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TRANSITIVE OPERATOR ALGEBRAS ON THE n-FOLD DIRECT SUM OF A CYCLIC BANACH SPACE

HURIYE ÖNDER (ARIKAN) and MEHMET ORHON

1. INTRODUCTION

Let X be a Banach space and $L(X)$ denote the (bounded linear) operators on X. Let K be a compact Hausdorff space and $C(K)$ denote the complex valued continuous functions on K. By $\Pi: C(K) \to L(X)$ we denote a bounded, unital algebra homomorphism. X is said to be cyclic with respect to $\Pi(C(K))$ if there is a vector $x_0 \in X$ such that $X = \overline{\Pi(C(K))x_0}$. A subalgebra $\mathscr{A}$ of $L(X)$ is called transitive if the only common invariant (closed linear) subspaces of the operators in $\mathscr{A}$ are $\{0\}$ and X. X^n will denote the n-fold direct sum $X \oplus \ldots \oplus X$ (n-copies) of X. When $T \in L(X)$ by $T^{(n)} \in L(X^n)$ we mean $T \oplus \ldots \oplus T$ (n-copies). For a subset $\mathscr{S}$ of $L(X)$, let $\mathscr{S}^{(n)} = \{T^{(n)} : T \in \mathscr{S}\}$. In this paper we will prove the following result.

THEOREM 1. *Let X be a Banach space and K be a compact Hausdorff space. Suppose $\Pi: C(K) \to L(X)$ is a bounded unital algebra homomorphism with respect to which X is cyclic. Let $\mathscr{A}$ be a transitive subalgebra of $L(X^n)$. If the weak operator closure of $\mathscr{A}$ contains $\Pi(C(K))^{(n)}$ then $\mathscr{A}$ is dense in $L(X^n)$ with respect to the weak operator topology.*

Several cases of Theorem 1 are well known. When X is a Hilbert space and Π is a $*$-homomorphism, for $n = 1$, it is due to Arveson [1; 3.3]; for $n > 1$ it is the theorem of Douglas and Pearcy [2; 3.1]. Also when K is the Stone representation space of a complete Boolean algebra of projections $\mathscr{B}$ on X, for $n = 1$ the result was proved by Rosenthal and Sourour [8; 8], [9]. Note that in this case one assumes there is a countably additive spectral measure in $L(X)$ such that the strong operator closure of the range of the spectral measure is equal to $\mathscr{B}$. This means that K is hyperstonian and the homomorphism Π is (w*, weak-operator)-continuous [5].

Recently Rădulescu and Vasilescu [7] studied the spectral properties of a class of matrix operators that they called the (A, n)-scalar operators. In particular (without the restriction that X is cyclic) they showed that if $T \in L(X^n) \setminus \mathbf{C}I$ with $T = [\Pi(\Phi_{jk})]_{j,k=1}^n$ ($\Phi_{jk} \in C(K)$, $j, k = 1, \ldots, n$) then T has a non-trivial hyperinvariant subspace [7, 4.8].

Their result extends the theorem of Hoover [3] for n-normal operators when X is a Hilbert space and that of Omladic [4] for "n-spectral" operators when X is reflexive.

In the setting of Hilbert spaces, the equivalence of the theorem of Douglas and Pearcy [2] and that of Hoover [3] can be shown. Our purpose here is to show that the theorem of Rădulescu and Vasilescu [7] implies the transitivity result stated in Theorem 1. In Section 2 we represent a cyclic Banach space as a space of continuous functions on a suitable compact Hausdorff space K_X. This enables us to represent a densely defined linear transformation on X^n that commutes with the weak operator closure of $\Pi(C(K))^{(n)}$ as a closable "$(C(K_X), n)$-scalar" operator (Section 3). In Section 4 we give the proof of Theorem 1 using Rădulescu and Vasilescu's theorem [7] and Arveson's lemma [1], [6; 8.8].

Our basic reference for operator theory is [6] and for Banach lattices is [10].

