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A STRONG TRANSITIVE ALGEBRA PROPERTY OF OPERATORS

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1. INTRODUCTION

An algebra of operators on a Hilbert space is called transitive if it has no nontrivial invariant subspaces. The transitive algebra problem is the question of whether every transitive operator algebra must be strongly dense in the algebra of all operators. There are a number of partial solutions to this problem [1], [3], [4], [5], [7], [9], [10], [11] and [12] and in this paper we shall make a contribution to this list.

As in [1] we shall say that an operator $T \in \mathcal{L}(\mathcal{H})$ has the *transitive algebra property* (TAP) if every transitive operator algebra $\mathcal{U}$ containing T is strongly dense in $\mathcal{L}(\mathcal{H})$. It was shown by Arveson [5] that the unilateral shift S has the TAP. In the present paper we shall study finite rank perturbations of S.

DEFINITION. ([3]). An operator $T \in \mathcal{L}(\mathcal{H})$ has the *strong transitive algebra property* (strong-TAP), if both the operators T and $S + T$ have the TAP.

A study of the strong-TAP was begun in [3]. In Section 2 we prove that every finite rank operator of the form $\sum_{i=1}^{n} f_i \otimes g_i$, where $f_i \in H^2$ is arbitrary and g_i is a rational function with poles off the closed unit disc, has the strong-TAP. This generalizes the results of [3]. We conclude this section with the conjecture that every finite rank operator has the strong transitive algebra property.

Some of the results of Section 2 are valid for finite rank perturbations of operators other than the unilateral shift. In these situations, one would like to know when does the operator $A|\mathcal{N}$ have the TAP for some (all) non-zero invariant subspace(s) of A. In Section 3 we show that if A is a strictly cyclic operator which generates a semi-simple weakly closed algebra $\mathcal{W}(A)$, then $A|\mathcal{N}$ has the TAP for every non-zero invariant subspace of A.

In Section 4 we obtain a partial solution of the well-known reductive algebra problem [11, Chapter 9]. Indeed, we prove that if a unital weakly closed reductive operator algebra $\mathscr{V}$ (i.e. Lat $\mathscr{V}$ = Lat $\mathscr{V}^*$) contains the operator $S + \varphi \otimes 1$, where φ is a single Blaschke factor, then $\mathscr{V}$ is a von Neumann algebra.

We conclude by presenting an open question that arises from our work.

2. FINITE RANK PERTURBATIONS OF THE SHIFT

Throughout this paper, $\mathscr{H}$ denotes a complex separable Hilbert space with a fixed orthonormal basis $\{e_i\}_{i\geqslant 1}$, and S denotes the unilateral shift ($Se_i = e_{i+1}$).

LEMMA 2.1. *Let $\mathscr{A}$ be a collection of non-zero operators on $\mathscr{H}$ and let $\mathscr{M} = \bigvee_{A\in\mathscr{A}} A\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$ such that $\mathscr{M} \in \operatorname{Lat}(T)$ and if $T|\mathscr{M}$ has the* TAP, *then $\mathscr{L}(\mathscr{H})$ is the only strongly (weakly) closed transitive operator algebra which contains $\mathscr{A} \cup \{T\}$.*

Proof. The lemma follows from Theorem 2.1 of [1]. Indeed, every operator $A \in \mathscr{A}$ has the matrix form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}$ with respect to the decomposition $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. It follows from the hypothesis that

$$(+) \qquad \bigvee \left\{ A_{11}\mathscr{M} + A_{12}\mathscr{M}^{\perp} : \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \in \mathscr{A} \right\} = \mathscr{M}.$$

The condition (+) and the assumption that $T|\mathscr{M}$ has the TAP imply the hypothesis of Theorem 2.1 of [1], hence the lemma follows from this theorem. ∎

COROLLARY 2.2. *If $A \in \mathscr{L}(\mathscr{H})$, p is a polynomial such that $p(A) \neq 0$. and $A|\operatorname{ran} p(A)^-$ has the* TAP, *then A has the* TAP.

For Theorem 2.3 and the discussion in Section 3 it will be convenient to say that an operator $A \in \mathscr{L}(\mathscr{H})$ has the *property* (P) if $A|\mathscr{M}$ has the TAP, whenever $\mathscr{M} \in \operatorname{Lat} A$, $\mathscr{M} \neq (0)$.

The following operators have the property (P): injective compact operators, unilateral shifts of finite multiplicity [11, Chapter 8], and the Dirichlet shift [12]. Furthermore, in the next section we show that strictly cyclic operators T which generate a semi-simple weakly closed algebra $\mathscr{W}(T)$ have the property (P).

