}_n(A)$ such that $\|X - Y\| < \varepsilon$ and so $\|x - \hat{Y}\| < \varepsilon$. But for small ε one has $[x]_{\mathrm{K}_1(A/J)} = [\hat{Y}]_{\mathrm{K}_1(A/J)}$ and we conclude that $\partial[x] = \partial[\hat{Y}] = 0$.

To prove the converse, let X in A and $\varepsilon > 0$. Put $x = \hat{X}$; since $\mathrm{tsr}(A/J) = 1$, there exists an invertible element y in A/J such that $\|x - y\| < \varepsilon/2$. By Remark 2 there exists an invertible element Y in A such that $\hat{Y} = y$ (by Theorem 2 this is true if y is unitary but in the general case one works with the polar decomposition of y). This means $\|\widehat{X - Y}\| < \varepsilon/2$. The norm is computed in A/J, so there exists Z in J such that $\|X - Y - Z\| < \varepsilon/2$. Since $1 + Y^{-1}Z$ belongs to $\tilde{J}$ and $\mathrm{tsr}(J) = 1$, there exists an invertible element T in $\tilde{J}$ such that $\|1 + Y^{-1}Z - T\| < (\varepsilon/2)\|Y\|$. Finally we obtain $\|X - YT\| < \varepsilon$ and YT is of course invertible in A. Hence $\mathrm{tsr}(A) = 1$. (See [7] for the description of the case "$\mathrm{tsr}(\cdot) = 1$".)

To give further applications of Theorem 2 we shall fix an ideal J in a unital C^*-algebra A. This context being fixed, we say that an integer n satisfies property (α) if the following (normal) subgroup of $\mathrm{GL}_n(A/J)$ acts transitively on $\mathrm{Lg}_n(A/J)$:

$$\{x \in \mathrm{GL}_n(A/J) / \partial[x] = 0\}.$$

As we shall see, M. A. Rieffel's relations concerning trs's of A, J and A/J can be slightly improved, following his techniques and Theorem 2.

PROPOSITION 2. *Suppose n has property* (α) *and* $n \geqslant \max(\mathrm{tsr}(J), \mathrm{tsr}(A/J))$. *Then* $n \geqslant \mathrm{tsr}(A)$.

Proof. The proof will be along the same lines as Theorem 4.11 of [7]. Take $X = (X_1, \ldots, X_n)$ in A^n and $\varepsilon > 0$. Put $x = (\hat{X}_1, \ldots, \hat{X}_n)$; x belongs to $(A/J)^n$ and, since $n \geqslant \mathrm{tsr}(A/J)$, there exists y in $\mathrm{Lg}_n(A/J)$ such that $\|x - y\| < \varepsilon/2$. Denote by E the element $(1, 0, \ldots, 0)$ in $\mathrm{Lg}_n(A)$, and take $e = \hat{E}$ which is in $\mathrm{Lg}_n(A/J)$. Since n satisfies (α), one can find t in $\mathrm{GL}_n(A/J)$ such that $y = te$, and $\partial[t] = 0$. Use Theorem 2 (Remark 2) and get T in $\mathrm{GL}_n(A)$ such that $\hat{T} = t$. Put $Y = TE$. So $\|\widehat{X - Y}\| = \|x - y\| < \varepsilon/2$. As in the proof of Corollary 2, there exists Z in J^n such that $\|X - Y - Z\| < \varepsilon/2$. As $(T^{-1}Y + T^{-1}Z)\hat{} = t^{-1}y = e$, $T^{-1}Y + T^{-1}Z$ belongs to $\tilde{J}^n$. Since $n \geqslant \mathrm{tsr}(J)$, there exists U in $\mathrm{Lg}_n(\tilde{J})$ such that $\|T^{-1}Y + T^{-1}Z - U\| < (\varepsilon/2)\|T\|$. This gives $\|Y + Z - TU\| < \varepsilon/2$, hence ($\mathrm{Lg}_n(\tilde{J}) \subset \mathrm{Lg}_n(A)$) TU belongs to $\mathrm{Lg}_n(A)$ and $\|X - TU\| < \varepsilon$, which proves that $\mathrm{tsr}(A) \leqslant n$.

