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K-THEORY FOR DISCRETE SUBGROUPS OF THE LORENTZ GROUPS

DANIEL SCHWALBE

INTRODUCTION

For any discrete group, Γ, Paul Baum and Alain Connes [3], [4], have defined a map

$$\mathrm{H}_*(\Gamma, \mathrm{F}\Gamma) \to \mathrm{K}_*[C^*_r(\Gamma)] \otimes_{\mathbf{Z}} \mathbf{C}$$

which they conjecture to be an isomorphism. Here $\mathrm{H}_*(\Gamma, \mathrm{F}\Gamma)$ is the homology of the discrete group Γ with coefficients in the module $\mathrm{F}\Gamma$ whose elements are finite $\mathbf{C}$-linear combinations of elements of finite order in Γ.

There are two steps in defining the map. First, they have geometrically defined topological K-theory groups $\mathrm{K}^i(X, G)$ for a Lie group or a countable discrete group acting by diffeomorphisms on a C^∞ manifold, X. For a discrete group there is a Chern character

$$\mathrm{K}^i(X, \Gamma) \to \mathrm{H}(\hat{X} \times_\Gamma \mathrm{E}\Gamma).$$

Here $\hat{X}$ is the manifold defined by $\hat{X} = \{(x, \gamma) \in X \times \Gamma \mid \gamma \text{ is of finite order and } x\gamma = x\}$. Γ acts on $\hat{X}$ by $(x, \gamma)\alpha = (x\alpha, \alpha^{-1}\gamma\alpha)$. $\mathrm{E}\Gamma$ is the contractible space which is a principal Γ-bundle over the classifying space of Γ. $\mathrm{H}(\hat{X} \times_\Gamma \mathrm{E}\Gamma)$ is the homology with complex coefficients of the homotopy quotient $(\hat{X} \times_\Gamma \mathrm{E}\Gamma)$. For the case where X is a point, this group is isomorphic to $\mathrm{H}_*(\Gamma, \mathrm{F}\Gamma)$. For discrete groups Baum and Connes have shown that the Chern character gives a rational isomorphism of groups.

Baum and Connes also define an index homomorphism

$$\mathrm{K}^*(X, G) \to \mathrm{K}_*[C_0(X) \rtimes G]$$

which they conjecture to be an isomorphism. In this paper we use Marc Rieffel's work on Morita equivalence [17], [18], [19], to show that if the Baum-Connes conjecture is true for a Lie group, it is also true for discrete subgroups of that Lie group. For the case of a semisimple Lie group with finite center, the Final Object Theorem of Baum-Connes implies that the Baum-Connes conjecture is equivalent to the so called generalized Connes-Kasparov conjecture. We state and give a proof of a weaker version of the Final Object Theorem which suffices for the case of a discrete subgroup of a Lie group. Kasparov's proof of the generalized Connes-Kasparov conjecture for the Lorentz groups then implies the Baum-Connes conjecture is true for discrete subgroups of the Lorentz groups.

1. DEFINITION OF $K^i(X, G)$

The geometric K-theory, $K^i(X, G)$, $i = 0$ or 1, where G is a Lie group with possibly countably many connected components and X is a C^∞ manifold on which G acts by diffeomorphisms, is defined as follows [3], [4]:

Let $\mathscr{C}(X, G)$ be a category of pairs (W, ρ) such that

(i) W is a C^∞ manifold with a given proper action of G by diffeomorphisms

$$W \times G \to W, \quad (w, g) \mapsto wg.$$

(ii) ρ is a G-equivariant submersion from W to X.

A morphism from (W_1, ρ_1) to (W_2, ρ_2) in the category is a C^∞ G-equivariant map $f: W_1 \to W_2$ such that the following diagram commutes:

$$\begin{array}{ccc} W_1 & \xrightarrow{f} & W_2 \\ & {\scriptstyle \rho_1} \searrow \quad \swarrow {\scriptstyle \rho_2} & \\ & X & \end{array}.$$

We denote by τ_W the kernel of the induced map $d\rho: TW \to TX$. A morphism $f: W_1 \to W_2$ induces a map in K-theory

$$f!: K_i[C_0(\tau_{W_1}) \rtimes G] \to K_i[C_0(\tau_{W_2}) \rtimes G] \quad i = 0 \text{ or } 1.$$

If we have two morphisms $f_1: W_1 \to W_2$ and $f_2: W_2 \to W_3$, the push forward maps compose $f_2! \circ f_1! = (f_2 \circ f_1)!$ [11].

The geometric group $K^i(X, G)$, $i = 0$ or 1, is defined as

$$K^i(X, G) = \varinjlim_{\mathscr{C}(X,G)} K_i[C_0(\tau_W) \rtimes G]$$

where the direct limit over the category is defined using the maps $f!$.