2. CYCLIC BANACH SPACES

Given a bounded unital algebra homomorphism $\Pi\colon C(K) \to L(X)$, by passing to the equivalent norm $|||x||| = \sup\{\|\Pi(f)x\| : \|f\| \leqslant 1, f \in C(K)\}$ $(x \in X)$ on X, we may assume that Π is a contraction. Since the kernel of Π is a closed ideal, we may also assume that Π is one-to-one. Then Π is an isometry with the property: $|f| \leqslant |g|$ implies $\|\Pi(f)x\| \leqslant \|\Pi(g)x\|$ for all $x \in X$. When K is totally disconnected a proof of this is given in [11; VI.1.2]. It is not hard to adapt the given proof to an arbitrary compact Hausdorff space K.

Let X have a cyclic vector x_0 with respect to $\Pi(C(K))$. Then the order structure induced on $\Pi(C(K)_r)x_0$ from $C(K)_r$ (the real-valued continuous functions on K) renders $X_r = \overline{\Pi(C(K)_r)x_0}$ a real Banach lattice with topological order unit (quasi-interior element) x_0 such that $\Pi(C(K)_r)x_0$ is contained in the order ideal generated by x_0 [11; IV.1.3]. Since $X_r \cap iX_r = \{0\}$ and $X = X_r + iX_r$, we may regard X as a complex Banach lattice with topological order unit x_0 as defined in [10; II.11].

Let $C_\infty(K)$ denote the set of continuous functions $f\colon K \to \mathbf{C} \cup \{\infty\}$ (one point compactification of the complex numbers) such that

i) $f^{-1}\{\infty\}$ is a nowhere dense set and

ii) $(\mathrm{re} f)^+$, $(\mathrm{im} f)^+$, $(\mathrm{re} f)^-$, $(\mathrm{im} f)^-$ have continuous extensions to all of K with range in $\mathbf{R} \cup \{\infty\}$ (one point compactification of the real numbers).

Here $\mathrm{re} f$ and $\mathrm{im} f$ denote the real and imaginary parts of the finite part of a continuous function f on K into $\mathbf{C} \cup \{\infty\}$, i.e. $\mathrm{re} f = \mathrm{re}(f|f^{-1}\mathbf{C})$ and $\mathrm{im} f = \mathrm{im}(f|f^{-1}\mathbf{C})$. By $C_\infty(K)_r$ we denote the set of functions in $C_\infty(K)$ with ranges in $\mathbf{R} \cup \{\infty\}$. $C_\infty(K)_r$ coincides with the set of numerical continuous functions on K given in [10; p. 168] where a continuous numerical function is defined as a continuous map $K \to \overline{\mathbf{R}} = [-\infty, \infty]$ (two point compactification of the real numbers [10; p. 109]) that is finite except possibly on a nowhere dense set.

It follows from above and [10; III.4.5] that there exists a compact Hausdorff space K_X (with K as a quotient) such that X_r is lattice isomorphic to a vector sublattice $\hat{X}_r$ of $C_\infty(K_X)_r$. Moreover $\hat{X}_r$ contains $C(K_X)_r$ as a dense order ideal with respect to the topology transfered from X_r. By $x \to \hat{x}$ we will denote the representation of X_r in $\hat{X}_r$ (with $\hat{x}_0 = 1$). In fact $C(K_X)_r$ is lattice isomorphic to the order ideal generated by x_0 and K_X is obtained from the Kakutani-Krein theorem when the order ideal generated by x_0 is represented as an AM-lattice with order unit [10; II.7.2 Corollary]. Then $f \to (\Pi(f)x_0)\hat{}$ for all $f \in C(K)_r$ describes the embedding of $C(K)_r$ in $C(K_X)_r$. Thus X is lattice isomorphic to the complex vector sublattice $\hat{X} = \hat{X}_r + i\hat{X}_r$ of $C_\infty(K_X)$ such that $\hat{X}$ contains $C(K_X)$ as a dense order ideal.