THEOREM 2.3. *Let $A \in \mathscr{L}(\mathscr{H})$ have the property* (P) *and let $B \in \mathscr{L}(\mathscr{K})$ be algebraic. Let p be the minimal polynomial of B and suppose that $p(A) \neq 0$. Then*

$$T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

has the transitive algebra property for every $X \in \mathscr{L}(\mathscr{K}, \mathscr{H})$.

Proof. It follows from the assumption that $p(T) = \begin{pmatrix} p(A) & Y \\ 0 & 0 \end{pmatrix} \neq 0$ for some $Y \in \mathscr{L}(\mathscr{H}, \mathscr{H})$. Thus there exists a non-zero subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\mathscr{N} \oplus 0 = \operatorname{ran} p(T)^{-}$. For $x \in \mathscr{N}$ we have $T(x \oplus 0) = Ax \oplus 0$, thus $\mathscr{N} \in \operatorname{Lat} A$ and $T|\operatorname{ran} p(T)^{-}$ is unitarily equivalent to $A|\mathscr{N}$. Hence, $T|\operatorname{ran} p(T)^{-}$ has the TAP and Corollary 2.2 implies that T has the TAP. ▨

Recall that if $A \in \mathscr{L}(\mathscr{H})$ is a completely non-unitary (c.n.u.) contraction and if φ is a bounded analytic function on the unit disc,

$$\varphi(A) = \lim_{r \uparrow 1} \varphi(rA) \quad \text{(SOT)}.$$

Also, recall that a c.n.u. contraction A is called a C_0-operator if there exists an inner function φ such that $\varphi(A) = 0$.

THEOREM 2.4. *If $A \in \mathscr{L}(\mathscr{H})$ is a C_0-operator and if $T = \begin{pmatrix} S & X \\ 0 & A \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ is a completely non-unitary contraction, then T has the* TAP.

Proof. Let $\varphi \in H^\infty$ be an inner function such that $\varphi(A) = 0$. It follows from the Sz.-Nagy—Foiaş, functional calculus that the operator $\varphi(A)$ has the form $\begin{pmatrix} S_\varphi & * \\ 0 & 0 \end{pmatrix}$. An argument similar to the one in the proof of Theorem 2.3 implies that T has the TAP. ▨

The following corollary generalizes results of Section 3 of [3]. To make the connection to [3], let $\mathscr{H}_n$ denote the span of the first n vectors in the orthonormal basis $\{e_i\}_{i \geq 1}$ of $\mathscr{H}$. Then every finite rank operator of the form $A \oplus 0 \in \mathscr{L}(\mathscr{H}_n \oplus \mathscr{H}_n^{\perp})$ can be written as

$$A \oplus 0 = \sum_{i=1}^{n} Ae_i \otimes e_i,$$

where $x \otimes y$ denotes the rank one operator $z \to (z, y)x$. Hence, Corollary 2.5 contains the results of Section 3 of [3] as a special case.

COROLLARY 2.5. *Let S be the unilateral shift acting as multiplication by z on H^2 and let $n \in \mathbf{N}$. Let $\{g_i\}_{i=1}^n$ be a set of rational functions with poles off the closed unit disc and let $\{f_i\}_{i=1}^n \subseteq H^2$ be arbitrary. The finite rank operator $F = \sum_{i=1}^{n} f_i \otimes g_i$, $F \neq 0$, has the strong-*TAP.

Proof. Let $\mathcal{M}$ denote the smallest invariant subspace of S^* which contains the functions $\{g_i\}_{i=1}^n$. It is easy to see that $\dim \mathcal{M} < \infty$. For example, see [6]. If we let $\mathcal{N} = H^2 \ominus \mathcal{M}$, then $\mathcal{N} \subseteq \operatorname{Ker}(F)$. The operator $T = S + F$ has the matrix form $\begin{pmatrix} S & T_{12} \\ 0 & T_{22} \end{pmatrix}$ with respect to the decomposition $H^2 = \mathcal{N} \ominus \mathcal{N}^\perp$, where T_{22} is algebraic. It now follows from Theorem 2.3 that T has the TAP, i.e., F has the strong-TAP. ▨

Corollary 2.5 suggests the following:

CONJECTURE. *If $F \in \mathcal{L}(\mathcal{H})$ is an arbitrary finite rank operator, then F has the strong transitive algebra property.*

3. THE PROPERTY (P) AND STRICTLY CYCLIC OPERATORS

In this section we show that certain strictly cyclic operators satisfy the assumption of Theorem 2.3.