Baum and Connes define an index homomorphism $\mu_{X,G}: \mathrm{K}^i(X, G) \to \mathrm{K}_i[C_0(X) \rtimes G]$, which they conjecture to be an isomorphism. We describe the map in a later section. One of the main theorems of the paper (see Section 5) is that this map is compatible with induction of K-theory groups from a subgroup of a Lie group to the group.

2. MORITA EQUIVALENCE AND ANALYTIC K-THEORY

For Γ a discrete subgroup of a Lie group, work of Marc Rieffel [17], [18] [19], leads to an induction isomorphism of K-theory groups,

$$M_{G,X}: \mathrm{K}_i[C_0(X) \rtimes \Gamma] \to \mathrm{K}_i[C_0(\tilde{X}) \rtimes G],$$

where X is a Γ-manifold and $\tilde{X}$ is the induced G-space $X \times_\Gamma G$. We describe this map as an invertible KK element in $\mathrm{KK}(C_0(X) \rtimes \Gamma, C_0(\tilde{X}) \times G)$. We refer to [13] for definitions and properties of KK groups.

Rieffel [18] shows $C_c(G, X)$ is a left $C_c(\Gamma, X)$, right $C_c(G, \tilde{X})$ bimodule. Here $C_c(Y, Z)$ denotes compactly supported functions on the space $Y \times Z$, and $C_c(\Gamma, X)$ and $C_c(G, \tilde{X})$ are algebras using the convolution product. He defines $C_c(\Gamma, X)$ and $C_c(G, \tilde{X})$ valued "inner products" on $C_c(G, X)$ for which the completion of $C_c(G, X)$, denoted $C^*(G, X)$, is an A-C equivalence bimodule for $A = C_0(X) \rtimes \Gamma$ and $C = C_0(\tilde{X}) \rtimes G$. (See [18] for the definition of an A-C equivalence bimodule.)

To set notation we describe the A and C valued "inner products" and the left A and right C actions. When a group acts on a space we denote the corresponding group action on the function space by a superscripted group element. So if a group G acts on a space X, and $f \in C_c(X)$ then $f^g(x) = f(g^{-1}x)$ if G acts on the left, and $f^g(x) = f(xg)$ if G acts on the right. It is enough to define these formulas for the dense subspaces, $C_c(\Gamma, X)$, $C_c(G, \tilde{X})$, $C_c(G, X)$ and then extend by density.

For $f_1, f_2 \in C_c(G, X)$, define $\langle f_1, f_2 \rangle_C \in C_c(G, \tilde{X})$, and $\langle f_1, f_2 \rangle_A \in C_c(\Gamma, X)$ by

$$\langle f_1, f_2 \rangle_C(g, x, g') = \sum_{\gamma \in \Gamma} (\bar{f_1}(\gamma g'))^\gamma(x)(f_2(\gamma g' g))^\gamma(x)$$

$$\langle f_1, f_2 \rangle_A(\gamma, x) = \int_G f_1(s, x)\bar{f_2}(\gamma^{-1}s)(x)\mathrm{d}s.$$

We assume G is unimodular to simplify notation.

A left action of $C_c(\Gamma, X)$ on $C_c(G, X)$ is defined by

$$k[\gamma]f = kf^{\gamma^{-1}}$$

where $k \in C_c(X)$, and $C_c(\Gamma, X)$ is thought of as finite formal sums, and the formula is extended by linearity.

A right action of $C_c(G, \tilde{X})$ on $C_c(G, X)$ is defined by

$$(f\chi)(g, x) = \int_G f(sg, x)\chi(s, x, gs^{-1})ds$$

for $\chi \in C_c(G, \tilde{X})$.

Let $E = C^*(X, G)$, $A = C_0(X) \rtimes \Gamma$, and $C = C_0(\tilde{X}) \times G$. With the operations just defined, E is a Hilbert B-module for C. Its dual, denoted E', is a Hilbert B-module for A. Being an equivalence bimodule, E has more structure than that of a Hilbert B-module. In particular, if we let 0_E denote the zero operator on E, we have the following theorem.

THEOREM. *The pair $\alpha_{G,X} = (E, 0_E)$ is an invertible* KK *element in the group* KK(A, C), *with $\alpha'_{G,X} = (E', 0_{E'})$ as its inverse. That is, $\alpha_{G,X} \# \alpha'_{G,X}$ and $\alpha'_{G,X} \# \alpha_{G,X}$ are the identity elements for the Kasparov product in the groups* KK(A, A) *and* KK(C, C) *respectively where $a \# b$ denotes the Kasparov product of a and b.*

Proof. The modules E and E' are assumed to have the trivial grading. The operator in a KK pair must satisfy three compactness conditions. For $(E, T) \in \in \mathrm{KK}(A, C)$, the elements $[M_\varphi, T]$, $(T^2 - 1)M_\varphi$ and $(T - T^*)M_\varphi$ must be compact Hilbert B-module operators for all $\varphi \in A$ where M_φ denotes the operator arising from the left action of $\varphi \in A$ on E. In our case the operator is the zero operator, so the only thing to check is that M_φ is a compact operator for all $\varphi \in A$.