Given a Banach lattice E, the ideal center $Z(E)$ of E is the set of all operators $T \in L(E)$ such that for some $\lambda \geqslant 0$ (e.g. $\lambda = \|T\|$), $|Tx| \leqslant \lambda x$ for all $x \in E_+$ [12]. As above, let $x \to \hat{x}$ denote the lattice isomorphism between X and $\hat{X}$. Then the map $T \to (Tx_0)\hat{}$ from $Z(X)$ onto $C(K_X)$ shows that $Z(X)$ is isometric, lattice and algebra isomorphic to $C(K_X)$. $Z(X)$ acts on $\hat{X}$ as multiplication by the functions in $C(K_X)$ [12].

We summarize the above discussion in the following essentially known lemma.

LEMMA 2. *If X is a cyclic Banach space with respect to $\Pi(C(K))$ with a cyclic vector x_0, then there exists a compact Hausdorff space K_X unique up to an homeomorphism such that:*

(1) *With respect to an equivalent norm X is a complex Banach lattice with topological order unit x_0 and $\Pi(C(K))x_0$ is contained in the ideal generated by x_0* [11; IV.1.3].

(2) *X is lattice isomorphic to a complex vector sublattice $\hat{X}$ of $C_\infty(K_X)$ and $\hat{X}$ contains $C(K_X)$ as a dense order ideal with respect to the topology transfered from X* [10; III.4.5].

(3) *The ideal center $Z(X)$ is isometric, lattice and algebra isomorphic to $C(K_X)$* [12].

(4) *$Z(X)$ is the closure of $\Pi(C(K))$ in the weak operator topology in $L(X)$.*

Proof. We only need to prove (4). Each operator in $Z(X)$ leaves every closed order ideal invariant and the converse of this is also true (e.g. [12; 5.2]). Hence $Z(X)$ is weak operator closed. By (3) and the above discussion $\Pi(C(K))$ is contained in $Z(X)$. Let $T \in Z(X)_r$. Since x_0 is a cyclic vector, there is a sequence $\{f_n\}$ in $C(K)_r$ such that $\Pi(f_n)x_0 \to Tx_0$ in norm. Let $f = (Tx_0)\hat{}$ with $f \in C(K_X)$. By (3) the norm of f in $C(K_X)$ is equal to $\|T\|$. Then $|f_n \wedge \|T\| - f| \leqslant |f_n - f|$ in $C(K_X)$ implies $\|\Pi(f_n \wedge \|T\|)x_0 - Tx_0\| \leqslant \|\Pi(f_n)x_0 - Tx_0\|$. The sequence $\{f_n \wedge \|T\|\}$ is bounded above by $\|T\|$ in $C(K_X)$. Therefore there exists a sequence $\{g_n\}$ in $C(K)$ such that $\|\Pi(g_n)\| \leqslant \|T\|$ and $\Pi(g_n)x_0 \to Tx_0$ in norm. Since X is cyclic and $\{\|\Pi(g_n)\|\}$ is bounded, it follows that $\Pi(g_n) \to T$ in the strong operator topology. ▨

3. GRAPH TRANSFORMATIONS

Lemma 2 enables us to reduce to the case when $\Pi: C(K) \to L(X)$ is an isometric unital algebra homomorphism such that $\Pi(C(K))$ is weak operator closed. Moreover we consider X as a vector sublattice of $C_\infty(K)$ on which $\Pi(C(K)) = Z(X)$ acts as multiplication by the functions in $C(K)$. Therefore in the lemmas that follow we consider X as a $C(K)$-module with respect to pointwise operations and omit writing the homomorphism Π.

Let V be a $C(K)$-submodule of X; the set of $C(K)$-module homomorphisms of V into X will be denoted by $\mathrm{Hom}_{C(K)}(V, X)$. We define $\mathrm{Car}\, x = \{s \in K : 0 < |x(s)| < \infty\}$ for $x \in X$ (the carrier of x in K) and $\mathrm{Car}\, V = \bigcup\{\mathrm{Car}\, x : x \in V\}$.

