J. OPERATOR THEORY
22(1989), 73–84

© Copyright by INCREST, 1989

CONTINUITY PROPERTIES OF THE DISTANCE CONSTANT FUNCTION

FLORIN POP and DAN TUDOR VUZA

INTRODUCTION

The interest for distance estimates for operator algebras and subspaces goes back to W. B. Arveson's work ([1]) where it was proved that the distance constant of a nest algebra is equal to one.

Since then, a large variety of results were obtained in this direction and we refer the reader to [2], [6], [9], [10], [11], [12], [13], [14] for more information about these problems.

After K. R. Davidson and S. C. Power ([5]) and J. Kraus and D. R. Larson ([8]) showed that the distance constant can be infinite, it appeared necessary to study its behaviour, at least from the following point of view: how natural are the infinite value examples and how frequent are the finite values (especially $K = 1$)?

This paper originates in our attempt to get more insight to these problems. We prove several continuity properties for the distance constant function which show, in particular, that the infinite value is a consequence of a very natural fact, but that, however, the distance constant function is far from having a nice behaviour, even in the two-dimensional Hilbert space case.

We derive a general method for constructing reflexive subspaces with infinite distance constant and we study the properties of CSL-algebras in connection with their distance constant.

The regular subspaces introduced in Section III are shown to be points of continuity for the distance constant function provided that the Hilbert space is finite dimensional.

This enables us, in particular, to describe completely the distance constant function in the case $\dim \mathscr{H} = 2$.

1. PRELIMINARIES

Throughout this paper $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$, $\mathscr{C}^1(\mathscr{H})$, $\mathscr{P}(\mathscr{H})$ will denote a complex Hilbert space, the algebra of bounded operators on $\mathscr{H}$, the ideal of trace-class operators on $\mathscr{H}$ and the set of selfadjoint projections in $\mathscr{B}(\mathscr{H})$ respectively. The scalar product on $\mathscr{H}$ will be denoted by $(\cdot, \cdot)$. Recall that $\mathscr{C}^1(\mathscr{H})^* = \mathscr{B}(\mathscr{H})$.

A subspace $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ will always be assumed to be linear and uniformly closed.

A subspace $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is said to be reflexive if it contains every operator $T \in \mathscr{B}(\mathscr{H})$ with the property that Tx belongs to the closure of $\{Sx \ S \in \mathscr{S}\}$ for every $x \in \mathscr{H}$.

$\mathscr{S}$ is said to be hyperreflexive if there is $K \geqslant 1$ such that for every operator $T \in \mathscr{B}(\mathscr{H})$

$$\operatorname{dist}(T, \mathscr{S}) \leqslant K \sup\{\|PTQ\|;\ P, Q \in \mathscr{P}(\mathscr{H}),\ P\mathscr{S}Q = \{0\}\}.$$

The infimum of the constants K that occur above is called the distance constant of $\mathscr{S}$ and is denoted by $K(\mathscr{S})$. For subspaces that are not hyperreflexive define $K(\mathscr{S}) = \infty$. The distance constant function is by definition the function $\mathscr{S} \to K(\mathscr{S})$.

Note that in finite dimensional Hilbert spaces reflexivity and hyperreflexivity are equivalent, which is not the case in the infinite dimensional situation.

Note also that reflexive subspaces are $\sigma(\mathscr{B}(\mathscr{H}), \mathscr{C}^1(\mathscr{H}))$ (i.e. ultraweakly) closed.

The preannihilator of the subspace $\mathscr{S}$ is

$$\mathscr{S}_\perp = \{C \in \mathscr{C}^1(\mathscr{H}); \operatorname{trace} CS = 0 \ (\forall) S \in \mathscr{S}\}.$$

By ([9]), $\mathscr{S}$ is reflexive if and only if $\mathscr{S}_\perp$ is the $\|\cdot\|_1$-closed linear span of its rank one operators.

For a subspace $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ and operators $U, V \in \mathscr{B}(\mathscr{H})$ it is convenient to use the notations

$$U\mathscr{S}V = \{USV \mid S \in \mathscr{S}\} \quad \mathscr{S}^* = \{S^* \mid S \in \mathscr{S}\}.$$

For any two subspaces $\mathscr{S}$ and $\mathscr{T}$ in $\mathscr{B}(\mathscr{H})$ define the distance between them to be

$$\operatorname{dist}(\mathscr{S}, \mathscr{T}) = \max\{\sup_{\substack{S \in \mathscr{S} \\ \|S\| \leqslant 1}} \inf_{T \in \mathscr{T}} \|S - T\|,\ \sup_{\substack{T \in \mathscr{T} \\ \|T\| \leqslant 1}} \inf_{S \in \mathscr{S}} \|S - T\|\}.$$

As usual, $B(S, \varepsilon)$ denotes the open ball of radius $\varepsilon > 0$ and center $S \in \mathscr{B}(\mathscr{H})$.

2. A LOWER SEMICONTINUITY PROPERTY

We begin with a lemma which turns out to be useful in estimating distance constants.