The key point is that E has an A-valued inner product whose image is dense in A. The inner products also satisfy the relation $\langle x, y\rangle_A z = x\langle y, z\rangle_C$. Thus for $\varphi \in A$ there exist $\lambda_i, \lambda'_i \in E$ such that $\varphi = \sum \langle \lambda_i, \lambda'_i\rangle_A$, and therefore

$$M_\varphi(f) = \sum \langle \lambda_i, \lambda'_i\rangle_A f = \sum \lambda_i\langle \lambda'_i, f\rangle_C = \sum T_{\lambda_i,\lambda'_i}(f).$$

$T_{x,y}$ denotes the rank one operator defined by $T_{x,y}(z) = x\langle y, z\rangle_C$, for $x, y, z \in E$. Thus M_φ is a sum of rank one operators and is thus compact by definition. Rieffel also proves $E \otimes_C E' \cong A$ and $E' \otimes_A E \cong C$. It follows that the map $M_{G,X}\colon K_*[C_0(X) \rtimes \Gamma] \to \mathrm{K}_*[C_0(\tilde{X}) \rtimes G]$ given by $M_{G,X}(\eta) = \eta \# \alpha_{G,X}$ is an isomorphism. Further, $M^{-1}_{G,X}(\beta) = \beta \# \alpha'_{G,X}$.

3. MORITA EQUIVALENCE IN THE GEOMETRIC THEORY

We show there is also an induction isomorphism, $I_{G,X}\colon \mathrm{K}^t(X, \Gamma) \to \mathrm{K}^t(\tilde{X}, G)$, in the geometric theory. One of our main results is then that the isomorphisms, $M_{G,X}$ and $I_{G,X}$, are compatible with the homomorphisms, $\mu_{G,\tilde{X}}$ and $\mu_{\Gamma,X}$, i.e. that $\mu_{G,\tilde{X}} \circ I_{G,X} = M_{G,X} \circ \mu_{\Gamma,X}$.

LEMMA. *Induction of spaces and maps gives an equivalence of categories*

$$\tilde{I}_{G,X}\colon \mathscr{C}(X, \Gamma) \to \mathscr{C}(\tilde{X}, G).$$

Proof. For $(W, \rho) \in \mathscr{C}(X, \Gamma)$, define $\tilde{I}_{G,X}(W, \rho) = (\tilde{W}, \tilde{\rho}) \in \mathscr{C}(\tilde{X}, G)$ where $\tilde{\rho}$ denotes the induced G-equivariant map $\rho \times_\Gamma \mathrm{id} : W \times_\Gamma G \to X \times_\Gamma G$.

Let $\{e\}$ denote the identity coset of $\Gamma \backslash G$, and denote by π the map $X \times_\Gamma G \to \to \Gamma \backslash G$. Define $Z^r = (\pi \circ \eta)^{-1}\{e\}$. Then Z^r is a manifold and is a proper Γ space by restriction of the G-action to Γ. If $z \in Z^r$ and $\eta(z) = \{x, \gamma\}$, define $\eta^r(z) = x\gamma$. This defines a Γ-equivariant submersion $\eta^r\colon Z^r \to X$, and so $(Z^r, \eta^r) \in \mathscr{C}(X, \Gamma)$. We define $I^r_{G,X}(Z, \eta) = (Z^r, \eta^r)$. It is straightforward to show that $I^r_{G,X}$ is an inverse for the map $I^i_{G,X}$.

Let $\{(W, \rho), \zeta\}$ be a cocycle for $\mathrm{K}^*(X, \Gamma)$; that is, $(W, \rho) \in \mathscr{C}(X, \Gamma)$ and $\zeta \in \mathrm{K}_*[C_0(\tau_W) \rtimes \Gamma]$. Then define a cocycle $I_{G,X}\{(W, \rho), \zeta\}$ for $\mathrm{K}^*(\tilde{X}, G)$ by $\tilde{I}_{G,X}\{(W, \rho), \zeta\} = \{I^i_{G,X}(W, \rho), M_{G,X}(\zeta)\}$. Then by the lemma of this section and the theorem of the last we have proved:

THEOREM. *$I_{G,X}$ is an isomorphism.*

4. THE HOMOMORPHISM FROM THE GEOMETRIC TO THE ANALYTIC

Next we decribe the homomorphism from the geometric K-theory to the analytic K-theory, $\mu_{G,X}\colon \mathrm{K}^*(X, G) \to \mathrm{K}_*[C_0(X) \rtimes G]$. The map is defined at the level of cocycles, and at this level it is defined in terms of a KK element.