For $T \in \mathcal{L}(\mathcal{H})$, we let $\mathcal{W}(T)$ denote the smallest weakly closed subalgebra of $\mathcal{L}(\mathcal{H})$ which contains T and the identity operator. Recall that T is called *strictly cyclic*, if there exists a vector $x_0 \in \mathcal{H}$ such that $\mathcal{H} = \{Ax_0 : A \in \mathcal{W}(T)\}$.

It was shown by Lambert [9], that every strictly cyclic operator has the TAP. However, if T is strictly cyclic and $\mathcal{M} \in \operatorname{Lat} T$, then $T|\mathcal{M}$ does not have to be strictly cyclic (see the discussion at the end of this section). In view of Theorem 2.3 it would be interesting to know whether every strictly cyclic operator T has the property (P), i.e., does $T|\mathcal{M}$ have the TAP for every non-zero invariant subspace $\mathcal{M}$ of T?

In the following lemma we consider a special case of this question.

LEMMA 3.1. *If $T \in \mathcal{L}(\mathcal{H})$ is a strictly cyclic operator and if $\mathcal{M}$ is an invariant subspace of T such that $\mathcal{M} \nsubseteq \operatorname{ran}(T)^-$, then $T|\mathcal{M}$ has the* TAP.

Proof. The proof will use a consequence of Arveson's Lemma [11, Chapter 8]. We first collect a few facts about $\mathcal{M}$ and T.

We assumed that $\mathcal{M} \nsubseteq \operatorname{ran}(T)^-$, thus there exists a non-zero vector e_0, $e_0 \in \mathcal{M} \cap (T\mathcal{M})^\perp$ and $e_0 \notin \operatorname{ran}(T)^-$. Fix such an e_0, $\|e_0\| = 1$, and let x_0 be a strictly cyclic vector for T.

CLAIM 1. *If $A \in \mathcal{W}(T)$ such that $Ax_0 \in \mathcal{M}$, then e_0 is an eigenvector for $(A|\mathcal{M})^*$.*

Proof of Claim 1. First we note that if $B \in \mathcal{W}(T)$, $Bx_0 \in \mathcal{M}$, and $Bx_0 \perp e_0$, then for every polynomial p we have

$$0 = (p(T)Bx_0, e_0) = (p(T)x_0, B^*e_0),$$

i.e., $B^*e_0 = 0$.

Now suppose $Ax_0 \in \mathcal{M}$ and note that $(A|\mathcal{M})^* e_0 = \lambda e_0 + x$, where $\lambda \in \mathbf{C}$, $x \in \mathcal{M}$, and $x \perp e_0$. Since x_0 is a strictly cyclic vector for T, there exists an operator $B \in \mathcal{W}(T)$ such that $Bx_0 = x$. We now show that $x = 0$. Indeed,

$$\|x\|^2 = ((A|\mathcal{M})^* e_0 - \lambda e_0, x) = ((A|\mathcal{M})^* e_0, x) =$$

$$= (e_0, Ax) = (e_0, ABx_0) = (B^* e_0, Ax_0) = 0 \qquad \text{(since } B^* e_0 = 0\text{)}.$$

Thus $(A|\mathcal{M})^* e_0 = \lambda e_0$.

If e_0 and x_0 are as above, then there exists an operator $A_0 \in \mathcal{W}(T)$ such that $A_0 x_0 = e_0$.

CLAIM 2. $(e_0, A_0^* e_0) \neq 0$.

Proof of Claim 2. Since T is strictly cyclic, the set of polynomials in T is norm dense in $\mathcal{W}(T)$. Thus there exist sequences of polynomials $\{p_n\}$ and $\{q_n\}$ such that

$$p_n(T) = p_n(0) + Tq_n(T) \to A_0 \text{ in norm.}$$

Hence,

$$p_n(0) = (p_n(T)e_0, e_0) \to (A_0 e_0, e_0). \tag{1}$$

To finish the proof of Claim 2 we argue by contradiction, i.e., let $(e_0, A_0^* e_0) = (A_0 e_0, e_0) = 0$. It follows from this assumption and (1) that

$$p_n(0) \to 0$$

and

$$\|e_0 - Tq_n(T)x_0\| \leqslant \|e_0 - p_n(T)x_0\| + |p_n(0)| \, \|x_0\| =$$

$$= \|A_0 x_0 - p_n(T)x_0\| + |p_n(0)| \, \|x_0\| \to 0.$$

That is, $e_0 \in \operatorname{ran}(T)^-$ which is a contradiction to the choice of e_0, thus $(e_0, A_0^* e_0) \neq 0$.