Given a subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$, we shall denote by $[\mathcal{S}]$ the set of all pairs $[x, y] \in \mathcal{H} \times \mathcal{H}$ such that $(Sx, y) = 0$ for every $S \in \mathcal{S}$. The following notations will be also used:

$$[\mathcal{S}]_1 = \{[x, y] \in [\mathcal{S}] \,;\, \|x\| \leqslant 1, \ \|y\| \leqslant 1\}$$

$$[\mathcal{S}]_2 = \{[x, y] \in [\mathcal{S}] \,;\, \|x\| = \|y\| = 1\}.$$

2.1. LEMMA. *For every subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ and every $T \in \mathcal{B}(\mathcal{H})$ we have*

$$\sup\{\|PTQ\| \,;\, P, Q \in \mathcal{P}(\mathcal{H}), P\mathcal{S}Q = \{0\}\} = \sup\{|(Tx, y)| \,;\, [x, y] \in [\mathcal{S}]_1\} =$$

$$= \sup\{|(Tx, y)| \,;\, [x, y] \in [\mathcal{S}]_2\}.$$

Proof. The second equality being immediate, we shall prove only the first one.

"$\leqslant$". Let $P, Q \in \mathcal{P}(\mathcal{H})$ be such that $P\mathcal{S}Q = \{0\}$. Then $\|PTQ\| = \sup\{|(PTQu, v)| \,;\, u, v \in \mathcal{H}, \|u\| \leqslant 1, \|v\| \leqslant 1\} \leqslant \sup\{|(TQu, Pv)| \,;\, \|u\| \leqslant 1, \|v\| \leqslant 1\} \leqslant \sup\{|(Tx, y)| \,;\, [x, y] \in [\mathcal{S}]_1\}$ since $[Qu, Pv] \in [\mathcal{S}]_1$.

"$\geqslant$". For every $[x, y] \in [\mathcal{S}]_1$, let P_x and P_y be the orthogonal projections onto $\mathbf{C}x$ and $\mathbf{C}y$ respectively. Then

$$|(Tx, y)| = |(P_yTP_xx, y)| \leqslant \|P_yTP_x\| \leqslant \sup\{\|PTQ\| \,;\, P, Q \in \mathcal{P}(\mathcal{H}), \quad P\mathcal{S}Q = \{0\}\}$$

since $P_y\mathcal{S}P_x = \{0\}$.

2.2. THEOREM. *Let $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ be a ultraweakly closed subspace such that $\mathcal{S}_\perp$ is reflexive as a Banach space (i.e. $\mathcal{S}_\perp$ is canonically isomorphic with its second dual) and let $(\mathcal{S}_n)_{n\geqslant 1}$ be a sequence of hyperreflexive subspaces of $\mathcal{B}(\mathcal{H})$ such that*

$$\lim_{n\to\infty} \operatorname{dist}(\mathcal{S}_n, \mathcal{S}) = 0.$$

If the sequence $(K(\mathcal{S}_n))_{n\geqslant 1}$ is bounded, then $\mathcal{S}$ is hyperreflexive and

$$K(\mathcal{S}) \leqslant \limsup_{n\to\infty} K(\mathcal{S}_n).$$

Proof. Let $K = \limsup\limits_{n\to\infty} K(\mathcal{S}_n)$. By passing, if necessary, to a subsequence, we may assume that $K(\mathcal{S}_n) < K + 1/n$ for every $n \geqslant 1$. Given $T \in \mathcal{B}(\mathcal{H})$, Lemma 2.1

yields a sequence of pairs $[x_n, y_n] \in [\mathscr{S}_n]_1$ such that

$$\text{dist}(T, \mathscr{S}_n) \leqslant \left(K + \frac{1}{n} \right) |(Tx_n, y_n)|. \tag{1}$$

If $x \otimes y$ denotes the operator $z \to (z, y)x$ then $x_n \otimes y_n \in (\mathscr{S}_n)_\perp$ and $\|x_n \otimes y_n\|_1 \leqslant 1$. Since

$$\text{dist}((\mathscr{S}_n)_\perp, \mathscr{S}_\perp) = \text{dist}(\mathscr{S}_n, \mathscr{S})$$

([7], Proposition 2.9) there are operators $C_n \in \mathscr{S}_\perp$ such that

$$\lim_{n \to \infty} \|x_n \otimes y_n - C_n\|_1 = 0. \tag{2}$$

Since $\mathscr{S}_\perp$ is reflexive as a Banach space, its unit ball is $\sigma(\mathscr{C}^1(\mathscr{H}), \mathscr{B}(\mathscr{H}))$-compact; we may therefore assume that

$$\lim_{n \to \infty} C_n = C \in \mathscr{S}_\perp \tag{3}$$

in the $\sigma(\mathscr{C}^1(\mathscr{H}), \mathscr{B}(\mathscr{H}))$-topology.

Moreover, we may assume that

$$\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y \tag{4}$$

in the weak topology of $\mathscr{H}$.

It follows from (2)–(4) that $C = x \otimes y$ and that

$$\lim_{n \to \infty} (Tx_n, y_n) = (Tx, y).$$

By passing to the limit in (1) and taking into account that $[x, y] \in [\mathscr{S}]_1$ we obtain

$$\text{dist}(T, \mathscr{S}) \leqslant K |(Tx, y)| \leqslant K \sup\{|(Tu, v)|;\ [u, v] \in [\mathscr{S}]_1\};$$

hence $K(\mathscr{S}) \leqslant K$ and the proof is complete.