Let $\{(W, \rho), \zeta\}$ be a cocycle of $\mathrm{K}^*(X, G)$. Denote the projection of τ_W onto W by π_W. Then $\rho \circ \pi_W : \tau_W \to X$ is a K-oriented map. In fact $\ker d(\rho \circ \pi_W) \cong \tau_W \oplus \tau_W$ and is thus not only a spin^c vector bundle over τ_W, but is an almost complex vector bundle. Denote by $\mathscr{S}_{W,X}$ the irreducible module of the Clifford algebra of the complexification of $\ker d(\rho \circ \pi_W)$. For each $x \in X$ construct the Dirac operator D_x on the fiber of $\mathscr{S}_{W,X}$ over x. Let $S_c(\mathscr{S}_{W,X})$ denote the compactly supported sections of the bundle $\mathscr{S}_{W,X}$. Then the family of operators, D_x, defines a map $D\colon S_c(\mathscr{S}_{W,X}) \to S_c(\mathscr{S}_{W,X})$ by $D\sigma(w) = (D_{\rho(\pi(w))}\sigma)(w)$.

Several authors [11], [15], [20], have developed techniques to show D and $\mathscr{S}_{W,X}$ can be used to define an element in $\mathrm{KK}(C_0(\tau_W), C_0(X))$. For $\sigma_1, \sigma_2 \in S_c(\mathscr{S}_{W,X})$, define a $C_0(X)$ valued inner product by

$$\langle \sigma_1, \sigma_2 \rangle_X(x) = \int_{(\rho\circ\pi)^{-1}(x)} (\overline{\sigma_1(y)}, \sigma_2(y))^{1/2} \mathrm{d}y.$$

The completion of $S_c(\mathscr{S}_{W,X})$ using this inner product, denoted $\overline{\mathscr{S}_{W,X}}$, is a Hilbert B-module for $B = C_0(X)$.

Then $D_x^b = D_x(1 + D_x^* D_x)^{-1/2}$ is a family of bounded, pseudodifferential, elliptic operators which define a Kasparov operator, $D_{W,X}$, on the space $\overline{\mathscr{S}_{W,X}}$. Moreover, the pair $\beta_{W,X} = (\overline{\mathscr{S}_{W,X}}, D_{W,X})$ is a well defined element in the group $\mathrm{KK}(C_0(\tau_W), C_0(X))$. In fact, the operator $D_{W,X}$ can be constructed to be G-invariant. Also, the $C_0(X)$ valued inner product on $\overline{\mathscr{S}_{W,X}}$, can be made G-equivariant, since τ_W is a proper G-manifold ([15]). Thus, we may consider $\beta_{W,X}$ to be an element in the group, $\mathrm{KK}_G(C_0(\tau_W), C_0(X))$.

In [14], Kasparov defines an induction homomorphism

$$j_G : \mathrm{KK}_G(A, B) \to \mathrm{KK}(A \rtimes G, B \rtimes G).$$

To describe the image of $\beta_{W,X}$ under this map, define on $C_c(G, \overline{\mathscr{S}_{W,X}})$ a $C_c(G, X)$ valued inner product by the formula

$$\langle h_1, h_2 \rangle_{G,X}(g) = \int_G \langle h_1(s), h_2(sg) \rangle_X^s \, ds.$$

With this inner product, $C_c(G, \overline{\mathscr{S}_{W,X}})$ is a left $C_0(\tau_W) \rtimes G$, right $C_0(X) \rtimes G$ module. Its completion, $C^*(G, \overline{\mathscr{S}_{W,X}})$, is a Hilbert B-module. The operator $D_{W,X}$ extends to an operator on $C^*(G, \overline{\mathscr{S}_{W,X}})$. Kasparov sets $j_G(\beta_{W,X}) = (C^*(G, \overline{\mathscr{S}_{W,X}}), D_{W,X})$. This is the KK element we need, and we define $\mu_{G,X}\{(W, \rho), \zeta\} = \zeta \ \# \ j_G(\beta_{W,X})$.

5. COMPATIBILITY OF MORITA EQUIVALENCE MAPS

THEOREM. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{K}^*(X, \Gamma) & \xrightarrow{I_{G,X}} & \mathrm{K}^*(\tilde{X}, G) \\ {\scriptstyle \mu_{\Gamma,X}} \big\downarrow & & \big\downarrow {\scriptstyle \mu_{G,X}} \\ \mathrm{K}_*[C_0(X) \times \Gamma] & \xrightarrow{M_{G,X}} & \mathrm{K}^*[C_0(\tilde{X}) \times G]. \end{array}$$