We are now ready to prove Lemma 3.1. Let $\mathcal{A}$ be a transitive subalgebra of $\mathcal{L}(\mathcal{M})$ that contains the operator $T|\mathcal{M}$. By Lemma 8.15 of [11] we only need to show that every densely defined graph transformation for $\mathcal{A}$ has compression spectrum. To this end, let R be a graph transformation for $\mathcal{A}$ with a dense domain $\mathcal{D} \subseteq \mathcal{M}$. (For a definition, see [11, Chapter 8].)

It follows from Claim 2 and the density of $\mathcal{D}$ that there exists a vector $x_1 \in \mathcal{D}$ such that $(x_1, e_0) \neq 0$ and $(x_1, A_0^* e_0) \neq 0$. We set

$$\lambda = \frac{(Rx_1, e_0)}{(x_1, e_0)},$$

and we will conclude the proof by showing that $(Rx, e_0) = \lambda(x, e_0)$ for all $x \in \mathscr{D}$. To this end, choose $A_1 \in \mathscr{W}(T)$ such that $A_1 x_0 = x_1$.

It follows from Claim 1 that there exists $\mu \in \mathbf{C}$ such that $(A_1|\mathscr{M})^* e_0 = \mu e_0$. We will show that $\mu \neq 0$. Indeed,

$$\mu = \mu(e_0, e_0) = ((A_1|\mathscr{M})^* e_0, e_0) = (e_0, A_1 e_0) =$$

$$= (e_0, A_1 A_0 x_0) = (A_0^* e_0, x_1) \neq 0.$$

Now let $x \in \mathscr{D}$ and $B \in \mathscr{W}(T)$ such that $Bx_0 = x$. By Claim 1 we have $(B|\mathscr{M})^* e_0 = \beta e_0$ for some $\beta \in \mathbf{C}$. Thus

$$(Rx, e_0) = (RBx_0, e_0) = (1/\bar{\mu})(RBx_0, (A_1|\mathscr{M})^* e_0) =$$

$$= (1/\bar{\mu})(A_1 RBx_0, e_0) = (1/\bar{\mu})(Rx_1, (B|\mathscr{M})^* e_0) =$$

$$= (\bar{\beta}/\bar{\mu})(Rx_1, e_0) = \lambda(\bar{\beta}/\bar{\mu})(x_1, e_0) = \lambda/\bar{\mu}(x_1, (B|\mathscr{M})^* e_0) =$$

$$= \lambda/\bar{\mu}(Bx_0, (A_1|\mathscr{M})^* e_0) = \lambda(x, e_0).$$ ∎

For the next theorem it will be convenient to say that T is *strongly strictly cyclic* if $T|(T^n \mathscr{H})^-$ is strictly cyclic for every $n \geqslant 0$. In the special case where T is a weighted shift operator, this definition coincides with the one given in [13].

THEOREM 3.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be strictly cyclic. If either*

(a) $\bigcap_{\lambda \in \mathbf{C}} \operatorname{ran}(T - \lambda)^- = (0)$, *or*

(b) $\bigcap_{n \geqslant 0} (T^n \mathscr{H})^- = (0)$ *and T is strongly strictly cyclic,*

then $T|\mathscr{M}$ has the TAP *for every $\mathscr{M} \in \operatorname{Lat} T$, $\mathscr{M} \neq (0)$,* i.e., *T has the property* (P).

Proof. (a) Let $\mathscr{M} \in \operatorname{Lat} T$, $\mathscr{M} \neq (0)$. It follows from the assumption that there exists $\lambda \in \mathbf{C}$ such that $\mathscr{M} \not\subseteq \operatorname{ran}(T - \lambda)^-$. The operator $T - \lambda$ is strictly cyclic, thus by Lemma 3.1 $(T - \lambda)|\mathscr{M}$ has the TAP. From this it follows that $T|\mathscr{M}$ has the TAP. The proof of (b) is similar. ∎

We note that condition (a) of the previous theorem is satisfied if and only if the algebra $\mathscr{W}(T)$ is semi-simple, i.e., the intersection of all maximal ideals in $\mathscr{W}(T)$ is (0) (see [9] for the details).

To conclude this section, we make a few remarks about the hypothesis of Theorem 3.2. There are examples of strictly cyclic weighted shift operators that do not satisfy the assumptions of Theorem 3.2 (see [8]). However, it is quite easy to give examples of strictly cyclic operators T that satisfy both assumptions (a) and (b) and have invariant subspaces $\mathscr{M}$ such that $T|\mathscr{M}$ is not strictly cyclic, i.e., the conclusion of Theorem 3.2 does not directly follow from Lambert's theorem ([9]).