LEMMA 3. *Suppose* $\mathrm{Car}\, V$ *is dense in* K *and* $T \in \mathrm{Hom}_{C(K)}(V, X)$. *Then there is a closed nowhere dense set* K_0 *in* K *and* $\Phi \in C(K \setminus K_0)$ *such that* $Tx = \Phi x$ *for each* $x \in V$.

Proof. For each $x \in V$ let $W_x = \mathrm{Car}\, x \cap \{s \in K : Tx(s) \neq \infty\}$. We choose $K_0 = K \setminus \bigcup\{W_x : x \in V\}$. Define $\Phi \in C(K \setminus K_0)$ as $\Phi(s) = \dfrac{Tx(s)}{x(s)}$ where $s \in W_x$ for some $x \in V$. Let $s \in W_x \cap W_y$ for $x, y \in V$. By Tietze's extension theorem for a compact neighborhood $K(x, y, s)$ of s in $W_x \cap W_y$ there is an $f \in C(K)$ such that $f(t)x(t) = y(t)$ for each $t \in K(x, y, s)$. Let U be an open set containing s whose closure is in $\mathrm{int}\, K(x, y, s)$. Define a Urysohn function $u \in C(K)$ with $u(t) = 1$ on $\bar{U}$ and $u(t) = 0$ for $t \notin \mathrm{int}\, K(x, y, s)$ and $0 \leqslant u \leqslant 1$. We have $u(fx - y) \in V \subset C_\infty(K)$. Let $t \notin \{s \in K : \max(|x(s)|, |y(s)|) = \infty\}$ then by the choice of u and f, we have $u(fx - y)(t) = 0$. That is $u(fx - y)$ is zero on a dense open subset of K. Thus $u(fx - y) = 0$ and $Tu(fx - y) = 0$. In particular $0 = T(u(fx - y))(s) = u(fTx - Ty)(s) = u(s)(f(s)Tx(s) - Ty(s))$. Then $\dfrac{Tx(s)}{x(s)} = \dfrac{Ty(s)}{y(s)}$. So Φ is well defined and $\Phi \in C(K \setminus K_0)$. Thus $Tx(s) = \Phi(s)x(s)$ for all $s \in W_x$.

Since $(Tx)^{-1}\{\infty\}$ is a nowhere dense set, by a similar argument we can show that $Tx = 0$ on $\mathrm{int}(x^{-1}\{0\})$. Therefore $Tx = \Phi x$ on a dense open subset of K for each $x \in V$. ▨

Consider X^n with the norm $\|x\| = \sum_{j=1}^{n} \|x_i\|$ where $x = (x_1, \ldots, x_n)$ with $x_i \in X$, $i = 1, \ldots, n$. Since the cyclic Banach space X is a Banach $C(K)$-module, X^n is also a Banach $C(K)$-module where $fx = (fx_1, fx_2, \ldots, fx_n)$ for each $x = (x_1, x_2, \ldots, x_n) \in X^n$ and $f \in C(K)$. In fact this gives the embedding of $\Pi(C(K))^{(n)}$ in $L(X^n)$.

LEMMA 4. *Let V be a $C(K)$-submodule of X^n, and $T \in \mathrm{Hom}_{C(K)}(V, X)$. Suppose there is a nowhere dense closed set K_0 in K and there exists $\Phi_i \in C(K \setminus K_0)$ ($i = 1, 2, \ldots, n$) such that $Tx = \sum_{i=1}^{n} \Phi_i x_i$ for each $x = (x_1, \ldots, x_n) \in V$. Then T is closable.*