COROLLARY 3. *If the "index" homomorphism* $\partial : \mathrm{K}_1(A/J) \to \mathrm{K}_0(J)$ *is null, then* $\mathrm{tsr}(A) \leqslant \max(\mathrm{tsr}(J), \mathrm{tsr}(A/J), \mathrm{gsr}(A/J))$.

EXAMPLE. Suppose that the quotient A/J is isomorphic to a commutative C^*-algebra $C(X)$ with $\dim(X) \leqslant 4$ and the "index" homomorphism is null. Then $\mathrm{tsr}(A) = \max(\mathrm{tsr}(J), \mathrm{tsr}(C(X)))$. Indeed, if $\dim(X) \leqslant 4$, one has (see [3]) $\mathrm{gsr}(C(X)) \leqslant \mathrm{csr}(C(X)) \leqslant \mathrm{trs}(C(X \times [0, 1])) \leqslant 3$. Since $\mathrm{GL}_k(C(X))$ obviously acts transitively on $\mathrm{Lg}_k(C(X))$, $k = 1, 2$ one gets $\mathrm{gsr}(C(X)) = 1$.

PROPOSITION 3. *Suppose n has property* (α) *and* $n \geqslant \mathrm{gsr}(J)$. *Then* $\mathrm{GL}_n(A)$ *acts transitively on* $\mathrm{Lg}_n(A)$.

Proof. The proof will be similar to the relation between csr's (see [2]). We will use the same notations as in Proposition 2. Take $X = (X_1, \ldots, X_n)$ in $\mathrm{Lg}_n(A)$ and let $x = (\hat{X}_1, \ldots, \hat{X}_n)$. x belongs to $\mathrm{Lg}_n(A/J)$ and, by property (α), there exists t in $\mathrm{GL}_n(A/J)$ such that $tx = e$, and $\partial[t] = 0$. By Theorem 2 there exists T in $\mathrm{GL}_n(A)$ such that $\hat{T} = t$. Take $Y = TX$; Y is in $\mathrm{Lg}_n(A)$, but it is also an element of $\tilde{J}^n$, because $\widehat{TX} = tx = e$. In this case Y is actually in $\mathrm{Lg}_n(\tilde{J})$ (if $Y = (Y_1, \ldots, Y_n)$, $Y_1^*Y_1 + \ldots + Y_n^*Y_n$ is invertible in A or, equivalently in $\tilde{J}$). Since $n \geqslant \mathrm{gsr}(J)$, there exists S in $\mathrm{GL}_n(\tilde{J}) \subset \mathrm{GL}_n(A)$ such that $SY = E$. Consequently $U = ST$ belongs to $\mathrm{GL}_n(A)$ and $UX = E$ which ends the proof.

COROLLARY 4. $\mathrm{gsr}(A) \leqslant \max(\mathrm{gsr}(J), \mathrm{csr}(A/J))$.

REMARK 3. If one denotes, for fixed J and A,

$$\mathrm{gsr}_J(A) = \min\{n \in \mathbf{N} \mid \text{every } m \geqslant n \text{ has property } (\alpha)\},$$

one gets

$$\mathrm{gsr}(A/J) \leqslant \mathrm{gsr}_J(A) \leqslant \mathrm{csr}(A/J);$$

if $\partial = 0$, then

$$\mathrm{gsr}_J(A) = \mathrm{gsr}(A/J)$$

$$\mathrm{tsr}(A) \leqslant \max(\mathrm{tsr}(J), \mathrm{tsr}(A/J), \mathrm{gsr}_J(A))$$

$$\mathrm{gsr}(A) \leqslant \max(\mathrm{gsr}(J), \mathrm{gsr}_J(A)).$$

COROLLARY 4. *If the "index" homomorphism is null, then*

$$\mathrm{gsr}(A) \leqslant \max(\mathrm{gsr}(J), \mathrm{gsr}(A/J)).$$

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