Proof. For $(\tilde{X}, \tilde{\rho}) \in \mathscr{C}(\tilde{X}, G)$, we claim the KK element, $\beta_{\tilde{W},\tilde{X}} = (\overline{\mathscr{S}_{\tilde{W},\tilde{X}}}, D_{\tilde{W},\tilde{X}})$ is "induced" from the KK element, $\beta_{W,X} = (\overline{\mathscr{S}_{W,X}}, D_{W,X})$. To see this, first note that $\mathscr{S}_{\tilde{W},\tilde{X}} = \mathscr{S}_{W,X} \times_\Gamma G = \widetilde{S_{W,X}}$. We then define that $S_c(\mathscr{S}_{W,X}) \times_\Gamma G = \{f : G \to S_c(\mathscr{S}_{W,X}) \mid f$ has compact support, and $f(\gamma g)(w) = f(g)(w\gamma)\gamma^{-1}$ for all $\gamma \in \Gamma\}$. It follows from the fact that τ_W is a proper Γ-space that we have $S_c(\mathscr{S}_{W,X}) \times_\Gamma \times_\Gamma G = S_c(\mathscr{S}_{\tilde{W},\tilde{X}}) = S_c(\widetilde{\mathscr{S}_{W,X}})$. Since $D_{W,X}$ is Γ-invariant, it extends to an operator $\widetilde{D_{W,X}}$ on $S_c(\mathscr{S}_{\tilde{W},\tilde{X}})$ and so to an operator on $\overline{\mathscr{S}_{\tilde{W},\tilde{X}}}$. Further, we have the equality $\widetilde{D_{W,X}} = D_{\tilde{W},\tilde{X}}$, and thus $\beta_{\tilde{W},\tilde{X}} = (\overline{\mathscr{S}_{\tilde{W},\tilde{X}}}, \widetilde{D_{W,X}})$.

The next two lemmas give a proof of commutativity in the diagram by showing $M_{G,X} \circ \mu_{\Gamma,X} \circ I_{G,X}^{-1} = \mu_{G,\tilde{X}}$. At the level of cocycles, this reduces to the equality between KK elements, $\alpha_{G,\tau_W}^{-1} \# j_\Gamma(\beta_{G,X}) \# \alpha_{G,X} = j_G(\beta_{\tilde{W},\tilde{X}})$. Our next lemma asserts that these two KK elements have isomorphic Hilbert B-modules.

LEMMA. *The following map gives an isomorphism of Hilbert B-modules:*

$$R : C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(\Gamma, \overline{\mathscr{S}_{W,X}}) \underset{C_0(X) \rtimes \Gamma}{\otimes} C^*(G, X) \to C^*(G, \overline{\mathscr{S}_{\tilde{W},\tilde{X}}})$$

defined on dense subsets by

$$R(\omega \otimes h[\gamma] \otimes \chi)(g, g') = \sum_{\xi \in \Gamma} (\omega(\xi g'))^{\xi} h^{\xi}(\chi(\gamma^{-1}\xi g' g))^{\gamma^{-1}\xi}$$

where $h^{\xi}(w) = h(w\xi^{-1})\xi$ *and* $(\chi(g))^{\xi}(w) = (\chi(g))(\rho \circ \pi_W(w\xi^{-1}))$ *for* $h \in S_c(\mathscr{S}_{W,X})$ *and* $\chi \in C_c(G, X)$.

Proof. We must show that the map

a) is well defined

b) is a B-module homomorphism

c) preserves the B-valued inner product

d) has a dense image.

The verifications of a), b), and c), are straightforward calculations, and we include only the calculations which prove c).

First, we calculate the inner product, viewing the elements as lying in the tensor product.

$$\langle \omega_1 \otimes h_1[\gamma_1] \otimes \chi_1, \omega_2 \otimes h_2[\gamma_2] \otimes \chi_2 \rangle (g, g') =$$

$$= \langle \langle h_2, \langle \omega_2, \omega_1 \rangle h_1 \rangle \chi_1, \chi_2 \rangle (g, g') =$$

$$= \sum_{\tau} \overline{\langle h_2, \langle \omega_2, \omega_1 \rangle h_1 \rangle^{\tau} (\chi_1(\tau g'))^{\tau}} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \langle \overline{h_2}, \langle \overline{\omega_2}, \overline{\omega_1} \rangle \overline{h_1} \rangle^{\tau}(\varphi) (\overline{\chi_1(\varphi^{-1}\tau g')})^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \langle \overline{h_2^{\gamma_2}}, \langle \overline{\omega_2}, \overline{\omega_1} \rangle \overline{h_1})^{\gamma_2}(\gamma_2, \varphi) \rangle^{\tau} (\overline{\chi_1(\varphi^{-1}\tau g')})^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \left\langle h_2^{\gamma_2}, \overline{(\langle \omega_2, \omega_1 \rangle (\gamma_2 \varphi \gamma_1^{-1}))^{\gamma_2} h_1^{\gamma_1 \varphi^{-1}}} \right\rangle^{\tau} \overline{(\chi_1(\varphi^{-1}\tau g'))}^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \left\langle h_2^{\gamma_2}, \int_G \overline{(\omega_2(s))^{\gamma_2} (\omega_1(\gamma_1 \varphi^{-1} \gamma_2^{-1} s))^{\gamma_1 \varphi^{-1}}} \mathrm{d}s \, h_1^{\gamma_1 \varphi^{-1}} \right\rangle^{\tau} \overline{(\chi_1(\varphi^{-1}\tau g'))}^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \int_G \overline{\langle (\omega_2(s))^{\gamma_2} h_2^{\gamma_2}, (\omega_1(\gamma_1 \varphi^{-1} \gamma_2^{-1} s))^{\gamma_1 \varphi^{-1}} h_1^{\gamma_1 \varphi^{-1}} \rangle}^{\tau} \mathrm{d}s \overline{(\chi_1(\varphi^{-1}\tau g'))}^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau} =$$

