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SPECTRAL ANALYSIS OF CERTAIN OPERATOR FUNCTIONS

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1. INTRODUCTION

An elementary characterization of the range of the adjoint of a bounded linear operator in Banach space turns out to have interesting applications in spectral theory. In Section 1, the said characterization is given as Lemma 1.1, and various applications are presented. Examples are given in Section 2. We then turn in Section 3 to the discussion of the semisimplicity manifold of an arbitrary (possibly unbounded) operator with real spectrum in a reflexive Banach space. This concept was introduced in [4] for bounded operators, and was extended in [5], [6] to generators of groups of operators and in [8] to the general case. Here Lemma 1.1 is the basic tool for the construction of the semisimplicity manifold Z, which is maximal with the property that the given operator has on it a "spectral integral representation". The same tool can be used to obtain such representations for families of operators, such as one-parameter groups or cosine operator families. This is done in detail for the latter case in Section 4. The semisimplicity manifold Z for a given cosine family $C(\cdot)$ is constructed, and it is shown that there exists a spectral measure E on Z such that

$$C(t)x = \int_0^\infty \cos(ts)E(\mathrm{d}s)x \quad (t \in \mathbf{R},\ x \in Z).$$

The analysis of "local" cosine families of (unbounded) symmetric operators in Hilbert space is undertaken in Section 5, using an adaptation of the method used recently by Fröhlich [2] to deal with local semigroups. The main result in this section asserts the existence of a globally defined selfadjoint extension of $C(t)$ of the form $\cosh(t\sqrt{A^+}) + \cos(t\sqrt{A^-})$, defined by means of the operational calculus for a suitable selfadjoint operator A.

1. A GENERAL PRINCIPLE AND SOME SPECIAL CASES

Let X, Y be normed linear spaces, and let X^*, Y^* be their respective dual spaces. We denote by $\langle x, x^*\rangle$ the dual pairing of $x \in X$ and $x^* \in X^*$, and by $B(X, Y)$ the space of all bounded linear operators from X to Y. If $T \in B(X, Y)$, then $T^* \in B(Y^*, X^*)$ is defined by the relation $\langle Tx, y^*\rangle = \langle x, T^*y^*\rangle$ for all $x \in X$ and $y^* \in Y^*$. The following elementary characterization of the range of T^* in X^* is the fundamental tool of this paper.

1.1. LEMMA. *Let $T \in B(X, Y)$ and $x^* \in X^*$. Then $x^* = T^*y^*$ for some $y^* \in Y^*$ with $\|y^*\| \leq M$ if and only if*

$$(1) \qquad |\langle x, x^*\rangle| \leq M\|Tx\| \quad \textit{for all } x \textit{ in a dense subset of } X.$$

Proof. Of course, Condition (1) for all x in a dense subset of X is equivalent to the same condition valid for *all* $x \in X$. Now, if $x^* = T^*y^*$ with $\|y^*\| \leq M$, then for all $x \in X$

$$|\langle x, x^*\rangle| = |\langle x, T^*y^*\rangle| = |\langle Tx, y^*\rangle| \leq \|Tx\|\|y^*\| \leq M\|Tx\|.$$

Conversely, if (1) is valid (for all x), define $\varphi: TX \to \mathbf{C}$ by

$$(2) \qquad \varphi(Tx) = \langle x, x^*\rangle \quad (x \in X).$$

If $Tx_1 = Tx_2$ for some $x_1, x_2 \in X$, then by (1)

$$|\langle x_1, x^*\rangle - \langle x_2, x^*\rangle| = |\langle x_1 - x_2, x^*\rangle| \leq M\|T(x_1 - x_2)\| = 0,$$

so that indeed $\langle x_1, x^*\rangle = \langle x_2, x^*\rangle$, and φ is well-defined. It then follows immediately that φ is a linear functional on TX, with norm $\leq M$ (by(1)). The Hahn-Banach theorem implies that φ has an extension $y^* \in Y^*$ with $\|y^*\| = \|\varphi\| \leq M$. In particular,

$$\langle x, T^*y^*\rangle = \langle Tx, y^*\rangle = \varphi(Tx) = \langle x, x^*\rangle$$

for all $x \in X$, so that $x^* = T^*y^*$. Q.E.D.

With our applications in view, we consider now some special cases.

Let (S, Σ, σ) be a σ-finite positive measure space, and let Ω be a locally compact Hausdorff space. We take $X = L^1(S, \Sigma, \sigma)$ and identify X^* with $L^\infty(S, \Sigma, \sigma)$. We choose $Y = C_0(\Omega)$, the space of all complex continuous functions on Ω vanishing at infinity, with the supremum norm $\|\cdot\|_\infty$, and we identify Y^* with $M(\Omega)$, the space of all complex regular Borel measures on Ω with the total variation norm. Lemma 1.1 specializes as follows:

1.2. COROLLARY. *Let T be a bounded linear operator from $L^1(S, \Sigma, \sigma)$ to $C_r(\Omega)$ and let $\varphi \in L^\infty(S, \Sigma, \sigma)$. Then $\varphi = T^*\mu$ (as elements of $L^\infty(S, \Sigma, \sigma)$) for some $\mu \in M(\Omega)$ with $\|\mu\| \leqslant M$ if and only if $\left|\int_S f\varphi d\sigma\right| \leqslant M\|Tf\|_\infty$ for a dense set of f in $L^1(S, \Sigma, \sigma)$.*

Next, choose $X = A$, a commutative complex Banach algebra, and $Y = C_0(\Delta)$, where Δ is the regular maximal ideal space of A (with the Gelfand topology). Δ is a locally compact Hausdorff space, and we identify Y^* with $M(\Delta)$ as before. Let $T: A \to C_0(\Delta)$ be the Gelfand transform $Tx = \hat{x}$. Then we have:

1.3. COROLLARY. *Let $T^*: M(\Delta) \to A^*$ be the adjoint of the Gelfand transform, and let $x^* \in A^*$. Then $x^* = T^*\mu$ for some $\mu \in M(\Delta)$ with $\|\mu\| \leqslant M$ if and only if*

$$|\langle x, x^*\rangle| \leqslant M\|\hat{x}\|_\infty$$

for all X in a dense subset of A.

A special case of particular interest is the group algebra $A = L^1(G)$ of a locally compact abelian group G (with respect to Haar measure dt). Δ is identified with the dual group Γ of G, and the Gelfand transform reduces to the Fourier transform

$$Tf(\gamma) = \hat{f}(\gamma) = \int_G (t, \gamma) f(t) dt \quad (\gamma \in \Gamma)$$

where (t, γ) denotes the value of the character γ at t. It follows immediately from Fubini's theorem that for $\mu \in M(\Gamma)$,

$$T^*\mu(t) = \int_\Gamma (t, \gamma)\mu(d\gamma) \quad (t \in G)$$

that is, T^* is the Fourier-Stieltjes transform. The special case of Corollary 1.3 for the present situation gives the well-known Schoenberg-Eberlein characterization of Fourier-Stieltjes transforms:

1.4. COROLLARY. *A bounded continuous functions φ on G is the Fourier-Stieltjes transform of some $\mu \in M(\Gamma)$ with $\|\mu\| \leqslant M$ if and only if*

$$\left|\int_G f\varphi\, dt\right| \leqslant M\|\hat{f}\|_\infty$$

for all f in a dense subset of $L^1(G)$.