Proof. It is sufficient to show that for any sequence $\{x_p\}_{p=1}^{\infty}$ in V, $x_p \to 0$ and $Tx_p \to y$ imply $y = 0$. Let $O_m = \{s \in K : |\Phi_i(s)| < m$, for each $i = 1, \ldots, n\}$ for each positive integer m. For any $s \in O_m$ let F_1 and F_2 be two compact neighborhoods of s such that $F_1 \subset \operatorname{int} F_2 \subset F_2 \subset O_m$. Let $u \in C(K)$ be a Urysohn function with $u(t) = 1$ for $t \in F_1$ and $u(t) = 0$ when $t \notin \operatorname{int} F_2$ and $0 \leqslant u \leqslant 1$. Since $F_2 \cap K_0 = \emptyset$ and u is zero on the open set $K \setminus F_2 \supset K_0$, $u\Phi_i \in C(K)$ for each $i = 1, \ldots, n$. Multiplication by a function in $C(K)$ defines a bounded operator on X. $x_p = (x_1^p, \ldots, x_n^p)$ and $x_p \to 0$ imply $u\Phi_i x_i^p \to 0$ for each $i = 1, \ldots, n$. Since $u\left(\sum_{i=1}^{n} \Phi_i x_i^p\right) = uTx_p \to uy$ we have $uy = 0$. Hence y is identically zero on F_1. Since $\bigcup_{m=1}^{\infty} O_m = K \setminus K_0$ and K_0 is nowhere dense, we must have $y = 0$. ▨

LEMMA 5. *Let V be a dense $C(K)$-submodule of X^n. If $T \in \mathrm{Hom}_{C(K)}(V, X)$, there exists a closed nowhere dense subset K_0 in K and functions $\Phi_i \in C(K \setminus K_0)$, $i = 1, \ldots, n$ such that $Tx = \sum_{i=1}^{n} \Phi_i x_i$ for all $x = (x_1, \ldots, x_n) \in V$. Furthermore T is closable.*

Proof. We will give the proof by induction. For the case $n = 1$, it is sufficient to show that Car V is dense in K. Then the desired conclusion follows from Lemma 3 and Lemma 4.

Suppose $F = K \setminus \mathrm{Car}\, V$ and $\operatorname{int} F \neq \emptyset$. Then there is a non-zero $u \in C(K)$ such that the support of u is in $\operatorname{int} F$ and $0 \leqslant u \leqslant 1$. Clearly $ux = 0$ for all $x \in V$. Also $u, u^2 \in X$, since $C(K) \subset X$ (Lemma 2). Therefore $0 < \|u^2\| = \|u(u - x)\| \leqslant \|u - x\|$ and V is not dense in X. So Car V must be dense in K.

Assume for some $n \geqslant 1$, whenever V is a dense submodule of X^n and $T \in \mathrm{Hom}_{C(K)}(V, X)$ the representation in the statement of the lemma exists. Then in particular T is closable by Lemma 4.

Let V be a dense submodule of X^{n+1}. We will show there is $(x, 0, \ldots, 0) \in V$ for some $x \neq 0$. If not, the first component of each element of V is determined by the remaining n components. Let $\tilde{V} = \{(x_1, x_2, \ldots, x_n) : (x_0, x_1, \ldots, x_n) \in V$ for some $x_0 \in X\}$. Clearly, $\tilde{V}$ is a dense submodule of X^n. Define a module homomorphism $\tilde{T} : \tilde{V} \to X$ by $\tilde{T}(x_1, \ldots, x_n) = x_0$ whenever $(x_0, x_1, \ldots, x_n) \in V$. Let $0 \neq x \in X$. There is a sequence $x_p = (x_0^p, x_1^p, \ldots, x_n^p)$ in V that converges to $(x, 0, \ldots, 0)$. Since $\tilde{T}$ is closable by the induction hypothesis, we have $x = 0$. This is a contradiction.