2.3. Corollary. *The reflexive subspaces of $\mathscr{B}(\mathscr{H})$ whose preannihilators are reflexive as Banach spaces are points of lower semicontinuity for the distance constant function. In particular, if $\mathscr{H}$ is finite dimensional the distance constant function is lower semicontinuous.*

2.4. Remarks. Theorem 2.2 provides a method for constructing reflexive subspaces with infinite distance constant.

Indeed, let $\mathscr{H}$ be finite dimensional, let $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ be a non reflexive subspace and let $(\mathscr{S}_n)_{n\geqslant 1}$ be a sequence of reflexive subspaces of $\mathscr{B}(\mathscr{H})$ such that

$$\lim_{n\to\infty} \operatorname{dist}(\mathscr{S}_n, \mathscr{S}) = 0.$$

Then, by Theorem 2.2, $\lim_{n\to\infty} K(\mathscr{S}_n) = \infty$, hence the reflexive subspace $\bigoplus_{n\geqslant 1} \mathscr{S}_n$ of $\mathscr{B}(\bigoplus_{n\geqslant 1} \mathscr{H})$ has infinite distance constant.

3. SOME CONTINUITY PROPERTIES

3.1. DEFINITION. Let $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ be a subspace. A pair $[x, y] \in [\mathscr{S}]$ is called *regular* if the only operator $S \in \mathscr{S}$ satisfying $Sx = S^*y = 0$ is $S = 0$.

$\mathscr{S}$ is called *regular* if every $[x, y] \in [\mathscr{S}]$ with $x \neq 0$, $y \neq 0$ is the limit of a sequence of regular pairs in $[\mathscr{S}]$.

The main result in this section asserts that every regular subspace of $\mathscr{B}(\mathscr{H})$ ($\dim \mathscr{H} < \infty$) is a point of continuity for the distance constant function, consequently the necessity of knowing procedures for constructing regular subspaces becomes important. Some of these are provided by the next proposition. Recall that, given a subspace $\mathscr{S}$ of $\mathscr{B}(\mathscr{H})$, the algebra $\mathscr{A}(\mathscr{S})$ associated to $\mathscr{S}$ is by definition the subalgebra of $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ formed by all operators $\begin{pmatrix} \lambda I & S \\ 0 & \mu I \end{pmatrix}$ with $\lambda, \mu \in \mathbf{C}$ and $S \in \mathscr{S}$. (I denotes the identity map on $\mathscr{H}$.) It is known [8] that $\mathscr{A}(\mathscr{S})$ is reflexive, respectively hyperreflexive, if and only if $\mathscr{S}$ has the corresponding property.

Throughout 3.2–3.4, it will be assumed that $\dim \mathscr{H} < \infty$. It is known that in this situation, the set of all subspaces of $\mathscr{B}(\mathscr{H})$ is a compact space with respect to the metric introduced in Section 1, and that the set of all subspaces of $\mathscr{B}(\mathscr{H})$ of a given dimension is a connected component of that compact space.

3.2. PROPOSITION. (i) *One-dimensional subspaces of $\mathscr{B}(\mathscr{H})$ are regular.*

(ii) *If $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ is regular and $\dim \mathscr{S} \leqslant \dim \mathscr{H} - 1$, then $\mathscr{A}(\mathscr{S})$ is regular.*

(iii) *Let $\mathscr{H}_1$, $\mathscr{H}_2$ be finite-dimensional Hilbert-spaces and let $\mathscr{S}_i \subset \mathscr{B}(\mathscr{H}_i)$ be regular subspaces with $\dim \mathscr{S}_i \leqslant \dim \mathscr{H}_i$ $(i = 1, 2)$. Then the subspace $\mathscr{S}_1 \oplus \mathscr{S}_2$ of $\mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2)$ is regular.*

Proof. (i) Let $\mathscr{S} = \mathbf{C}T$ for some $T \in \mathscr{B}(\mathscr{H})$, $T \neq 0$. If $[x, y] \in [\mathscr{S}]$ is not regular, then $Tx = T^*y = 0$. If we choose $u \in \mathscr{H}$ so that $Tu \neq 0$, it follows that $([x + n^{-1}u, y])_{n\geqslant 1}$ is a sequence of regular pairs in $[\mathscr{S}]$ which converges to $[x, y]$.

(ii) $[\mathscr{A}(\mathscr{S})]$ is formed by those pairs $[s \oplus t, u \oplus v]$ for which $(s, u) = (t, v) = 0$ and $[t, u] \in [\mathscr{S}]$. Let $\mathscr{C}$ denote the closure of the set of all regular pairs in $[\mathscr{A}(\mathscr{S})]$. Consider $[s \oplus t, u \oplus v] \in [\mathscr{A}(\mathscr{S})]$ with $s \oplus t \neq 0 \oplus 0$, $u \oplus v \neq 0 \oplus 0$ and distinguish four cases.