An interesting general consequence of Lemma 1.1 is the following:

1.5. COROLLARY. *For any $T \in B(X, Y)$, the restriction of T^* to the closed M-ball in Y^* has weak*-closed range.*

Proof. Let $\{x_\alpha^*\}$ be a net in X^* such that $x_\alpha^* \to x^*$ weak*, and $x_\alpha^* = T^* y_\alpha^*$ with $\|y_\alpha^*\| \leqslant M$. Taking the limit in the inequalities $|\langle x, x_\alpha^*\rangle| \leqslant M\|Tx\|$, we obtain $|\langle x, x^*\rangle| \leqslant M\|Tx\|$, and therefore, by Lemma 1.1, $x^* = T^* y^*$ for some $y^* \in Y^*$ with $\|y^*\| \leqslant M$.

In the setting of Corollary 1.2, if $\varphi_n = T^*\mu_n$ with $\|\mu_n\| \leqslant M$ for all n, and if $\varphi_n \to \varphi$ *pointwise a.e.*, then $\|\varphi_n\| = \|T^*\mu_n\| \leqslant \|T^*\|\|\mu_n\| \leqslant \|T^*\|M$, so that $\varphi_n \to \varphi$ weak*, by the Lebesgue dominated convergence theorem. Hence $\varphi = T^*\mu$ for some $\mu \in M(\Omega)$ with $\|\mu\| \leqslant M$, and in case T^* is one-to-one, μ is the w*-limit of μ_n. This is true in particular in the setting of Corollary 1.4. Thus, if the continuous function φ on G is the *pointwise* limit of $\hat{\mu}_n$, where $\mu_n \in M(\Gamma)$ and $\|\mu_n\| \leqslant M$, then $\varphi = \hat{\mu}$ for $\mu \in M(\Gamma)$ with $\|\mu\| \leqslant M$, and $\mu = \text{w*-lim}\, \mu_n$.

We state this generalized version of the "continuity theorem" as

1.6. COROLLARY. *Let T be a bounded linear operator from $L^1(S, \Sigma, \sigma)$ to $C_0(\Omega)$. Let $\varphi_n = T^*\mu_n$ with $\|\mu_n\| \leqslant M$ $(n = 1, 2, \ldots)$ and $\varphi_n \to \varphi$ pointwise* a.e. *Then $\varphi = T^*\mu$ for some $\mu \in M(\Omega)$ with $\|\mu\| \leqslant M$. If T^* is one-to-one, then $\mu = \text{w*-lim}\, \mu_n$.*

A special case of Corollary 1.2 gives a necessary and sufficient condition for a bounded complex sequence to be the moments sequence of a measure on $\Omega = [0,1]$ or $[-1, 1]$. Indeed, consider the operator $T: \ell^1 \to C[0, 1]$ (or $C[-1, 1]$) defined by

$$T\xi = \sum_{n=0}^{\infty} \xi_n t^n \quad \xi = \{\xi_n\}_0^\infty \in \ell^1.$$

(The series is majorized by $\sum |\xi_n| = \|\xi\|_1 < \infty$, and so defines a continuous function indeed.)

Let $[a, b]$ be either $[0, 1]$ or $[-1, 1]$. A simple calculation shows that $T^*: M([a, b]) \to \ell^\infty$ is given by

$$T^*\mu = \left\{\int_a^b t^n \mu(\mathrm{d}t)\right\}_{n=0}^{\infty} \qquad (\mu \in M([a, b])).$$

Therefore we have the following special case of Corollary 1.2.

1.7. COROLLARY. *$\varphi = \{\varphi_n\} \in \ell^\infty$ is the moments sequence of some $\mu \in M([a, b])$ with $\|\mu\| \leqslant M$ if and only if*

$$\left|\sum \xi_n \varphi_n\right| \leqslant M \max_{[a,b]} \left|\sum \xi_n t^n\right|$$

for any finite set of complex numbers ξ_j.

The corresponding version for the trigonometric moments problem follows by an analogous specialization of Corollary 1.2.

2. EXAMPLES

2.1. INTEGRAL OPERATORS. Let $K\colon \mathbf{R}^2 \to \mathbf{C}$ be a *bounded* function such that

(i) $K(\cdot, t) \in C(\mathbf{R})$ for each $t \in \mathbf{R}$;

(ii) $K(s, \cdot)$ is measurable on $\mathbf{R}$, for each $s \in \mathbf{R}$;

(iii) $\lim\limits_{|s|\to\infty} \int_a^b K(s, t)\mathrm{d}t = 0$, for each $a, b \in \mathbf{R}$.

For $f \in L^1(\mathbf{R})$, set $(T_K f)(s) = \int_{\mathbf{R}} K(s, t) f(t)\mathrm{d}t$. It follows easily from Lebesgue's dominated convergence theorem that $T_K f$ is continuous on $\mathbf{R}$. Since

$$\|T_K f\|_\infty \leqslant \sup_{\mathbf{R}^2} |K| \cdot \|f\|_1$$

and $T_K f \xrightarrow[|s|\to\infty]{} 0$ by (iii) when f is the characteristic function of any interval, it follows that $T_K f$ vanishes at infinity for any $f \in L^1(\mathbf{R})$ (because the said characteristic functions are fundamental in $L^1(\mathbf{R})$). Hence $T_K \in B(L^1(\mathbf{R}), C_0(\mathbf{R}))$. By Fubini's theorem, for $\mu \in M(\mathbf{R})$,

$$(T_K^* \mu)(t) = \int_{\mathbf{R}} K(s, t)\mu(\mathrm{d}s) \quad \text{(for a.a. } t \in \mathbf{R}).$$

We then have the following special case of Corollary 1.2.

2.2. COROLLARY. *Let* $K\colon \mathbf{R}^2 \to \mathbf{C}$ *be a bounded function satisfying* (i) — (iii), *and let* $\varphi \in L^\infty(\mathbf{R})$. *Then* $\varphi(t) = \int_{\mathbf{R}} K(s, t)\, \mu(\mathrm{d}s)$ (a.a. $t \in \mathbf{R}$) *for some* $\mu \in M(\mathbf{R})$ *with* $\|\mu\| \leqslant M$ *if and only if*

$$\left|\int_{\mathbf{R}} f\varphi \,\mathrm{d}t\right| \leqslant M\|T_K f\|_\infty \quad \textit{for all } f \textit{ in a dense subset of } L^1(\mathbf{R}).$$

Note that 2.2 is valid in particular when K is a bounded separately continuous kernel satisfying Condition (iii). If $k\colon \mathbf{R} \to \mathbf{C}$ is a bounded continuous function such that $(Jk)(t) = \int_0^t k(u)\mathrm{d}u$ is also bounded on $\mathbf{R}$, then the kernel $K(s, t) = k(st)$ satisfies the above conditions.