Let $G_0 = \bigcup\{\mathrm{Car}\, x : (x, 0, \ldots, 0) \in V\}$. We will show G_0 is dense in K. Choose a non-empty open set U such that $\bar{U} \subset \mathrm{int}(K \setminus G_0)$. Let $u \in C(K)$ be a Urysohn function such that $0 \leqslant u \leqslant 1$, $u(t) = 1$ when $t \in \bar{U}$ and $u(t) = 0$ when $t \notin \mathrm{int}(K \setminus G_0)$. Then $(x, 0, \ldots, 0) \in V$ implies $ux = 0$. Therefore the first component of any element in uV is determined by the remaining n components. Define a module homomorphism $\tilde{T} : \tilde{V} \to X$ by $\tilde{T}(x_1, x_2, \ldots, x_n) = ux_0$ whenever $(x_0, x_1, \ldots, x_n) \in V$. Let $x_p = (x_0^p, x_1^p, \ldots, x_n^p)$ be a sequence in V that converges to $(u, 0, \ldots, 0) \in X^{n+1}$. Since $\tilde{T}$ is closable by the induction hypothesis, we have $u^2 = 0$. Again we obtain a contradiction.

Similarly $G_i = \bigcup\{\mathrm{Car}\, x : (0, \ldots, 0, x, 0, \ldots, 0) \in V$ for some $x \in X$ in the $(i + 1)^{\text{th}}$-coordinate$\}$ is a dense open subset of K for each $i = 1, 2, \ldots, n$.

Define $V_i = \{x \in X : (0, \ldots, 0, x, 0, \ldots, 0) \in V$ for some $x \in X$ in the $(i + 1)^{\text{th}}$-coordinate$\}$ for each $i = 0, 1, \ldots, n$ and let $T\colon V \to X$ be a module homomorphism. For each $x \in V_i$ $(i = 0, 1, \ldots, n)$, define $T_i(x) = T(0, \ldots, 0, x, 0, \ldots, 0$ where x is in the $(i + 1)^{\text{th}}$-coordinate. By Lemma 3 there is a nowhere dense closed subset K_0 of K and $\Phi_i \in C(K \setminus K_0)$ such that $T_i x = \Phi_i x$ for each $x \in V_i$ $(i = 0, 1, \ldots, n)$.

For each $x = (x_0, x_1, \ldots, x_n) \in V$, let $N(x_i) = \mathrm{int}\{s \in K : x_i(s) = 0\}$ and $Q(x) = \bigcap_{i=0}^{n} (\mathrm{Car}(x_i) \cap N(x_i))$. Let $G = \bigcap_{i=0}^{n} G_i$. Then $G \cap Q(x)$ is a dense open set in K. For each $s \in G \cap Q(x)$ and for each $i = 0, 1, \ldots, n$ there are two mutually exclusive possibilities:

(1) $s \in \mathrm{Car}\, x_i$. Let $y_i \in V_i$ with $s \in \mathrm{Car}\, y_i$. Choose a suitable compact neighborhood F_i and an open neighborhood U_i of s in $G \cap Q(x)$ such that $\bar{U}_i \subset \mathrm{int} F_i$. By Tietze's extension theorem and Urysohn's lemma choose two functions f_i, $u_i \in C(K)$ such that $f_i y_i = x_i$ on F_i and $0 \leqslant u_i \leqslant 1$ with $u_i(t) = 1$ when $t \in \bar{U}_i$ and $u_i(t) = 0$ when $t \notin \mathrm{int} F_i$. Then $u_i x_i = u_i f_i y_i \in V_i$.

(2) $s \in N(x_i)$. Choose a compact neighborhood F_i and an open neighborhood U_i of s in $N(x_i)$ such that $\bar{U}_i \subset \mathrm{int} F_i$. Let $u_i \in C(K)$ be a Urysohn function such that $0 \leqslant u_i \leqslant 1$, $u_i(t) = 1$ when $t \in \bar{U}_i$ and $u_i(t) = 0$ when $t \notin \mathrm{int} F_i$. Then $u_i x_i = 0 \in V_i$.