If $t \neq 0$ and $u \neq 0$, there is by hypothesis a sequence $([t_n, u_n])_{n\geqslant 1} \subset [\mathscr{S}]$ which converges to $[t, u]$. We may assume that $(t, t_n) \neq 0$ and $(u, u_n) \neq 0$. If we let

$$s_n = s - \frac{(s, u_n)}{(u, u_n)} u \quad \text{and} \quad v_n = v - \frac{(v, t_n)}{(t, t_n)} t,$$

then $([s_n \oplus t_n, u_n \oplus v_n])_{n\geqslant 1}$ is a sequence of regular pairs in $[\mathscr{A}(\mathscr{S})]$ which converges to $[s \oplus t, u \oplus v]$.

Suppose now that $t \neq 0$ and $u = 0$. If $\dim \mathscr{S}t \leqslant \dim \mathscr{H} - 2$, choose $u' \neq 0$ so that $u' \perp (\mathscr{S}t + \mathbf{C}s)$; it follows that $([s \oplus t, n^{-1}u' \oplus v])_{n\geqslant 1}$ is a sequence in $[\mathscr{A}(\mathscr{S})]$ which converges to $[s \oplus t, u \oplus v]$. By the first case, $[s \oplus t, n^{-1}u' \oplus v] \in \mathscr{C}$; hence, $[s \oplus t, u \oplus v] \in \mathscr{C}$. On the other side, if $\dim \mathscr{S}t = \dim \mathscr{H} - 1$, the map $S \to St$ $(S \in \mathscr{S})$ is one-to-one (because $\dim \mathscr{S} \leqslant \dim \mathscr{H} - 1$). Let (s_n) be a sequence in $\mathscr{H}$ such that $s_n \notin \mathscr{S}t$ and $s = \lim\limits_{n\to\infty} s_n$. It follows that $([s_n \oplus t, u \oplus v])_{n\geqslant 1}$ is a sequence of regular pairs in $[\mathscr{A}(\mathscr{S})]$ which converges to $[s \oplus t, u \oplus v]$.

The case $t = 0$, $u \neq 0$ is similarly treated.

Finally, suppose that $t = u = 0$. If $\dim \mathscr{H} = 1$ (and hence $\mathscr{S} = \{0\}$), then $[s \oplus t, u \oplus v]$ is already regular, as we must have $s \neq 0$ and $v \neq 0$. On the other hand, if $\dim \mathscr{H} \geqslant 2$, choose $t' \neq 0$ so that $(t', v) = 0$. It follows that $([s \oplus n^{-1}t', u \oplus v])_{n\geqslant 1}$ is a sequence in $[\mathscr{A}(\mathscr{S})]$ which converges to $[s \oplus t, u \oplus v]$. By the second case, $[s \oplus n^{-1}t', u \oplus v] \in \mathscr{C}$; hence, $[s \oplus t, u \oplus v] \in \mathscr{C}$.

The proof of (ii) is complete.

(iii) $[x_1 \oplus x_2, y_1 \oplus y_2] \in [\mathscr{S}_1 \oplus \mathscr{S}_2]$ iff $[x_i, y_i] \in [\mathscr{S}_i]$ for $i = 1, 2$; moreover, if $[x_1, y_1]$ and $[x_2, y_2]$ are regular, then $[x_1 \oplus x_2, y_1 \oplus y_2]$ is regular. Thus, (iii) will be a consequence of the above remarks and of the following assertion:

Let $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ be a regular subspace with $\dim \mathscr{S} \leqslant \dim \mathscr{H}$. Then every pair in $[\mathscr{S}]$ is the limit of a sequence of regular pairs.

Indeed, let $[x, y] \in [\mathscr{S}]$ be given. If $x \neq 0$ and $y \neq 0$, $[x, y]$ is the limit of a sequence of regular pairs by hypothesis.

Suppose that $x \neq 0$ and $y = 0$. If $\dim \mathscr{S}x \leqslant \dim \mathscr{H} - 1$, choose $y' \neq 0$ so that $y' \perp \mathscr{S}x$. It follows that $([x, n^{-1}y'])_{n\geqslant 1}$ is a sequence in $[\mathscr{S}]$ which converges to $[x, y]$; as every $[x, n^{-1}y']$ belongs to the closure of the set of regular pairs in $[\mathscr{S}]$, we get that $[x, y]$ belongs to that closure too. On the other side, if $\dim \mathscr{S}x = \dim \mathscr{H}$, the map $S \to Sx$ $(S \in \mathscr{S})$ is one-to-one (because $\dim \mathscr{S} \leqslant \dim \mathscr{H}$) and hence, $[x, y]$ is regular.

The cases $x = 0$, $y \neq 0$ and $x = y = 0$ are similarly treated.

The next lemma will be used in the proof of the continuity theorem.

3.3. LEMMA. *Let $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ be a subspace and let $[x_0, y_0]$ be a regular pair in $[\mathscr{S}]$. Then there are $\varepsilon > 0$, a basis $(S_1, \ldots, S_n)$ in $\mathscr{S}$ and two functions x, y: $B(S_1, \varepsilon) \times$*

$\ldots \times B(S_n, \varepsilon) \to \mathscr{H}$ *satisfying:*

(i) $x(T_1, \ldots, T_n)$ *(respectively* $y(T_1, \ldots, T_n)$*) is analytic (respectively anti-analytic) with respect to the entries of* $T_1, \ldots, T_n$.