$$= \sum_{\tau} \sum_{\varphi} \int_G \langle (\omega_1(\gamma_1 \varphi^{-1} \gamma_2^{-1} s))^{\gamma_1 \varphi^{-1}} h_1^{\gamma_1 \varphi^{-1}}, (\omega_2(s))^{\gamma_2} h_2^{\gamma_2} \rangle^{\tau} \mathrm{d}s \overline{(\chi_1(\varphi^{-1}\tau g'))}^{\varphi^{-1}\tau} (\chi_2(\tau g' g))^{\tau}.$$

Next, we calculate the inner product under the image of the map R.

$$\langle R(\omega_1 \otimes h_1[\gamma_1] \otimes \chi_1), R(\omega_2 \otimes h_2[\gamma_2] \otimes \chi_2) \rangle (g, g') =$$

$$= \int_G \langle R(\omega_1 \otimes h_1[\gamma_1] \otimes \chi_1)(a, g's^{-1}), R(\omega_2 \otimes h_2[\chi_2] \otimes \gamma_2)(sg, g's^{-1}) \rangle \mathrm{d}s =$$

$$= \int_G \left\langle \sum_{\xi} (\omega_1(\xi s))^{\xi} h_1^{\xi} (\chi_1(\gamma_1^{-1} \xi g'))^{\gamma_1^{-1}\xi}, \sum_{\eta} (\omega_2(\eta s))^{\eta} h_2^{\eta} (\chi_2(\gamma_2^{-1} \eta g' g))^{\gamma_2^{-1}} \right\rangle \mathrm{d}s =$$

$$\eta \mapsto \gamma_2 \eta$$
$$\xi \mapsto \gamma_1 \xi^{-1} \gamma_2^{-1} \eta$$
$$s \mapsto \eta^{-1} \gamma_2^{-1} s$$

$$= \int_G \sum_{\xi} \sum_{\eta} \langle (\omega_1(\gamma_1 \xi^{-1} \gamma_2^{-1} s))^{\gamma_1 \xi^{-1}} h_1^{\gamma_1 \xi^{-1}}, (\omega_2(s))^{\gamma_2} h_2^{\gamma_3} \rangle^{\eta} \overline{(\chi_1(\xi^{-1} \eta g'))}^{\xi^{-1}\eta} (\chi_2(\eta g' g))^{\eta} \mathrm{d}s.$$

To prove density, we can give $C_c(G, \overline{\mathscr{S}_{W,X}})$, the structure of a left $C_0(\tau_W) \rtimes \Gamma$, right $C_0(\tilde{X}) \rtimes G$ module, with a $C_0(\tilde{X}) \rtimes G$ valued inner product. We denote its completion by $C^*(G, \overline{\mathscr{S}_{W,X}})$. By calculations similar to those just completed, one can show that we have the following isomorphism of Hilbert B-modules:

$$C^*(\Gamma, \overline{\mathscr{S}_{W,X}}) \underset{C_0(X) \rtimes \Gamma}{\otimes} C^*(G, X) \cong C^*(G, \overline{\mathscr{S}_{W,X}})$$

$$h[\gamma] \otimes \chi \mapsto h(\chi^{\gamma^{-1}}(g))^{\gamma^{-1}}.$$

The map R collapses to a map

$$C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(G, \overline{\mathcal{S}_{W,X}}) \to C^*(G, \overline{\mathcal{S}_{\tilde{W},\tilde{X}}}).$$

Locally this map is the same as the map Rieffel defines,

$$C_c(G, \tau_W) \times C_c(G, \tau_W) \to C_c(G, \widetilde{\tau_W}),$$

in situation 10 of [17]. His result on density applies and proves the image of R is dense in $C^*(G, \overline{\mathcal{S}_{\tilde{W},\tilde{X}}})$.

Next we want to show that the two operators of our KK elements give equivalent KK elements.