The Fourier kernel $K(s, t) = e^{-ist}$ and the cosine kernel $K(s, t) = \cos(st)$ are of this type. Denote $\tilde{f}(s) = \int_{\mathbf{R}} \cos(st)f(t)\mathrm{d}t$. Then formally (for later application):

2.3. COROLLARY. *A bounded continuous function φ on $\mathbf{R}$ is the cosine-Stieltjes transform* $\left(\varphi(t) = \int_{\mathbf{R}} \cos(st)\mu(\mathrm{d}s)\right)$ *of some $\mu \in M(\mathbf{R})$ with $\|\mu\| \leqslant M$ iff* $\left|\int_{\mathbf{R}} f\varphi \,\mathrm{d}t\right| \leqslant M\|\tilde{f}\|_\infty$ *for all f in a dense subset of $L^1(\mathbf{R})$.*

2.4. CONVOLUTION KERNELS. Suppose $k \in C_b(\mathbf{R})$ (bounded continuous complex function on $\mathbf{R}$), and the improper Riemann integral $\int_{-\infty}^{\infty} k(u)\mathrm{d}u$ converges. Set $K(s, t) = k(t - s)$. Then $\int_a^b K(s, t)\mathrm{d}t = \int_{a-s}^{b-s} k(u)\mathrm{d}u \to 0$ as $|s| \to \infty$, and therefore K satisfies all the conditions of Corollary 2.2. Denoting convolution as usual ($k * \mu(t) = \int_{\mathbf{R}} k(t - s)\mu(\mathrm{d}s)$) and $\check{k}(u) = k(-u)$, we have the following:

2.5. COROLLARY. *Let $k \in C_b(\mathbf{R})$ have convergent improper Riemann integral on $\mathbf{R}$, and let $\varphi \in C_b(\mathbf{R})$. Then $\varphi = k * \mu$ for some $\mu \in M(\mathbf{R})$ with $\|\mu\| \leqslant M$ iff*

$$\left|\int_{\mathbf{R}} f\varphi \,\mathrm{d}t\right| \leqslant M\|\check{k} * f\|_\infty$$

for all f in a dense subset of $L^1(\mathbf{R})$.

Moreover, if $\int_{-\infty}^{\infty} k\mathrm{d}u$ converges ***absolutely*** *and $\hat{k} \neq 0$ on $\mathbf{R}$, then the representation $\varphi = k * \mu$ is* ***unique*** *(indeed, if $k * \mu = 0$, then $\hat{k}\hat{\mu} = 0$, hence $\hat{\mu} = 0$, and so $\mu = 0$).*

The Dirichlet and Fejér kernels $\frac{\sin u}{u}$ and $\left(\frac{\sin u}{u}\right)^2$ are classical kernels atisfying the conditions of Corollary 2.5, so that we have criteria for representation of functions as Dirichlet or Fejér integrals of measures. The Poisson kernel $k_\varepsilon(u) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + u^2}$ (for $\varepsilon > 0$ *fixed*) is in $C_b(\mathbf{R}) \cap L^1(\mathbf{R})$ ($\|k_\varepsilon\|_1 = 1$) and $\hat{k}_\varepsilon(t) = ce^{-\varepsilon|t|} \neq 0$. Therefore, by Corollary 2.5, a bounded continuous function φ on $\mathbf{R}$ is the Poisson integral of a measure $\mu \in M(\mathbf{R})$ with $\|\mu\| \leqslant M$, that is

$$(*) \qquad \varphi(t) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\varepsilon}{\varepsilon^2 + (t-s)^2}\,\mu(\mathrm{d}s) \quad (t \in \mathbf{R})$$

if and only if

$$\left|\int_{\mathbf{R}} f\varphi \,\mathrm{d}t\right| \leqslant M\|\tilde{f}_\varepsilon\|_\infty \quad \text{for all } f \text{ in a dense subset of } L^1(\mathbf{R}).$$

Here $\tilde{f}_\varepsilon(s) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\varepsilon}{\varepsilon^2 + (t-s)^2} f(t)\mathrm{d}t$ is the Poisson transform of f, and the representation (*) is *unique* (M may depend on ε).

3. APPLICATIONS TO SPECTRAL THEORY: SINGLE OPERATOR

Let T be an operator with real spectrum with domain $\mathrm{D}(T)$ in a reflexive Banach space X. Let $\mathrm{R}(\lambda, T) = (\lambda I - T)^{-1}$.

3.1. DEFINITION. The *semisimplicity manifold* Z for T is the set of all vectors $x \in X$ satisfying the following two conditions:

(i) $\lim\limits_{\lambda\to\infty} \mathrm{R}(\lambda; T)x = 0$;

and

(ii) $|||x||| < \infty$,

where

$$|||x||| = \sup\left\|\int_{\mathbf{R}} f(t)\frac{1}{2\pi \mathrm{i}}[\mathrm{R}(t-\mathrm{i};\ T) - \mathrm{R}(t+\mathrm{i};\ T)]x\mathrm{d}t\right\|,$$

and the supremum is taken over all f in (a dense subset of) $L^1(\mathbf{R})$ with $\|\tilde{f}\|_\infty \leqslant 1$; here we denote

$$\tilde{f}(s) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{1 + (s-t)^2} f(t)\mathrm{d}t, \quad s \in \mathbf{R}.$$

The integrals in the definition of $|||x|||$ make sense because of (i). Z is a linear manifold. Define as usual the restriction $T|Z$ to be the operator T with domain

$$D(T|Z) = \{x \in D(T)\,;\; x,\ Tx \in Z\}.$$

3.2. DEFINITION. Denote by $T(Z)$ the algebra of all linear transformations of X with domain Z and range in Z. Let $\mathscr{B}(\mathbf{R})$ be the Borel σ-algebra of $\mathbf{R}$. A *spectral measure on* Z is an algebra homomorphism

$$E\colon \mathscr{B}(\mathbf{R}) \to T(Z)$$

such that $E(\cdot)x$ is a (regular) σ-additive vector measure for each $x \in Z$.

3.3. LOCAL SPECTRAL THEOREM. *Let T be an operator with real spectrum, with domain $D(T)$ in a reflexive Banach space X, and let Z be its semisimplicity manifold. Then there exists a spectral measure E on Z such that*

(i) *for each $\delta \in \mathscr{B}(\mathbf{R})$, $E(\delta)$ commutes with every $U \in B(X)$ which commutes with T;*

(ii) $D(T|Z) = \left\{x \in Z\ ;\ \int_{\mathbf{R}} sE(\mathrm{d}s)x \text{ “exists” and belongs to } Z\right\};$

and

(iii) $Tx = \int_{\mathbf{R}} sE(\mathrm{d}s)x$ *for all* $x \in D(T|Z)$.

Note that the “existence” of $\int_{\mathbf{R}} sE(\mathrm{d}s)x$ means that the integrals $\int_a^b sE(\mathrm{d}s)x$ converge (strongly) in X $\left(\text{to} \int_{\mathbf{R}} sE(\mathrm{d}s)x\right)$ as $a \to -\infty$ and $b \to \infty$.

Proof. For each $x \in Z$ and $x^* \in X^*$, consider the bounded continuous function

$$\varphi(t) = \frac{1}{2\pi \mathrm{i}}\, x^*[R(t - \mathrm{i};\ T) - R(t + \mathrm{i};\ T)]x$$

(it is *bounded* because $R(\lambda; T)x \to 0$ as $\lambda \to \infty$, for each $x \in Z$). Then, for all $f \in L^1(\mathbf{R})$,

$$\left|\int_{\mathbf{R}} f(t)\varphi(t)\mathrm{d}t\right| \leqslant \|x^*\|\, |||x|||\, \|\hat{f}\|_\infty.$$

By our next to last example, it follows that there exists a *unique* (regular) *measure* $\mu(\cdot|x, x^*) \in M(\mathbf{R})$ with

$$\|\mu(\cdot|x, x^*)\| \leqslant \|x^*\| \, |||x|||$$

such that

$$\varphi(t) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{1}{1 + (t - s)^2}\, \mu(\mathrm{d}s|x, x^*). \tag{1}$$

It then follows as usual from the reflexivity of X, that there exists a unique linear transformation $E(\delta)\colon Z \to X$ (for each $\delta \in \mathscr{B}(\mathbf{R})$) such that

$$\mu(\delta|x, x^*) = x^* E(\delta)x \quad (x \in Z, x^* \in X^*)$$

and

$$\|E(\delta)x\| \leqslant |||x||| \quad (\delta \in \mathscr{B}(\mathbf{R}), x \in Z).$$