(ii) $x(S_1, \ldots, S_n) = x_0$, $y(S_1, \ldots, S_n) = y_0$.

(iii) $(T_i x(T_1, \ldots, T_n), y(T_1, \ldots, T_n)) = 0$ *for* $1 \leqslant i \leqslant n$ *and* $(T_1, \ldots, T_n) \in \in B(S_1, \varepsilon) \times \ldots \times B(S_n, \varepsilon)$.

Proof. Let $(S_1, \ldots, S_n)$ be a basis of $\mathscr{S}$ such that $S_1, \ldots, S_q$ is a basis of $\{S \in \mathscr{S} \ ; \ Sx_0 = 0\}$ (if the latter space is $\{0\}$, take $q = 0$).

Consider the system of equations

$$f(T_i, \lambda_1, \ldots, \lambda_n) = 0 \quad 1 \leqslant i \leqslant n$$

where

$$f(T, \lambda_1, \ldots, \lambda_n) = \left(T\left(x_0 + \sum_{j=1}^{q} \lambda_j S_j^* y_0\right), y_0 + \sum_{j=q+1}^{n} \lambda_j S_j x_0\right)$$

is an analytic function of $\lambda_1, \ldots, \lambda_n$ and of the entries of T.

The lemma will be a direct consequence of the implicit function theorem, provided that the determinant of the matrix

$$\left(\frac{\partial f}{\partial \lambda_j}(S_i, 0, \ldots, 0)\right)_{1 \leqslant i, j \leqslant n}$$

is different from 0.

But the above determinant is equal, up to sign, to the product of the Gram determinant of the vectors $S_1^* y_0, \ldots, S_q^* y_0$ and the Gram determinant of the vector $S_{q+1} x_0, \ldots, S_n x_0$. The former is nonzero by the regularity of $\mathscr{S}$, while the latter is nonzero by the choice of $S_1, \ldots, S_n$. The lemma is proved.

3.4. THEOREM. *Regular reflexive subspaces of* $\mathscr{B}(\mathscr{H})$ *(*$\dim \mathscr{H} < \infty$*) are points of continuity for the distance constant function. In particular, such a subspace is an interior point in the set of reflexive subspaces of* $\mathscr{B}(\mathscr{H})$.

Proof. Let $\mathscr{S} \subset \mathscr{B}(\mathscr{H})$ be reflexive, regular, $\dim \mathscr{S} = n$ and let $\varepsilon > 0$ be arbitrary.

Since $[\mathscr{S}]_2$ is compact, choose $[x_1, y_1], \ldots, [x_p, y_p]$ in $[\mathscr{S}]_2$ such that for every $[x, y] \in [\mathscr{S}]_2$ there is $1 \leqslant i \leqslant p$ with

$$\|x - x_i\| \leqslant \varepsilon, \quad \|y - y_i\| \leqslant \varepsilon.$$

By hypothesis, the $[x_i, y_i]$'s can be taken to be regular. By Lemma 3.3 applied to each $[x_i, y_i]$, $1 \leqslant i \leqslant p$, there are $\varepsilon_i > 0$, a basis $(S_1^i, \ldots, S_n^i)$ in $\mathscr{S}$, and functions $x^i, y^i : B(S_1^i, \varepsilon_i) \times \ldots \times B(S_n^i, \varepsilon_i) \to \mathscr{H}$ as in Lemma 3.3.

We may also assume that $\|S_k^i\| = 1$ for $1 \leqslant i \leqslant p$, $1 \leqslant k \leqslant n$, and that the ε_i's are small enough so that any system $(T_1^i, \ldots, T_n^i) \in B(S_1^i, \varepsilon_i) \times \ldots \times B(S_n^i, \varepsilon_i)$ is linearly independent and verifies

$$\|x^i(T_1^i, \ldots, T_n^i) - x_i\| \leqslant \varepsilon/2$$

$$\|y^i(T_1^i, \ldots, T_n^i) - y_i\| \leqslant \varepsilon/2.$$

Define $\delta(\varepsilon) = \min\{\varepsilon, \varepsilon_1, \ldots, \varepsilon_p\}$.

Let now $\mathscr{T} \subset \mathscr{B}(\mathscr{H})$ be an n-dimensional subspace such that $\mathrm{dist}(\mathscr{T}, \mathscr{S}) < \delta(\varepsilon)$.

Choose $T_k^i \in \mathscr{T}$ such that $\|S_k^i - T_k^i\| < \delta(\varepsilon)$ for $1 \leqslant i \leqslant p$, $1 \leqslant k \leqslant n$. Let

$$\bar{x}_i = \frac{x^i(T_1^i, \ldots, T_n^i)}{\|x^i(T_1^i, \ldots, T_n^i)\|}, \quad \bar{y}_i = \frac{y^i(T_n^i, \ldots, T_n^i)}{\|y^i(T_1^i, \ldots, T_n^i)\|},$$

hence $[\bar{x}_i, \bar{y}_i] \in [\mathscr{T}]_2$ and $\|\bar{x}_i - x_i\| \leqslant \varepsilon$, $\|\bar{y}_i - y_i\| \leqslant \varepsilon$.