LEMMA. *We have the following equality of* KK *elements,*

$$\alpha_{G,\tau_W}^{-1} \# j_\Gamma(\beta_{W,X}) \# \alpha_{G,X} =$$

$$= (C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(\Gamma, \overline{\mathcal{S}_{W,X}}) \underset{C_0(X) \rtimes \Gamma}{\otimes} C^*(G, X), R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R).$$

Proof. The Kasparov product of the two right elements is given by

$$j_\Gamma(\beta_{W,X}) \# \alpha_{G,X} = \big(C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(G, \overline{\mathcal{S}_{W,X}}), D_{W,X} \otimes \mathrm{id}\big).$$

To compute $\alpha_{G,\tau_W}^{-1} \# (j_\Gamma(\beta_{W,X}) \# \alpha_{G,X})$, we use Connes and Skandalis' "simplified" approach to the Kasparov product, [11]. We claim $R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R$ is a $D_{W,X} \otimes \mathrm{id}$ connection. For $U \in C^*(G, \tau_W)$, let T_U to be the B-module homomorphism

$$T_U : C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(G, \overline{\mathcal{S}_{W,X}}) \to$$

$$\to C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} C^*(\Gamma, \overline{\mathcal{S}_{W,X}}) \underset{C_0(X) \rtimes \Gamma}{\otimes} C^*(G, X)$$

defined by $T_U(h[\gamma] \otimes \chi) = U \otimes h[\gamma] \otimes \chi$. To prove our claim, we must show that the operator, $T = (R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R) \circ T_U - T_U \circ (D_{W,X} \otimes \mathrm{id})$, is a compact Hilbert B-module operator.

One sees that $R \circ T_U$ is compact by the following computation:

$$R \circ T_U(h[\gamma] \otimes \chi)(g, g') =$$

$$= \sum_{\xi \in \Gamma} D_{\tilde{W},\tilde{X}}(U^{\xi}(\xi g')h^{\xi}\chi^{\gamma^{-1}\xi}(\gamma^{-1}\xi g'g)) - U^{\xi}(\xi g')D_{\tilde{W},\tilde{X}}(h^{\xi}\chi^{\gamma^{-1}\xi}(\gamma^{-1}\xi g'g)) =$$

$$= \sum_{\xi \in \Gamma} (D_{W,X}(U^{\xi}(\xi g')h^{\xi}) - U^{\xi}(\xi g')D_{W,X}(h^{\xi}))\chi^{\gamma^{-1}\xi}(\gamma^{-1}\xi g'g).$$

Then since $D_{W,X}$ is a Kasparov operator, this last expression is a sum of compact operators and is thus compact.

It follows that the operator T is compact as we wished to show. Similiarly, the operator $T^* = T_U^* \circ (D_{W,X} \otimes \mathrm{id})^* - (R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R)^* \circ T_U^*$ is a compact Hilbert B-module operator. Thus in the language of Connes-Skandalis, $(R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R)$ is a $(D_{W,X} \otimes \mathrm{id})$ connection. Since the pair

$$\left(C^*(G, \tau_W) \underset{C_0(\tau_W) \rtimes \Gamma}{\otimes} \overline{C^*(\Gamma, \mathscr{S}_{W,X})} \underset{C_0(X) \rtimes \Gamma}{\otimes} C^*(G, X),\ R^{-1} \circ D_{\tilde{W},\tilde{X}} \circ R\right)$$

is a Kasparov element, the uniqueness theorem of Connes-Skandalis [11] implies it is the Kasparov product, $\alpha_{G,\tau_W}^{-1} \# j_\Gamma(\beta_{W,X}) \# \alpha_{G,X} = j_G(\beta_{\tilde{W},\tilde{X}})$.

Combining this lemma with the previous one proves the diagram is commutative.

6. PROOF THAT $\mu_{\Gamma,X}$ IS AN ISOMORPHISM

If Γ is a discrete subgroup of a semisimple Lie group with finite center, the existence of a final object in the category $\mathscr{C}(X, \Gamma)$ allows one to determine the geometric group, $K^*(X, \Gamma)$.

FINAL OBJECT THEOREM [Baum, Connes]. *Let Γ be a discrete subgroup of a semisimple Lie group G with finite center, and let H be the maximal compact subgroup of G. For W any proper Γ-manifold, there exists a unique (up to homotopy) Γ-equivariant map $f_W : W \to H \backslash G$.*

Proof. Take $p \in W$ and let θ_p be the orbit of p, under the Γ-action on W. Let Φ be the isotropy group of p. Φ is finite by the assumption that W is a proper Γ-manifold. Further, we have $\theta_p \cong \Phi \backslash \Gamma$.

Φ is G conjugate to a subgroup of H. Choose $g \in G$ so that $g\Phi g^{-1} \subset H$. $H \backslash G$ has a natural Γ-action. The isotropy group Φ' of $Hg \in H \backslash G$ under this action contains Φ because if $\gamma \in \Phi$, then $\gamma = g^{-1}hg$ for some $h \in H$. And so $Hg\gamma = Hgg^{-1}hg = Hg$, giving us a Γ-equivariant map of θ_p to the orbit of $Hg \in H \backslash G$.