Let $u = u_0 u_1 \ldots u_n \in C(K)$. u is 1 on the neighborhood $\bigcap_{i=0}^{n} \bar{U}_i$ of s and vanishes outside the open set $\bigcap_{i=0}^{n} \mathrm{int} F_i$ in $G \cap Q(x)$. Moreover $ux_i \in V_i$ for each $i = 0, 1, \ldots, n$. Then

$$uTx = T(ux) = \sum_{i=0}^{n} T_i(ux_i) = \sum_{i=0}^{n} u\Phi_i x_i.$$

For each $i = 0, 1, \ldots, n$, $u\Phi_i x_i \in C(K \setminus K_0)$ and $uT(x) \in C(K \setminus K_0)$. So $Tx = \sum_{i=0}^{n} \Phi_i x_i$ on the dense open set $G \cap Q(x) \setminus K_0$ when $x = (x_0, x_1, \ldots, x_n) \in V$. Lemma 4 completes the proof. ▨

COROLLARY 6. *Let $T \in \mathrm{Hom}_{C(K)}(V, X^n)$ for some dense $C(K)$-submodule V of X^n. Then there is a closed nowhere dense set K_0 in K and there are $\Phi_{jk} \in C(K \setminus K_0)$ such that $T_j x = P_j(Tx) = \sum_{k=1}^{n} \Phi_{jk} x_k$, $j = 1, \ldots, n$, where $x = (x_1, \ldots, x_n) \in V$ and $P_j: X^n \to X$ is the projection onto the j^{th} coordinate for each $j = 1, \ldots, n$. Moreover T is closable.*

Given an algebra $\mathscr{A} \subset L(X)$ and a dense $\mathscr{A}$-invariant linear submanifold V of X a transformation $T \in \mathrm{Hom}_{\mathscr{A}}(V, X)$ is called a *graph transformation of $\mathscr{A}$* in [6]. In the general case Corollary 6 implies that the graph transformations of the weak-operator closure of $\Pi(C(K))^{(n)}$ on X^n can be represented as closable "$(C(K_X), n)$-scalar" operators [7; 1.2] with respect to an extension $\hat{\Pi}: C(K_X) \to L(X)$ of Π for a suitable compact Hausdorff space K_X.

4. PROOF OF THEOREM 1

Suppose Π is as in Section 3 and $\mathscr{A}$ is weak-operator closed. To show $\mathscr{A} = L(X^n)$ by Arveson's lemma [1], it is sufficient to prove that every graph transformation of $\mathscr{A}$ is a scalar multiple of the identity.

Let V be any dense $\mathscr{A}$ invariant linear manifold in X^n and let $T \in \mathrm{Hom}_{\mathscr{A}}(V, X^n)$. In particular V is a $C(K)$-submodule of X^n and T is a $C(K)$-module homomorphism. Therefore we may assume T has the matrix representation given in Corollary 6.

For each $m = 1, 2, \ldots,$ define $O_m = \{s \in K : |\Phi_{jk}(s)| < m,\ j, k = 1, \ldots, n\}$. Let $s \in O_m$ and choose an open set U and compact sets F_0 and F such that $s \in U \subset \subset \bar{U} \subset \operatorname{int} F_0 \subset F_0 \subset \operatorname{int} F \subset F \subset O_m$. Let u, $v \in C(K)$ be Urysohn functions such that $0 \leqslant u \leqslant 1$, $u(t) = 1$ when $t \in F_0$, $u(t) = 0$ when $t \notin \operatorname{int} F$; $0 \leqslant v \leqslant 1$ $v(t) = 1$ when $t \in \bar{U}$, $v(t) = 0$ when $t \notin \operatorname{int} F_0$. Since $F \cap K_0 = \emptyset$, we have $u\Phi_{jk} \in \in C(K)$ and $uT \in L(X^n)$ is $(C(K), n)$-scalar operator. Let $Y = \overline{vX}$ and $Y^n = \overline{vX^n}$. Then

(1) $uy = y$ for each $y \in Y^n$,

(2) $uT(Y^n) \subset Y^n$,

(3) $\tilde{\Pi}: C(K) \to L(Y): : \tilde{\Pi}(f) = \Pi(f)|Y$ is a bounded unital algebra homomorphism,

(4) $\tilde{T} = (uT)|Y^n$ is a $(C(K), n)$-scalar operator with respect to $\tilde{\Pi}$ on Y^n.