For each $x \in Z$, $E(\cdot)x$ is an X-valued countably additive function on $\mathscr{B}(\mathbf{R})$, so that we may rewrite (1) in the form

$$[\mathrm{R}(t - \mathrm{i};\ T) - \mathrm{R}(t + \mathrm{i};\ T)]x = \int_{\mathbf{R}} \left[\frac{1}{t - \mathrm{i} - s} - \frac{1}{t + \mathrm{i} - s} \right] E(\mathrm{d}s)x. \tag{2}$$

Define $F(z)x = \int_{\mathbf{R}} \frac{1}{z - s} E(\mathrm{d}s)x$ for $x \in Z$ and $z \in \mathbf{C}\setminus\mathbf{R}$. $F(\cdot)x$ is a well-defined analytic function in $\mathbf{C}\setminus\mathbf{R}$, and vanishes for $z \to \infty$. Consider $G(z)x = F(z)x - \mathrm{R}(z;\ T)x$ for $z \in \mathbf{C}\setminus\mathbf{R}$ and $x \in Z$ fixed. $G(\cdot)x$ is analytic and $G(z)x \to 0$ for $z \to \infty$ (since $x \in Z$). We may rewrite the relation (2) in the form:

$$G(t + \mathrm{i})x = G(t - \mathrm{i})x \quad (\forall t \in \mathbf{R}). \tag{3}$$

Fix $x \in Z$ and $x^* \in X^*$, and consider the complex analytic function

$$\Phi(z) = x^* G(z)x + \overline{x^* G(\bar{z})x} \quad (z \in \mathbf{C}^+).$$

$\Phi(z) \to 0$ for $z \to \infty$. On the line $z = t + \mathrm{i}$, it follows from (3) that

$$\Phi(t + \mathrm{i}) = 2\mathrm{Re}\, x^* G(t + \mathrm{i})x.$$

Therefore, given $\varepsilon > 0$ arbitrary, choose $K_\varepsilon > 0$ such that $|\Phi(z)| < \varepsilon$ for z with $\operatorname{Im} z \geqslant 1$ and $|z| = K > K_\varepsilon$. Then the *real harmonic* function $\operatorname{Im}\Phi(z)$ has absolute value less than ε on the contour

$$\Gamma = \{t + i; -K \leqslant t \leqslant K\} \cup \{z \in \mathbf{C}; |z - i| = K, \operatorname{Im} z > 1\},$$

and therefore $|\operatorname{Im}\Phi(z)| < \varepsilon$ for all z within the contour (by the maximum-minimum principles for harmonic functions). Since K is arbitrary ($K > K_\varepsilon$), and then $\varepsilon > 0$ is arbitrary as well, we conclude that $\operatorname{Im}\Phi(z) = 0$ in the half-plane $\operatorname{Im} z \geqslant 1$. Therefore $\Phi(z) = 0$ in this half-plane. Since Φ is analytic in $\operatorname{Im} z > 0$ it follows that $\Phi = 0$ in $\mathbf{C}^+$.

Using the same argument with the function

$$\Psi(z) = x^* G(z)x - \overline{x^* G(\bar{z})x}$$

and the real harmonic function $\operatorname{Re}\Psi(z)$, we obtain similarly that $\Psi = 0$ in $\mathbf{C}^+$. Therefore $x^* G(z)x = (1/2)(\Phi(z) + \Psi(z)) = 0$ in $\mathbf{C}^+$, for each $x \in Z$ and $x^* \in X^*$. This means that

$$R(z; T)x = \int_{\mathbf{R}} \frac{1}{z - s} E(ds)x \quad (x \in Z)$$

for all $z \in \mathbf{C}^+$ (and similarly for $z \in \mathbf{C}^-$). This is relation (5) in our paper [8], from which we deduce there Theorem 3.3.

The "maximal-uniqueness" property (2.4 in [8]) follows trivially from the Fubini theorem and the uniqueness property of the Stieltjes transform. In particular, the present definition of Z must coincide with that of [8].

4. APPLICATIONS TO SPECTRAL THEORY: COSINE OPERATOR FUNCTIONS

Let $C(\cdot)\colon \mathbf{R} \to B(X)$ be a strongly continuous cosine operator function, that is $C(0) = I$ and

$$C(t + s) + C(t - s) = 2C(t)C(s) \quad (s, t \in \mathbf{R}).$$

4.1. DEFINITION. The *semisimplicity manifold* Z for $C(\cdot)$ is the set of all $x \in X$ for which $C(\cdot)x$ is *bounded*, and

$$|||x||| := \sup \left\| \int_{\mathbf{R}} f(t)C(t)x\,dt \right\| < \infty,$$

where the supremum is taken over all $f \in L^1(\mathbf{R})$ with $\|\tilde{f}\|_\infty \leqslant 1$, and $\tilde{f}$ is the cosine transform of f:

$$\tilde{f}(s) = \int_{\mathbf{R}} \cos(st) f(t) \mathrm{d}t \quad (f \in L^1(\mathbf{R})).$$

We shall assume that the Banach space X is *reflexive*. For each $x \in Z$ and $x^* \in X^*$, we consider the bounded continuous function on $\mathbf{R}$

$$\varphi(t) = x^* C(t) x.$$

Clearly $\left| \int_{\mathbf{R}} f\varphi \, \mathrm{d}t \right| \leqslant \|x^*\| \, |||x||| \, \|\tilde{f}\|_\infty$ for all $f \in L^1(\mathbf{R})$, and therefore (cf. Corollary 2.3), there exists a measure $\mu(\cdot \mid x, x^*) \in M(\mathbf{R})$ with $\|\mu(\cdot \mid x, x^*)\| \leqslant |||x||| \, \|x^*\|$ such that

$$\varphi(t) \equiv x^* C(t) x = \int_{\mathbf{R}} \cos(ts) \mu(\mathrm{d}s \mid x, x^*)$$

for all $t \in \mathbf{R}$, $x \in Z$ and $x^* \in X^*$. Using the even-ness of the cosine, we rewrite this relation as

(1) $$x^* C(t) x = \int_0^\infty \cos(ts) \, \alpha(\mathrm{d}s \mid x, x^*)$$

where $\alpha \in M([0, \infty))$ and $\|\alpha(\cdot \mid x, x^*)\| \leqslant 2|||x||| \, \|x^*\|$.

The cosine transform $\alpha \to \int_0^\infty \cos(ts) \alpha(\mathrm{d}s)$ acting on $M([0, \infty))$ is one-to-one. Indeed, if α belongs to its kernel, then for all $u > 0$,

$$0 = \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/4u} \int_0^\infty \cos(ts) \alpha(\mathrm{d}s) \mathrm{d}t = \int_0^\infty \frac{1}{\sqrt{\pi u}} \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/4u} \cos(ts) \mathrm{d}t \alpha(\mathrm{d}s) =$$

$$= 2 \int_0^\infty \mathrm{e}^{-us^2} \alpha(\mathrm{d}s) = 2 \int_0^\infty \mathrm{e}^{-u\sigma} \alpha_1(\mathrm{d}\sigma)$$

(where $\sigma = s^2$ and $\alpha_1(\delta^2) = \alpha(\delta)$ for each $\delta \in \mathscr{B}([0, \infty))$). By the uniqueness property of the Laplace transform, it follows that $\alpha_1 = 0$ and therefore $\alpha = 0$.