By Palais' Slice Theorem [16] we can find a tubular neighborhood ν_p of θ_p which is mapped to itself by Γ. Thus the projection $\nu_p \to \theta_p$ is a Γ-equivariant map. Combining this with our previous map, $\theta_p \to H\backslash G$, we get a Γ-equivariant map, $f_p : \nu_p \to H\backslash G$. The main point of the proof is that for a semisimple Lie group G with finite center, $H\backslash G$ has a G-invariant Riemannian metric of non-positive sectional curvature. It follows there is a unique geodesic joining any two points of $H\backslash G$. For $a, b \in H\backslash G$, let $\Omega\{a, b\}(0) = a$ and $\Omega\{a, b\}(1) = b$.

Choose a countable set of points $\{p_i\}$ in W so that $\bigcup \nu_{p_i}$ covers W and is a locally finite cover. Let U_i be the image under the map $\pi: W \to W/\Gamma$ of the set ν_{p_i}. Then the U_i cover the space W/Γ. Choose a partition of unity $\{\varphi_i\}$ on W/Γ subordinate to the cover U_i in such a way that $\varphi_i \circ \pi: \nu_{p_i} \to [0, 1]$ is a C^∞ map for each i.

A Γ-equivariant map $f_W: W \to H\backslash G$ is constructed by induction on the cover $\{\nu_{p_i}\}$. On ν_{p_1} we take f_{p_1} as our map. Suppose a map f_n has been constructed on the union of the first n sets of the cover. Define f_{n+1} by

$$f_{n+1}(x) = \begin{cases} f_{p_{n+1}}(x) & \text{if } x \in \nu_{p_{n+1}},\ x \notin \bigcup_{i=1}^{n} \nu_{p_i} \\ f_n(x) & \text{if } x \in \bigcup_{i=1}^{n} \nu_{p_i},\ x \notin \nu_{p_{n+1}} \\ \Omega\{f_{p_{n+1}}(x), f_n(x)\}(\varphi_{n+1}(\pi(x))) & \text{otherwise.} \end{cases}$$

Then f_{n+1} is Γ-equivariant since the action of G takes geodesics to geodesics. Since our cover is locally finite, it makes sense to set $f_W = \lim f_i$. Finally, for any two Γ-equivariant maps f_0 and f_1, we can construct a Γ-equivariant homotopy f_t between them by letting $f_t(w)$ = point on the geodesic between $f_0(w)$ and $f_1(w)$ the fraction t of the length of the geodesic segment.

The Final Object Theorem implies $(X \times H\backslash G, \pi_1) \in \mathscr{C}(X, \Gamma)$ is a final object in the category where π_1 is projection onto the first factor. Then since $\ker d\pi_1 =$ $= X \times T(H\backslash G)$, we have $K^*(X, \Gamma) = K_*[C_0(X \times T(H\backslash G)) \rtimes \Gamma]$.

There is an identification of G-spaces, $(X \times H\backslash G) \times_\Gamma G \cong \tilde{X} \times H\backslash G$. Thus by our lemma on induction of categories, $(\tilde{X} \times H\backslash G, \pi_1)$ is a final object in the category $\mathscr{C}(\tilde{X}, G)$. It then follows that $K^*(\tilde{X}, G) \cong K_*[C_0(\tilde{X} \times T(H\backslash G)) \rtimes G]$.

Next, we have Kasparov's result.

THEOREM [Kasparov, 12]. *For G a Lorentz group*, SO$(n, 1)$, *H the maximal compact subgroup of G, and Y any G-space, there exists an invertible element*

$$\alpha \in KK(C_0(Y \times \mathfrak{h}\backslash\mathfrak{g}) \rtimes H,\ C_0(Y) \rtimes G)$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively.

From this theorem we deduce that for Y a proper G-space, using the identification

$$(Y \times \mathfrak{h} \backslash \mathfrak{g}) \times_\Gamma G \cong (T(H \backslash G) \times Y) \times G,$$

we have by Morita equivalence an invertible element,

$$\tau \in \mathrm{KK}(C_0(T(H \backslash G) \times Y) \rtimes G, C_0(Y \times \mathfrak{h} \backslash \mathfrak{g}) \rtimes H).$$

Letting $Y = \tilde{X}$, it follows that $\mu_{G,X} = \tau \# \alpha$. Thus by Kasparov's theorem, $\mu_{G,X}$ is an isomorphism for G a Lorentz group. Commutativity in the diagram then implies that for Γ a discrete subgroup of a Lorentz group, $\mu_{\Gamma,X}$: $\mathrm{K}^*(X, \Gamma) \to \mathrm{K}_*[C_0(X) \rtimes \Gamma]$ is an isomorphism.

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DANIEL SCHWALBE
Department of Mathematics
St. Louis University,
St. Louis, MO 63103,
U.S.A.

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