If $A \in \mathscr{B}(\mathscr{H})$ is arbitrary, then

$$\mathrm{dist}(A, \mathscr{T}) \leqslant \mathrm{dist}(A, \mathscr{S}) + 2\|A\|\mathrm{dist}(\mathscr{S}, \mathscr{T}) \leqslant$$

$$\leqslant K(\mathscr{S})\sup\{|(Ax, y)| \; ; \; [x, y] \in [\mathscr{S}]_2\} + 2\varepsilon\|A\| \leqslant$$

$$\leqslant K(\mathscr{S}) \sup_{1 \leqslant i \leqslant p} \{|(Ax_i, y_i)|\} + 4\varepsilon K(\mathscr{S})\|A\| \leqslant$$

$$\leqslant K(\mathscr{S}) \sup_{1 \leqslant i \leqslant p} \{|(A\bar{x}_i, \bar{y}_i)|\} + 6\varepsilon K(\mathscr{S})\|A\| \leqslant$$

$$\leqslant K(\mathscr{S})\sup\{|(Ax, y)|, \; [x, y] \in [\mathscr{T}]_2\} + 6\varepsilon K(\mathscr{S})\|A\|.$$

Replacing in the above inequality A by $A - T$ with $T \in \mathscr{T}$ we get $\mathrm{dist}(A, \mathscr{T}) \leqslant \leqslant K(\mathscr{S})\sup\{|(Ax, y)| \; ; \; [x, y] \in [\mathscr{T}]_2\} + 6\varepsilon K(\mathscr{S})\mathrm{dist}(A, \mathscr{T})$, hence $K(\mathscr{T}) \leqslant \dfrac{K(\mathscr{S})}{1 - 6\varepsilon K(\mathscr{S})}$ if $\mathrm{dist}(\mathscr{T}, \mathscr{S}) < \delta(\varepsilon)$. This implies that $\mathscr{T}$ is reflexive and that $\mathscr{S}$ is a point of upper semicontinuity for the distance constant function. Since the lower semicontinuity was already established, the proof is complete.

We return now to the case when the Hilbert space $\mathscr{H}$ is arbitrary. A CSL algebra is a reflexive subalgebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ with commutative invariant subspace lattice $\mathrm{Lat}\,\mathscr{A} = \{P \in \mathscr{P}(\mathscr{H}) \; ; \; (I - P)\mathscr{A}P = \{0\}\}$.

3.5. Proposition. *Every* CSL *algebra is a point of continuity for the restriction of the distance constant function to the class of Banach subalgebras of* $\mathscr{B}(\mathscr{H})$.

Proof. The main ingredient is the following result due to M. D. Choi and K. R. Davidson ([3] and [4]):

"*If* $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ *is a* CSL-*algebra then there is a constant* $c > 0$ *such that for enough small* $\varepsilon > 0$ *and for every Banach subalgebra* $\mathscr{B} \subset \mathscr{B}(\mathscr{H})$, $\mathrm{dist}(\mathscr{A}, \mathscr{B}) \leqslant \varepsilon$ *entails the existence of a lattice isomorphism* $\varphi\colon \mathrm{Lat}\,\mathscr{A} \to \mathrm{Lat}\,\mathscr{B}$ *with* $\|P - \varphi(P)\| \leqslant c\varepsilon$ *for every* P *in* $\mathrm{Lat}\,\mathscr{A}$".

From this point the proof is similar to the last part of the proof of 3.4, so we shall omit it.

4. SOME EXAMPLES

4.1. It was proved in [9] that one-dimensional subspaces of $\mathscr{B}(\mathscr{H})$ are hyperreflexive. Hence, Theorem 3.4 shows that the distance constant function is continuous on the set of one dimensional subspaces of $\mathscr{B}(\mathscr{H})$, at least in the situation $\dim \mathscr{H} < \infty$. It seems to be not known whether there are one-dimensional subspaces $\mathscr{S}$ with $K(\mathscr{S}) \neq 1$.

The following notion turns out to be useful in the study of the continuity points of the distance constant function.

4.2. DEFINITION. Call two subspaces $\mathscr{S}, \mathscr{T} \subset \mathscr{B}(\mathscr{H})$ *equivalent* if there are two unitary operators U and V such that either $\mathscr{T} = U\mathscr{S}V$ or $\mathscr{T} = U\mathscr{S}^*V$.

4.3. PROPOSITION. (i) *A subspace equivalent to a reflexive subspace is reflexive.*

(ii) *Equivalent reflexive subspaces have the same distance constant.*

The proof is easily obtained from Lemma 2.1.

4.4. As an example of equivalence, consider the following situation. In [8] Kraus and Larson showed that the subspace $\mathscr{S} = \bigoplus_{n\geqslant 1} \mathscr{S}_n$ of $\mathscr{B}(\bigoplus_{n\geqslant 1} \mathbf{C}^2)$ where $\mathscr{S}_n$ consists of all matrices

$$\begin{pmatrix} 0 & \lambda \\ \mu & -n(\lambda + \mu) \end{pmatrix} \quad \text{with } \lambda, \mu \in \mathbf{C}$$

is a reflexive subspace with infinite distance constant. $\mathscr{S}$ itself is not a subalgebra; it is however equivalent to a subalgebra which turns out to be, by virtue of Proposition 4.1 a reflexive algebra with infinite distance constant.