Now the uniqueness of the representation (1) and the reflexivity of X provide as usual linear transformations $E(\delta)\colon Z \to X$ for $\delta \in \mathscr{B}([0, \infty))$ such that $\alpha(\delta, x, x^*) = x^*E(\delta)x$ and $\|E(\delta)x\| \leqslant 2|||x|||$ for all $x \in Z$, $x^* \in X^*$, and $\delta \in \mathscr{B}([0, \infty))$. $E(\cdot)x$ is then a strongly countably additive regular vector measure on $\mathscr{B}([0, \infty))$, for each $x \in Z$, and we may rewrite (1) as

$$C(t)x = \int_0^\infty \cos(ts)E(\mathrm{d}s)x \quad (t \in \mathbf{R};\ x \in Z). \tag{2}$$

Taking $t = 0$, we see that $E([0, \infty)) = I|Z$. If $B \in B(X)$ commutes with $C(t)$ for all $t \in \mathbf{R}$, then for each $x \in Z$, $C(t)(Bx) = BC(t)x$ is a bounded function of t, and for $f \in L^1(\mathbf{R})$,

$$\left\| \int_{\mathbf{R}} f(t)C(t)Bx\,\mathrm{d}t \right\| = \left\| B \int_{\mathbf{R}} f(t)C(t)x\,\mathrm{d}t \right\| \leqslant \|B\|\, |||x|||\, \|\hat{f}\|_\infty,$$

so that $|||Bx||| \leqslant \|B\|\, |||x|||$. Therefore $BZ \subset Z$, and B is bounded as an operator in $(Z, |||\cdot|||)$, with norm $\leqslant \|B\|$.

Taking in particular $B = C(u)$ for $u \in \mathbf{R}$ fixed, we have

$$C(u)Z \subset Z \tag{3}$$

and

$$|||C(u)x||| \leqslant \|C(u)\|\, |||x||| \quad (u \in \mathbf{R},\ x \in Z).$$

Note that $|||\cdot||| \geqslant \|\cdot\|$ on Z, so that $(Z, |||\cdot|||)$ is a normed space. Indeed, if $\{f_n\} \subset L^1(\mathbf{R})$ is an approximate identity for $L^1(\mathbf{R})$ at 0, then $\hat{f}_n \to 1$ uniformly on $\mathbf{R}$ and $\int_{\mathbf{R}} f_n(t)C(t)x\,\mathrm{d}t \to C(0)x = x$ in the norm of X. Passing to the limit as $n \to \infty$ in the inequalities

$$\left\| \int_{\mathbf{R}} f_n(t)C(t)x\,\mathrm{d}t \right\| \leqslant |||x|||\, \|\hat{f}_n\|_\infty \quad (\text{for } x \in Z),$$

we obtain $\|x\| \leqslant |||x|||$.

Moreover, $(Z, |||\cdot|||)$ is *complete*. Indeed, let $\{x_n\} \subset Z$ be $|||\cdot|||$-Cauchy. It is then $\|\cdot\|$-Cauchy, and therefore $x_n \to x$ in X. Let $K = \sup_{n,m} |||x_n - x_m|||$. Then for each $f \in L^1(\mathbf{R})$,

$$\left\| \int_{\mathbf{R}} f(t)C(t)(x_n - x_m)\,\mathrm{d}t \right\| \leqslant |||x_n - x_m|||\, \|\hat{f}\|_\infty \leqslant K\|f\|_1, \tag{4}$$

so that $\|C(t)(x_n - x_m)\| \leqslant K$ for all n, m. In particular, letting $m \to \infty$, we have $\|C(t)(x_n - x)\| \leqslant K$ for all n and t. Also, for each n, $f(t)C(t)(x_n - x_m) \to f(t)C(t)(x_n - x)$ as $m \to \infty$. It then follows from Lebesgue's dominated convergence theorem that for each $n = 1, 2, \ldots$ and $f \in L^1(\mathbf{R})$

$$\int_{\mathbf{R}} f(t)C(t)(x_n - x_m)\mathrm{d}t \xrightarrow[m\to\infty]{} \int_{\mathbf{R}} f(t)C(t)(x_n - x)\mathrm{d}t$$

in X-norm.

Given $\varepsilon > 0$, choose n_ε such that $|||x_n - x_m||| < \varepsilon$ for $n, m > n_\varepsilon$. Then, for $n > n_\varepsilon$, letting $m \to \infty$ in the inequalities

$$\left\| \int_{\mathbf{R}} f(t)C(t)(x_n - x_m)\mathrm{d}t \right\| \leqslant |||x_n - x_m|||\, \|\tilde{f}\|_\infty < \varepsilon \|\tilde{f}\|_\infty,$$

we obtain $\left\| \int_{\mathbf{R}} f(t)C(t)(x_n - x)\mathrm{d}t \right\| \leqslant \varepsilon \|\tilde{f}\|_\infty$ for all $f \in L^1(\mathbf{R})$. Thus $|||x_n - x||| \leqslant \varepsilon$ for $n > n_\varepsilon$. In particular $x_n - x \in Z$, so that $x = x_n - (x_n - x) \in Z$, and $|||x_n - x||| \to 0$.

Returning to the integral representation (2), if $B \in B(X)$ commutes with $C(\cdot)$, we saw that $BZ \subset Z$, so that

$$\int_0^\infty \cos(ts)E(\mathrm{d}s)Bx = \int_0^\infty \cos(ts)BE(\mathrm{d}s)x \quad (t \in \mathbf{R},\ x \in Z),$$

and the uniqueness implies that

$$BE(\delta)x = E(\delta)Bx \quad (x \in Z, \delta \in \mathscr{B}([0, \infty)).$$

In particular, for each $u \in \mathbf{R}$

(5) $$C(u)E(\delta)x = E(\delta)C(u)x \quad (x \in Z,\ \delta \in \mathscr{B}([0, \infty)).$$

For $x \in Z$ and $t, u \in \mathbf{R}$,

$$\int_0^\infty \cos(ts)E(\mathrm{d}s)C(u)x = C(t)C(u)x = \frac{1}{2}C(t+u)x + \frac{1}{2}C(t-u)x =$$

$$= \int_0^\infty \left[\frac{1}{2}\cos(t+u)s + \frac{1}{2}\cos(t-u)s\right]E(\mathrm{d}s)x = \int_0^\infty \cos(ts)\cos(us)E(\mathrm{d}s)x.$$

By uniqueness and by (5), for each $\delta \in \mathscr{B}([0, \infty))$ etc.,

$$C(u)E(\delta)x = E(\delta)C(u)x = \int_0^\infty \cos(us)I_\delta(s)E(ds)x, \tag{6}$$

where I_δ is the characteristic function of δ.

In particular, $C(\cdot)E(\delta)x$ is a bounded function,

$$\|C(u)E(\delta)x\| \leqslant \|E(\delta)x\| \leqslant 2\|\|x\|\|, \tag{7}$$

and for each $f \in L^1(\mathbf{R})$,

$$\left\| \int_{\mathbf{R}} f(u)C(u)E(\delta)x \, du \right\| = \left\| \int_{\mathbf{R}} f(u) \int_0^\infty \cos(us)I_\delta(s)E(ds)x \, du \right\| =$$

$$= \left\| \int_0^\infty \tilde{f}(s)I_\delta(s)E(ds)x \right\| \leqslant 2\|\|x\|\| \, \|\tilde{f}\|_\infty$$

(recall that $\|x^*E(\cdot)x\| = \|\alpha(\cdot \mid x, x^*)\| \leqslant 2\|\|x\|\| \, \|x^*\|$).