EXAMPLE 3.3. Consider the space H_1^2 of analytic functions f in the open unit disc such that $f' \in H^2$. A norm on H_1^2 is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} (n+1)^2 |\hat{f}(n)|^2, \quad \text{where } f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n.$$

It is easy to check that $T: f \to zf$ is a weighted shift operator with the weight sequence $\{(n+2)/(n+1)\}$. Furthermore, it is well known that T is strictly cyclic (see [13, p. 99]) and T satisfies both the assumptions (a) and (b) of Theorem 3.2. The functions in H_1^2 extend to be continuous on the closed unit disc, thus $\mathcal{M} = \{f \in H_1^2 : : f(1) = 0\}$ is a proper invariant subspace of T. One can show that $\mathcal{N} = (z-1)^2 H_1^2$ is a dense linear manifold in $\mathcal{M}$ (see [13, pp. 108—109]). We now define the closable linear transformation R on $\mathcal{N}$ by $g \to g/(z-1)$ and we note that R commutes with T on the dense subset $\mathcal{N}$ of $\mathcal{M}$. If $T|\mathcal{M}$ were strictly cyclic, then it would follow from [9] that R is bounded. This is clearly impossible, thus it follows that $T|\mathcal{M}$ cannot be strictly cyclic.

4. A RANK ONE PERTURBATION OF THE UNILATERAL SHIFT AND THE REDUCTIVE ALGEBRA PROBLEM

The reductive algebra problem is the question: must every unital weakly closed reductive operator algebra $\mathcal{V}$ (i.e., $\mathrm{Lat}\,\mathcal{V} = \mathrm{Lat}\,\mathcal{V}^*$) be a von Neumann algebra? This question is still open. An affirmative answer to the reductive algebra problem provides an affirmative answer to the transitive algebra problem, i.e., every operator has the TAP.

THEOREM 4.1. *Let $\mathcal{V}$ be a unital weakly closed reductive subalgebra of $\mathcal{L}(\mathcal{H})$. If $\varphi(z) = \dfrac{\xi - z}{1 - \bar{\xi} z}$, $|\xi| < 1$ and if $S + \varphi \otimes 1 \in \mathcal{V}$, then $\mathcal{V}$ is a von Neumann algebra.*

Proof. We show that the operator $T = S + \varphi \otimes 1$ is similar to $S \oplus A$, where A is a finite dimensional completely non-unitary contraction. To this end, let $f(z) = \xi - z$ and let $g(z) = 1 - \bar{\xi} z$. The function g is invertible in H^∞, the algebra of bounded analytic functions on the unit disc. Hence, the analytic Toeplitz operator T_g ($T_g \psi = g\psi$, $\psi \in H^2$) is invertible and $T_g S T_{1/g} = S$.

For $\psi \in H^2$ we have:

$$(T_g \varphi \otimes 1 T_{1/g})(\psi) = (T_g \varphi \otimes 1)(\psi/g) =$$

$$= T_g \left(\frac{\psi(0)}{g(0)} \right) \varphi = \psi(0) g \varphi = \psi(0) f = (f \otimes 1)(\psi).$$

Thus, the operators T and $S + f \otimes 1$ are similar. We conclude by noting that the operator $S + f \otimes 1$ is unitarily equivalent to $S \oplus \xi$, where ξ is a one dimensional operator. The fact that $|\xi| < 1$ together with Theorem 1 of [2] imply that $\mathscr{V}$ is a von Neumann algebra. ▨

5. AN OPEN QUESTION

It follows from Corollary 2.4 that if $A \in \mathscr{L}(\mathscr{H})$, $\mathscr{M} = \mathrm{ran}(A)^-$, and $A|\mathscr{M}$ has the TAP, then A has the TAP. Furthermore, if A is bounded below, then one has a converse, because A and $A|\mathscr{M}$ are similar operators (the similarity is given by $A : \mathscr{H} \to \mathscr{M}$). However, even for strictly cyclic operators, it is not clear whether the converse holds in general (see Section 3).

QUESTION. If A has the TAP, must $A|\mathrm{ran}(A)^-$ have the TAP?

If the answer to this question is affirmative, then one can prove the converse of Corollary 2.2; and the assumption in Theorem 2.3 "A has the property (P)" can be replaced by the weaker hypothesis "A has the TAP".

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