The equivalence is obtained by remarking that $\mathscr{S}_n = U_n\mathscr{A}_n$ where $\mathscr{A}_n$ is the algebra of operators

$$\begin{pmatrix} \lambda & \mu \\ 0 & \lambda + \dfrac{1}{n}\mu \end{pmatrix}, \quad \lambda, \mu \in \mathbf{C} \quad \text{and} \quad U_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The fact that $\mathscr{A}$, and hence $\mathscr{S}$, is not hyperreflexive can also be obtained without resorting to any distance estimate, directly from 2.2.

4.5. In order to illustrate the general theory exposed in Sections 2 and 3, we shall investigate the points of continuity of the distance constant function defined on the set of all subspaces of $\mathscr{B}(\mathbf{C}^2)$. By virtue of 4.1, only the cases $\dim \mathscr{S} = 2$ and $\dim \mathscr{S} = 3$ are to be considered. It follows from Theorem 2.2 that the function under consideration is continuous at every $\mathscr{S} \subset \mathscr{B}(\mathbf{C}^2)$ with $K(\mathscr{S}) = \infty$; hence we have to study its behaviour only at the reflexive subspaces of $\mathscr{B}(\mathbf{C}^2)$.

4.6. Every reflexive three-dimensional subspace $\mathscr{S}$ of $\mathscr{B}(\mathbf{C}^2)$ is equivalent to the algebra $\mathscr{A}$ of all matrices $\begin{pmatrix} \lambda & \mu \\ 0 & \nu \end{pmatrix}$ with $\lambda, \mu, \nu \in \mathbf{C}$; consequently, $K(\mathscr{S})=1$ by 4.3 and [1]. If $\mathscr{A}_n$ denotes the non-reflexive subspace of all matrices $\begin{pmatrix} \lambda & \mu \\ \mu/n & \nu \end{pmatrix}$ with $\lambda, \mu, \nu \in \mathbf{C}$, then $\lim\limits_{n\to\infty} \operatorname{dist}(\mathscr{A}_n, \mathscr{A}) = 0$. Hence, the distance constant function is discontinuous at every reflexive three-dimensional subspace of $\mathscr{B}(\mathbf{C}^2)$; however, its restriction to the set of reflexive three-dimensional subspace is constant, and hence continuous on that set.

4.7. In order to study the case $\dim \mathscr{S} = 2$, we introduce the subspaces $\mathscr{S}_{\alpha,\beta}$ for α, β positive real numbers. $\mathscr{S}_{\alpha,\beta}$ is the set of all matrices

$$\begin{pmatrix} \lambda & \mu \\ 0 & \alpha\lambda + \beta\mu \end{pmatrix} \quad \lambda, \mu \in \mathbf{C}.$$

It is not difficult to see that every reflexive two-dimensional subspace of $\mathscr{B}(\mathbf{C}^2)$ is equivalent to one and only one of the subspaces listed below:

a) the subspaces $\mathscr{S}_{\alpha,\beta}$ with $\alpha \geqslant 0$, $\beta > 0$;

b) the subspace $\mathscr{S}_{0,0}$;

c) the subalgebra of all matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda, \mu \in \mathbf{C}$. All subspaces occuring in a) and c) are regular. Consequently, the distance constant function is continuous at these subspaces by Theorem 3.4, and hence, it is continuous at every subspace equivalent to one of the above indicated, by 4.3.

On the other side, the distance constant function is discontinuous at $\mathscr{S}_{0,0}$, and hence, at every subspace equivalent to it. This happens because $\lim\limits_{n\to\infty} \operatorname{dist}(\mathscr{S}_{0,0}, \mathscr{S}_{n^{-1},0}) = 0$ and $K(\mathscr{S}_{n^{-1},0}) = \infty$. We shall see that, unlike the case of three-dimensional subspaces, the restriction of the distance constant function to the set of two dimensional reflexive subspaces of $\mathscr{B}(\mathbf{C}^2)$ is discontinuous at $\mathscr{S}_{0,0}$. Indeed, $\lim\limits_{n\to\infty} \operatorname{dist}(\mathscr{S}_{0,0}, \mathscr{S}_{n^{-1},n^{-2}}) = 0$ while $\lim\limits_{n\to\infty} K(\mathscr{S}_{n^{-1},n^{-2}}) = \infty$, as follows from the next lemma.

4.8. Lemma. *$K(\mathscr{S}_{\alpha,\beta}) \geqslant ((\alpha+1)^2 + \beta^2)^{-1/2} \min \left\{ \left(1 + \frac{\beta^2}{\alpha^2}\right)^{1/2} \left(1 + \frac{1}{\beta^2}\right)^{1/2}, \frac{\alpha}{\beta|\alpha - 1|} \right\}$ for $\alpha, \beta > 0$, $\alpha \neq 1$.*

Proof. Let e_1, e_2 be the canonical basis in $\mathbf{C}^2$.