Hence

$$\|\|E(\delta)x\|\| \leqslant 2\|\|x\|\| \quad (\delta \in \mathscr{B}([0, \infty)), \ x \in Z), \tag{8}$$

that is, $E(\delta)Z \subset Z$ (for each δ) and $E(\delta)$ is a bounded operator in the Banach space $(Z, \|\|\cdot\|\|)$ (with norm $\leqslant 2$).

Returning to (6), since $E(\delta)x \in Z$ for $x \in Z$, we have by (2)

$$\int_0^\infty \cos(us)E(ds)E(\delta)x = \int_0^\infty \cos(us)I_\delta(s)E(ds)x,$$

and by uniqueness,

$$E(\sigma)E(\delta)x = \int_0^\infty I_\sigma(s)I_\delta(s)E(ds)x = E(\sigma \cap \delta)x \tag{9}$$

for all $\sigma, \delta \in \mathscr{B}([0, \infty))$ and $x \in Z$.

We conclude that E is a spectral measure on Z (cf. Definition 3.2). Collecting the main facts established above, we have:

4.2. THEOREM. *Let $C(\cdot)$ be a cosine operator function on the reflexive Banach space X, and let Z be its semisimplicity manifold. Then there exists a spectral measure E on Z with the following properties:*

(1) E commutes with every $B \in B(X)$ which commutes with $C(\cdot)$;

(2) E is uniformly bounded as an operator function on the Banach space $(Z, |||\cdot|||)$; and

$$C(t)x = \int_0^\infty \cos(ts)E(\mathrm{d}s)x \quad (t \in \mathbf{R}, x \in Z), \tag{3}$$

where the integral is a strong integral in X.

4.3. MAXIMAL-UNIQUENESS. The pair (Z, E) associated with $C(\cdot)$ is *maximal-unique* in the following sense. If Z' is any subset of X with the property that, for each $x \in Z'$, there exists a regular strongly countably additive vector measure $E'(\cdot)x$ on $\mathscr{B}([0, \infty))$ such that (3) is valid with E', then $Z' \subset Z$ and $E'(\cdot)x = E(\cdot)x$ for all $x \in Z'$.

Proof. The vector measure $E'(\cdot)x$ is necessarily bounded (cf. [1, p. 319]). Let $\|E'(\delta)x\| \leqslant M_x$ $(x \in Z', \ \delta \in \mathscr{B}([0, \infty)))$. Then $\left\|\int_0^\infty \varphi(t)E'(\mathrm{d}t)x\right\| \leqslant 4M_x\|\varphi\|_\infty$ for each bounded Borel function φ. In particular, (3) for E' implies that $C(t)x$ is a bounded function on $\mathbf{R}$ for each $x \in Z'$. Also, by Fubini's theorem, for each $f \in L^1(\mathbf{R})$ and $x \in Z'$,

$$\left\|\int_{\mathbf{R}} f(t)C(t)x\mathrm{d}t\right\| = \left\|\int_{\mathbf{R}} f(t)\int_0^\infty \cos(ts)E'(\mathrm{d}s)x\mathrm{d}t\right\| =$$

$$= \left\|\int_0^\infty \hat{f}(s)E'(\mathrm{d}s)x\right\| \leqslant 4M_x\|\hat{f}\|_\infty$$

so that $|||x||| \leqslant 4M_x$, and therefore $Z' \subset Z$.

The uniqueness property of the cosine transform now implies that $E'(\cdot)x = E(\cdot)x$ for all $x \in Z'$.

4.4. It follows easily from (8), the closed graph theorem and the uniform boundedness theorem that the following statements (i)—(iii) are equivalent:

(i) $Z = X$;

(ii) $C(\cdot)$ is uniformly bounded and the norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are equivalent;

(iii) $E(\cdot)$ is a spectral measure in the usual sense on X.

When (i) (and therefore (ii) and (iii)) is satisfied, the operators $C(t)$ are scalar-type spectral operators with spectrum in $[0, \infty)$, with "common" integral representation

$$C(t) = \int_0^\infty \cos(ts)E(ds) = \cos(tS) \quad (t \in \mathbf{R}),$$

where $Sx = \int_0^\infty sE(ds)x = \lim_{n\to\infty} \int_0^n sE(ds)x$ is the scalar-type spectral operator with resolution of the identity $E(\cdot)$ and with domain $D(S)$ consisting of all vectors $x \in X$ for which the above limit exists in X.

4.5. Set $Z_0 = \{x \in Z\,;\, s^2 \text{ is } E(\cdot)x\text{-integrable on } [0, \infty)\}$. For each $t \in \mathbf{R}$, let

$$\varphi_t(s) = \begin{cases} \dfrac{2}{t^2}\,(1 - \cos(ts)) & t \neq 0 \\ s^2 & t = 0. \end{cases}$$

Then $0 \leqslant \varphi_t(s) = s^2 \left[\dfrac{\sin(ts/2)}{(ts/2)}\right]^2 \leqslant s^2$ and $\varphi_t(s) \to s^2$ as $t \to 0$. By the Dominated Convergence Theorem (see version for vector measures in [1; p. 328]), we have for $x \in Z_0$

$$\frac{2}{t^2}\,[I - C(t)]x = \int_0^\infty \frac{2}{t^2}\,(1 - \cos(ts))E(ds)x \xrightarrow[t\to 0]{} \int_0^\infty s^2E(ds)x,$$

where the limit exists in X. Therefore $Z_0 \subset D(A)$ where $D(A)$ denotes the domain of the *infinitesimal generator* A of the cosine operator function $C(\cdot)$, and

$$-Ax = \int_0^\infty s^2E(ds)x \quad (x \in Z_0).$$

5. FAMILIES OF UNBOUNDED OPERATORS

In this section, we demonstrate versions of Theorem 4.2 for certain cosine families of *unbounded* operators in Hilbert space. The spectral representation of one-parameter groups and semigroups of unbounded operators was studied by Nussbaum [10], Fröhlich [2], and Klein-Landau [9] in Hilbert space, and by Kantorovitz-Hughes [7] in Banach space.

5.1. DEFINITION. A *local cosine operator family* $\{C(t)\}_{t\in\mathbf{R}}$ consists of a one-parameter family of closed linear operators $C(t)$ acting in a given complex Hilbert space H, such that

(i) for each x in a dense linear manifold D in H, there exists $\varepsilon(x) > 0$ such that $x \in \mathrm{Dom}(C(t))$ for all t with $|t| < \varepsilon(x)$;

(ii) $C(t)x$ is strongly continuous for $|t| < \varepsilon(x)$, and $C(0)x = x$; and

(iii) if $|s|$, $|t|$, and $|t \pm s|$ are all less than $\varepsilon(x)$, then $C(s)x \in \mathrm{Dom}(C(t))$ and

$$C(t+s)x + C(t-s)x = 2C(t)C(s)x. \tag{$*$}$$

The local cosine family $\{C(t)\}$ is *bounded below* if

$$(C(t)x, x) \geqslant \|x\|^2 \tag{$**$}$$

for all $x \in \mathrm{Dom}\, C(t)$ and $t \in \mathbf{R}$.

Of course ($**$) implies the symmetry property

$$(C(t)x,\ y) = (x,\ C(t)y) \qquad x,\ y \in \mathrm{Dom}\, C(t),\ t \in \mathbf{R}. \tag{$***$}$$

5.2. THEOREM. *Let $\{C(t)\}_{t\in\mathbf{R}}$ be a bounded below local cosine family on the complex Hilbert space H. Then there exists a unique positive selfadjoint operator A such that*

$$C(t)x = \cosh(tA^{1/2})x$$

for all $x \in D$ and $|t| < \varepsilon(x)$.