Suppose $\tilde{T}$ is not a scalar multiple of the identity. Then by the Rădulescu and Vasilescu theorem [7; 4.8], $\tilde{T}$ has a proper hyperinvariant subspace Z in Y^n. For each $S \in \mathscr{A}$, define $\tilde{S} \in L(Y^n)$ by $\tilde{S}(y) = vS(y)$ for each $y \in Y^n$. Let $x \in V$. Then $\tilde{T}\tilde{S}(vx) = uTv(Svx) = vST(vx) = vST(uvx) = \tilde{S}\tilde{T}(vx)$. Since V is dense in X^n, vV is dense in Y^n. Therefore $\tilde{T}$ commutes with each $\tilde{S}$ whenever $S \in \mathscr{A}$. Let $0 \neq y \in Z$ and $Z_0 = \overline{\mathscr{A}y}$ in X^n. Z is a proper subspace of Y^n and $vZ_0 \subset Z$. So Z_0 is a proper subspace of X^n. Since Z_0 is an invariant subspace of the transitive algebra $\mathscr{A}$, this is not possible. Therefore $\tilde{T}$ is a scalar multiple of the identity on Y^n.

Now there is a complex number $\lambda(s)$ such that $T(x)(t) = \lambda(s)x(t)$ for all $x \in V$ and $t \in U$. It is clear that $\lambda(s)$ is independent of our choice of the sets U, F_0, F and the functions u, v that we used to find $\lambda(s)$. Therefore, given two distinct points $s_1, s_2 \in O_m$, choose disjoint compact neighborhoods F_1, F_2 of s_1, s_2 respectively. Now choose the functions u_1, v_1 for s_1 with supports in $\operatorname{int} F_1$, and u_2, v_2 for s_2 with supports in $\operatorname{int} F_2$ to find $\lambda(s_1)$ and $\lambda(s_2)$ respectively. That is when $Y_1^n = \overline{v_1 X^n}$ and $Y_2^n = \overline{v_2 X^n}$, we suppose that $(u_1 T)|Y_1^n = \lambda(s_1)I$, $(u_2 T)|Y_2^n = \lambda(s_2)I$. Now let $Y^n = \overline{(v_1 + v_2)X^n}$. Then the proof in the above paragraph shows $((u_1 + u_2)T)|Y^n$ is a scalar multiple of the identity. Since Y_1^n and Y_2^n are subspaces of Y^n and $u_1 v_2 = u_2 v_1 = 0$, we have

$$((u_1 + u_2)T)|Y_1^n = (u_1 T)|Y_1^n = \lambda(s_1)I,$$

$$((u_1 + u_2)T)|Y_2^n = (u_2 T)|Y_2^n = \lambda(s_2)I.$$

Therefore $\lambda(s_1) = \lambda(s_2)$. That is, there is a constant $\lambda(m)$ such that $Tx(s) = \lambda(m)x(s)$ for all $x \in V$ and $s \in O_m$. But $O_m \subset O_{m+1}$ implies $\lambda(m) = \lambda(m+1)$. So there is a constant λ such that $Tx(t) = \lambda x(t)$ for all $x \in V$, $t \notin K_0$. Since K_0 is a closed nowhere dense set, T is a scalar multiple of the identity on V. ▨

REMARK 7. In [5] it was proved that if $\mathscr{B}$ is a complete Boolean algebra of projections on a Banach space Y with finite multiplicity n (i.e. there exist cyclic subspaces $X_1, \ldots, X_n$ with pairwise zero intersection such that $Y = \overline{X_1 + \ldots + X_n}$), then a weak-operator closed transitive subalgebra $\mathscr{A}$ of $L(Y)$ containing $\mathscr{B}$ is equal to $L(Y)$. Does the result remain true if we replace $\mathscr{B}$ by a unital bounded algebra homomorphism $\Pi : C(K) \to L(Y)$?

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HURIYE ÖNDER (ARIKAN) and MEHMET ORHON
Mathematics Department,
Middle East Technical University, Ankara,
Turkey.

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