Any couple of vectors in $[\mathscr{S}_{\alpha\,\beta}]$ is equal, up to multiplication by scalars, to $[e_1, e_2]$ or to

$$[\alpha(\alpha^2+\beta^2)^{-1/2}e_1+\beta(\alpha^2+\beta^2)^{-1/2}e_2,\ \beta(1+\beta^2)^{-1/2}e_1-(1+\beta^2)^{-1/2}e_2].$$

It $T=\begin{pmatrix}0 & 0\\ c & 1\end{pmatrix}$ where $c=\beta-\dfrac{\beta}{\alpha}$ then

$$\sup\{|(Tx,y)|;\,[x,y]\in[\mathscr{S}_{\alpha,\beta}]\}=\max\left\{\left(1+\frac{\beta^2}{\alpha^2}\right)^{-1/2}\left(1+\frac{1}{\beta^2}\right)^{-1/2},\ \frac{\beta|\alpha-1|}{\alpha}\right\},$$

$$\operatorname{dist}(T,\mathscr{S}_{\alpha,\beta})=\sup\{|\operatorname{trace}XT|\,;\,X\in(\mathscr{S}_{\alpha,\beta})_\perp\ \|X\|_1\leqslant 1\}\geqslant$$

$$\geqslant\left|\operatorname{trace}\begin{pmatrix}0 & 0\\ c & 1\end{pmatrix}\begin{pmatrix}-\alpha & 0\\ -\beta & 1\end{pmatrix}\right|\left(\left\|\begin{pmatrix}-\alpha & 0\\ -\beta & 1\end{pmatrix}\right\|_1\right)^{-1}=((\alpha+1)^2+\beta^2)^{-1/2},$$

which concludes the proof.

4.9. We have seen (3.5) that the restriction of the distance constant function to the set of all subalgebras of $\mathscr{B}(\mathscr{H})$ is continuous at every CSL-algebra.

In general, a CSL-algebra is not a point of continuity for the (unrestricted) distance constant function. An example is provided by 4.6. Note that the CSL-algebra considered there is actually a nest algebra.

There are, however, CSL-algebras which are points of continuity for the unrestricted distance constant function. For instance, the algebra $\mathscr{D}_n$ of all diagonal $n\times n$ matrices is a regular CSL-algebra of $\mathscr{B}(\mathbf{C}^n)$, hence a point of continuity by Theorem 3.4.

By taking into account 3.2, it is not difficult to see that actually every subalgebra of $\mathscr{D}_n$ (not necessarily unital) is also regular.

On the other hand, there are, at least in the case $\dim\mathscr{H}<\infty$, subalgebras of $\mathscr{B}(\mathscr{H})$ of arbitrary large dimension which are points of continuity for the (unrestricted) distance constant function, although they are not CSL-algebras.

This is a consequence of Theorem 3.4 and Proposition 3.2 applied repeatedly to the algebras $\mathscr{S}_{1,\beta}$ $(\beta\neq 0)$ introduced in 4.7.

REFERENCES

1. ARVESON, W., Interpolation problems in nest algebras, *J. Funct. Anal.*, 20(1975), 208–233.
2. ARVESON, W., *Ten lectures on operator algebras*, CBMS Regional Conf. Ser. Math., No. 55, Amer. Math. Soc., Providence, 1984.

3. CHOI, M. D.; DAVIDSON, K. R., Perturbations of matrix algebras, *Michigan Math. J.*, 33(1986), 273–287.
4. DAVIDSON, K. R., Perturbations of reflexive operator algebras, *J. Operator Theory*, 15(1986), 289–306.
5. DAVIDSON, K. R.; POWER, S. C., Failure of the distance formula, *J. London Math. Soc.*, 32(1985), 157–165.
6. GILFEATHER, F.; LARSON, D. R., Nest-subalgebras of von Neumann algebras: commutants modulo compacts and distance estimates, *J. Operator Theory*, 7(1982), 279–302.
7. KATO, T., *Perturbation theory for linear operators*, Springer Verlag, 1976.
8. KRAUS, J.; LARSON, D. R., Some applications of a technique for constructing reflexive operator algebras, *J. Operator Theory*, 13(1985), 227–236.
9. KRAUS, J.; LARSON, D. R., Reflexivity and distance formulae, *Proc. London Math. Soc.*, 53(1986), 340–356.
10. LANCE, E. C., Cohomology and perturbations of nest algebras, *Proc. London Math. Soc.*, 43(1981), 334–356.
11. LARSON, D. R., Annihilators of operators algebras, in *Topics in modern operator theory*, Birkhäuser Verlag, Basel, 1982, pp. 119–130.
12. POP, F., Perturbations of nest-subalgebras of von Neumann algebras *J. Operator Theory*, 21(1989), 139–144.
13. POWER, S. C., The distance to upper triangular operators, *Math. Proc. Cambridge Philos. Soc.*, 88(1980), 327–329.
14. ROSENOER, S., Distance estimates for von Neumann algebras, *Proc. Amer. Math. Soc.*, 86(1982), 248–252.

FLORIN POP and DAN TUDOR VUZA
Department of Mathematics,
INCREST
Bdul Păcii 220, 79622 Bucharest,
Romania.

Received October 19, 1987; revised December 12, 1988.