REMARK. The family $\{\cosh(tA^{1/2})\}_{t\in\mathbf{R}}$ is a cosine family of selfadjoint operators that extends the local family $C(\cdot)$.

Proof. Let $\{\delta_n\}$ be an approximate delta distribution, that is, δ_n are non-negative C^∞-functions on $\mathbf{R}$ with $\mathrm{supp}\,\delta_n \subset [-1/n, 1/n]$ and $\int_{\mathbf{R}} \delta_n(t)\,\mathrm{d}t = 1$. For $x \in D$, fix $n(x) > 1/\varepsilon(x)$, and set

$$x_n = \int_{\mathbf{R}} \delta_n(s)C(s)x\,\mathrm{d}s \quad (n \geqslant n(x)), \tag{1}$$

where the integral is well-defined as a strong integral because of (i) and (ii). Clearly $x_n \to x$ strongly. Set

$$D_0 = \{x_n ; x \in D,\ n \geqslant n(x)\}.$$

Then D_0 is dense in H (because D is dense and $x_n \to x$ for each $x \in D$). If $y \in D_0$, so that $y = x_n$ for some $x \in D$ and $n \geqslant n(x)$, and if $|t| < \varepsilon'(y) = \varepsilon(x) - 1/n$ (note that $1/n \leqslant 1/n(x) < \varepsilon(x)$), then Condition (iii) implies that $C(s)x \in \operatorname{Dom} C(t)$ for all s with $|s| \leqslant 1/n$. Also $C(t)C(s)x = C(t+s)x/2 + C(t-s)x/2$ is strongly continuous for $|s| < 1/n$ (because $|t \pm s| < \varepsilon(x)$). Since $C(t)$ is closed, it follows from Theorem 3.3.2 in [3] that

$$y \in \operatorname{Dom} C(t) \tag{2}$$

and

$$C(t)y = \int_{\mathbf{R}} \delta_n(s) C(t) C(s) x \, ds.$$

Suppose $h > 0$ is such that $|t \pm h| < \varepsilon'(y)$. Using (2) for t and $t \pm h$, we obtain (using (*) four times):

$$[C(t+h) + C(t-h) - 2C(t)]y = \int_{\mathbf{R}} \delta_n(s)[C(t+h) + C(t-h) - 2C(t)]C(s)x\,ds =$$

$$= \int_{\mathbf{R}} \delta_n(s)C(s)[C(t+h) + C(t-h) - 2C(t)]x\,ds = \int_{\mathbf{R}} \delta_n(s)C(s)[2C(t)C(h) - 2C(t)]x\,ds =$$

$$= \int_{\mathbf{R}} \delta_n(s)C(t)[2C(s)C(h) - 2C(s)]x\,ds = \int_{\mathbf{R}} \delta_n(s)C(t)[C(s+h) + C(s-h) - 2C(s)]x\,ds =$$

$$= \int_{\mathbf{R}} [\delta_n(u-h) + \delta_n(u+h) - 2\delta_n(u)]C(t)C(u)x\,du.$$

For the variables u and t considered, Condition (*) is applicable, showing that $C(t)C(u)x$ is strongly continuous, and it follows therefore that for $|t| < \varepsilon'(y)$:

$$\operatorname*{s-lim}_{h\to 0+} h^{-2}[C(t+h) + C(t-h) - 2C(t)]y = \int_{\mathbf{R}} \delta_n''(u)C(t)C(u)x\,du. \tag{3}$$

Set $D_1 = \{C(t)y\,;\, y \in D_0,\ 0 \leqslant t < \varepsilon'(y)\}$.

Then $D_0 \subseteq D_1$, and as before, if $u \in D_1$, there exists $\varepsilon''(u) > 0$ such that $u \in \operatorname{Dom} C(t)$ for $|t| < \varepsilon''(u)$. By Condition (**),

$$2h^{-2}([C(h) - I]u, u) \geqslant 0 \quad (|h| < \varepsilon''(u)). \tag{4}$$

However

$$2h^{-2}[C(h) - I]u = h^{-2}[2C(h)C(t)y - 2C(t)y] =$$

$$= h^{-2}[C(t+h) + C(t-h) - 2C(t)]y$$

(with $y \in D_0$ and $|t| < \varepsilon'(y)$). By (3), $\operatorname{s-lim}_{h \to 0+} 2h^{-2}[C(h) - I]u$ exists; call it $A_0 u$. A^0 is defined as a linear transformation with the *dense* domain D_1 ($\supseteq D_0$), and it follows from (4) that A_0 is *positive*, that is

$$(A_0 u, u) \geqslant 0 \quad (u \in D_1).$$

Let A be the Friedrichs extension of A_0; A is a positive selfadjoint operator. Denote by E its resolution of the identity; let $E_m = E([0, m])$ $(m = 1, 2, \ldots)$, and $A_m = E_m A$. Note that A_m is a *bounded* positive selfadjoint operator. For $y \in D_0$, denote (cf. (2)):

$$y_m(t) = E_m C(t)y \quad 0 \leqslant t < \varepsilon'(y).$$

By (3) and the definition of A_0,

$$d^2 y_m(t)/dt^2 = E_m A_0 C(t)y = A_m y_m(t) \quad (|t| < \varepsilon'(y)).$$

Therefore

$$y_m(t) = \cosh(tA_m^{1/2})y_m(0) =$$

$$= E_m \cosh(tA^{1/2})y_m(0) = \cosh(tA^{1/2})y_m(0).$$

When $m \to \infty$, $y_m(t) \to C(t)y$ strongly (for each $|t| < \varepsilon'(y)$). In particular, $y_m(0) \to y$.

Also $\cosh(tA^{1/2})y_m(0) \to C(t)y$ strongly (by the preceding relation). Since $\cosh(tA^{1/2})$ is closed, it follows that

$$D_0 \subset \operatorname{Dom}(\cosh(tA^{1/2}))$$

and

$$C(t)y = \cosh(tA^{1/2})y \quad (y \in D_0, \ |t| < \varepsilon'(y)). \tag{5}$$

The relation (5) determines A uniquely, that is, if A_1 is a positive slefadjoint operator such that (5) holds with A_1 replacing A, then $A_1 = A$. Indeed, we then obtain

$$(\cosh(tA_1^{1/2})y,\ z) = (\cosh(tA^{1/2})y,\ z)$$

for all $y \in D_0$, $|t| < \varepsilon'(y)$, and $z \in H$. Fix y and z. Both sides extend analytically to the strip $\{t \in \mathbf{C}\ ; |\mathrm{Re}\, t| < \varepsilon'(y)\}$ and coincide on the segment $-\varepsilon'(y) < t < \varepsilon'(y)$; hence they coincide throughout the strip, and in particular on the imaginary axis. Thus

$$(\cos(tA_1^{1/2})y,\ z) = (\cos(tA^{1/2})y,\ z)$$

for *all* $t \in \mathbf{R}$, $z \in H$ and y in the dense subset D_0 of H. Therefore $\cos(tA_1^{1/2}) = \cos(tA^{1/2})$ $(t \in \mathbf{R})$, since these are *bounded* operators. Let

$$T(t)x = (\pi t)^{-1/2} \int_0^\infty e^{-s^2/4t} \cos(sA^{1/2})x\, ds, \quad t > 0.$$

Then $T(\cdot)$ is a C_0-semigroup with generator A. The semigroup $T_1(\cdot)$ defined by means of A_1 coincides with $T(\cdot)$ and has generator A_1. Hence $A_1 = A$ by the uniqueness of semigroup generators. Returning to (5) and recalling that $y = x_n$ with $x \in D$ *arbitrary* and $n \geqslant n(x)$, we have $x_n \in \mathrm{Dom}(\cosh(tA^{1/2}))$ and $\cosh(tA^{1/2})x_n = C(t)x_n$, for each fixed t with $|t| < \varepsilon(x)$ and n large enough. We saw that $x_n \to x$ strongly as $n \to \infty$. Also by (2)

$$C(t)x_n = \int_{\mathbf{R}} \delta_n(s) C(t) C(s) x\, ds =$$

$$= (1/2) \int_{\mathbf{R}} \delta_n(s)[C(t+s) + C(t-s)]x\, ds \to C(t)x$$

strongly as $n \to \infty$. Since $\cosh(tA^{1/2})$ is closed, it follows that $x \in \mathrm{Dom}(\cosh(tA^{1/2}))$ and

$$C(t)x = \cosh(tA^{1/2})x \tag{6}$$

for all $x \in D$ and $|t| < \varepsilon(x)$. Q.E.D.

We consider next a local cosine family $C(\cdot)$ satisfying the weaker condition $({*}{*}{*})$ instead of $({*}{*})$ (a local cosine family of symmetric operators).

5.3. THEOREM. *Let $\{C(t)\}_{t\in\mathbf{R}}$ be a local cosine family of symmetric operators. Then there exists a selfadjoint operator A such that*

$$C(t)x = \cosh(t\sqrt{A^+})x + \cos(t\sqrt{A^-})x$$

for all $x \in D$ and $|t| < \varepsilon(x)$.

Note again that the right hand side provides a globally defined cosine family of selfadjoint operators that extends the local family $C(\cdot)$. However A is not uniquely determined, as can be seen by modifying Fröhlich's example in an obvious way.

$A^\pm$ are the positive operators $f^\pm(A)$ where $f^+(t) = \max(t, 0)$ and $f^-(t) = -\min(t, 0)$.

Proof. Identify H with $H \oplus \{0\} \subset H \oplus H = \mathbf{H}$. With A_0 as in the proof of Theorem 5.2, consider $\mathbf{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & -A_0 \end{pmatrix}$ acting on $\mathbf{H}$. $\mathbf{A}_0$ has a selfadjoint extension $\mathbf{A}$ (cf. [2]) with resolution of the identity $\mathbf{E}$. Set

$$\mathbf{E}_m^+ = \mathbf{E}([0, m]); \quad \mathbf{E}_m^- = \mathbf{E}([-m, 0]),$$

and

$$\pm \mathbf{A}_m^\pm = \mathbf{E}_m^\pm \mathbf{A} \quad (m = 1, 2, \ldots).$$

For $y \in D_0$, define

$$\bar{y}_m^\pm(t) = \mathbf{E}_m^\pm \begin{pmatrix} C(t)y \\ 0 \end{pmatrix}$$

for $|t| < \varepsilon'(y)$ (cf. preceding proof and (2)).

Then for $|t| < \varepsilon'(y)$,

$$d^2\bar{y}_m^\pm(t)/dt^2 = \mathbf{E}_m^\pm \begin{pmatrix} A_0 C(t)y \\ 0 \end{pmatrix} = \mathbf{E}_m^\pm \mathbf{A}_0 \begin{pmatrix} C(t)y \\ 0 \end{pmatrix} =$$

$$= \mathbf{E}_m^\pm \mathbf{A} \begin{pmatrix} C(t)y \\ 0 \end{pmatrix} = \mathbf{A}_m^\pm \bar{y}_m^\pm(t).$$

Therefore $\bar{y}_m^\pm(t) = \cosh(t\sqrt{\mathbf{A}_m^\pm})\bar{y}_m^\pm(0)$ (since $\mathbf{A}_m^\pm$ is a bounded positive selfadjoint operator).

Consider also the positive selfadjoint operators $\mathbf{A}^+ = \mathbf{E}([0, \infty))\mathbf{A}$ and $\mathbf{A}^- = -\mathbf{E}((-\infty, 0])\mathbf{A}$. Then

$$\bar{y}_m^{\pm}(t) = \mathbf{E}_m^{\pm} \cosh(t\sqrt{\pm\mathbf{A}^{\pm}})\bar{y}_m^{\pm}(0) =$$

$$= \cosh(t\sqrt{\pm\mathbf{A}^{\pm}})\bar{y}_m^{\pm}(0) \quad (|t| < \varepsilon'(y)).$$

When $m \to \infty$, $\bar{y}_m^{\pm}(t) \to \bar{y}^{\pm}(t)$ strongly, where $\bar{y}^{+}(t) = \mathbf{E}([0, \infty))\begin{pmatrix} C(t)y \\ 0 \end{pmatrix}$ etc. Since $\cosh(t\sqrt{\pm\mathbf{A}^{\pm}})$ is a closed operator, it follows that $\bar{y}^{\pm}(0) \in \mathrm{Dom}(\cosh(t\sqrt{\pm\mathbf{A}^{\pm}}))$, and

$$\bar{y}^{\pm}(t) = \cosh(t\sqrt{\pm\mathbf{A}^{\pm}})\bar{y}^{\pm}(0). \tag{7}$$

Let $\mathbf{P}$ denote the projection of $\mathbf{H}$ onto $H = H \oplus \{0\}$. Then $\mathbf{P}$ commutes with $\mathbf{A}_0$, hence with $\mathbf{A}$, and so $A = \mathbf{PA}$ is a selfadjoint operator on H with resolution of the identity $E = \mathbf{PE}$. Identifying $\begin{pmatrix} y \\ 0 \end{pmatrix}$ with y, one sees immediately (using the usual spectral integrals) that

$$C(t)y = \cosh(t\sqrt{A^{\pm}})y + \cos(t\sqrt{A^{-}})y \tag{8}$$

for all $y \in D_0$ and $|t| < \varepsilon'(y)$. It then follows as at the end of the preceding proof that (8) is valid for all $y \in D$ and $|t| < \varepsilon(y)$.

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REFERENCES

1. DUNFORD, N.; SCHWARTZ, J. T., *Linear operators. Vol. 1*, Wiley-Interscience, New York, 1958.
2. FRÖHLICH, J., Unbounded symmetric semigroups on a separable Hilbert space are essentially selfadjoint, *Adv. Appl. Math.*, **1**(1980), 237–256.
3. HILLE, E.; PHILLIPS, R. S., *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. **31**, Providence, R.I., 1957.
4. KANTOROVITZ, S., The semi-simplicity manifold of arbitrary operators, *Trans. Amer. Math. Soc.*, **123**(1966), 241–252.
5. KANTOROVITZ, S., Spectrality criteria for unbounded operators with real spectrum, *Math. Ann.*, **256**(1981), 19–28.

6. KANTOROVITZ, S., *Spectral theory of Banach space operators*, Springer-Verlag, Lecture Notes in Mathematics, **1012**, 1983.
7. KANTOROVITZ, S.; HUGHES, R., Spectral representation of local semigroups, *Math. Ann.*, **259**(1982), 455--470.
8. KANTOROVITZ, S.; HUGHES, R., Spectral representations for unbounded operators with real spectrum, *Math. Ann.*, **282**(1988), 535—544.
9. KLEIN, A; LANDAU, L., Construction of a unique self-adjoint generator for a symmetric local semigroup, *J. Funct. Anal.*, 44(1981), 121--137.
10. NUSSBAUM, A. E., Spectral representation of certain one-parameter families of symmetric operators in Hilbert space, *Trans. Amer. Math. Soc.*, **152**(1970), 419—429.

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