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THE ORDER BIDUAL OF LATTICE ORDERED ALGEBRAS. II

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1. INTRODUCTION

Recall that for any (associative, but not necessarily commutative) algebra A a multiplication can be introduced in the algebraic bidual A^{**} of A (the so-called Arens multiplication [2, 3]). This is accomplished in three steps: given $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$, we define $f \cdot a \in A^*$, $G \cdot f \in A^*$ and $F \cdot G \in A^{**}$ by the equations

(1) $$(f \cdot a)(b) = f(ab)$$

(2) $$(G \cdot f)(a) = G(f \cdot a)$$

(3) $$(F \cdot G)(f) = F(G \cdot f).$$

It is straightforward to show that A^{**} is an associative algebra with respect to the Arens multiplication as defined in (3), but even if the initial multiplication in A is commutative, the Arens multiplication in A^{**} need not be.

As usual, A can be embedded canonically in A^{**} as follows: for each $a \in A$, the element $a'' \in A^{**}$ is defined by $a''(f) = f(a)$ for all $f \in A^*$. The mapping $\sigma : A \to A^{**}$ defined by $\sigma(a) = a''$ for all $a \in A$ is an injective algebra homomorphism, so if A and $\sigma(A)$ are identified, A is embedded in A^{**} as a subalgebra. The following properties of the Arens multiplication are easily deduced:

(i) the Arens multiplication in A^{**} extends the original multiplication in A (i.e., $a'' \cdot b'' = (ab)''$ for all $a, b \in A$).

(ii) if A has a unit element e, then e'' is the unit element of A^{**}.

(iii) if A is commutative, then $a'' \cdot f = f \cdot a$ for all $a \in A$, $f \in A^*$ (indeed, $a'' \cdot f)(b) = a''(f \cdot b) = (f \cdot b)(a) = f(ba) = f(ab) = (f \cdot a)(b)$ for all $b \in A$).

Now, let A be a (real) Riesz algebra (or lattice ordered algebra) i.e., A is a Riesz space (also called a vector lattice) which is simultaneously an associative (not necessarily commutative) algebra such that $a, b \in A^+$ (where A^+ stands for the positive cone of A) implies $ab \in A^+$. The first order dual of A is denoted by A' and the second order dual by A''. It is shown in [6, Theorem 4.1] that the order bidual A'' is again a Riesz algebra with respect to the Arens multiplication.

The band of all order bounded, order continuous linear functionals on A' is denoted by $(A')'_n$ and its disjoint complement in A'' by $(A')'_s$, the band of all order bounded, singular linear functionals on A', so $(A')'_s = \{(A')'_n\}^d$. It is well-known that (under the assumption that every point set has a non-measurable cardinal) the Dedekind complete Riesz space A' has the property that every σ-order continuous linear functional on A' is order continuous (see e.g. [12, Corollary 87.10]).

Observe that

$$A'' = (A')'_n \ominus (A')'_s,$$

an order direct sum, as A'' is Dedekind complete (for terminology on Riesz spaces and Riesz algebras not explained in this paper we refer to [1], [9], [12]). It is shown in [6, Theorem 4.1] that $(A')'_n$ is a Riesz algebra with respect to the Arens multiplication as well and even more is true as will be explained in the next paragraph.

From now on, A denotes an f-algebra (i.e., A is a Riesz algebra which satisfies the extra requirement that $a \wedge b = 0$ implies

$$(ac) \wedge b = (ca) \wedge b = 0$$

for all $c \in A^+$) such that A' separates the points of A, so $f(a) = 0$ for all $f \in A'$ implies $a = 0$. Notice that the latter condition implies that A is Archimedean. Indeed, if $a, b \in A^+$ and $0 \leqslant na \leqslant b$ $(n = 1, 2, \ldots)$, then $0 \leqslant nf(a) \leqslant f(b)$ $(n = 1, 2, \ldots)$ for all $f \in (A')^+$, so $f(a) = 0$ for all $f \in (A')^+$ and hence $a = 0$. This shows that A is automatically commutative and associative (see e.g. [7, Theorem 2.1]). The main result of [6] states that in this case $(A')'_n$ is a (Dedekind complete and thus Archimedean) f-algebra with respect to the Arens multiplication ([6, Theorem 4.4]). It follows immediately that the Arens product in $(A')'_n$ is Arens regular (for details on this notion we refer to [4, Sections 9 and 28].

At that time (1984) we were not able to prove the more general result that the whole order bidual A'' is also an f-algebra with respect to the Arens multiplication apart from some special cases to be mentioned now.

(a) if A has a multiplicative unit element e, then $A'' = (A')'_n$ is an f-algebra with respect to the Arens multiplication ([6, Corollary 4.5]).

(b) also, if A does not have a unit element it may happen that A'' is an f-algebra with respect to the Arens multiplication. Actually, it was shown in [5, Example 5] that this holds true for $A = \ell_1$ with the coordinatewise multiplication. It was conjectured in the same paper that A'' is an f-algebra with respect to the Arens multiplication and also that

$$F \cdot G = G \cdot F = 0$$

for all $F \in A''$, $G \in (A')'_s$.

(c) Recently, E. Scheffold showed ([11, Theorem 2.2]) that for every Banach-f-algebra A (so in particular for $A = \ell_1$) the second order dual A'' is again a Banach-f-algebra with respect to the Arens multiplication.

It is the main goal of the present paper to demonstrate that, more generally, the order bidual A'' of every f-algebra with point separating order dual A', is again a (Dedekind complete, hence Archimedean, commutative and Arens regular) f-algebra with respect to the Arens multiplication and that the other conjecture mentioned in (b) above is also true. However, the proofs are quite different from [11], mainly because we do not have a norm at our disposal.

2. THE MAIN RESULT

Throughout this section A denotes an f-algebra with point separating order dual A' (so A is Archimedean and commutative). The proof of the main theorem is divided in several steps.

LEMMA 2.1.

$$(ba - na)^+ \leqslant \frac{1}{n} b^2 a \quad (n = 1, 2, \ldots)$$

for all $a, b \in A^+$.

Proof.

$$(na - ba \wedge na) \wedge (ba - ba \wedge na) = 0 \quad (n = 1, 2, \ldots).$$

Multiplying the left hand term with $\frac{1}{n} b$ we get

$$\left(ba - b\left(\frac{1}{n} ba \wedge a\right)\right) \wedge (ba - ba \wedge na) = 0 \quad (n = 1, 2, \ldots),$$

and thus

$$\left(b\left(\frac{1}{n} ba \wedge a\right) - ba\right) \vee (ba \wedge na - ba) = 0 \quad (n = 1, 2, \ldots).$$

Hence,

$$ba = b\left(\frac{1}{n} ba \wedge a\right) \vee (ba \wedge na) \quad (n = 1, 2, \ldots).$$

This yields

$$0 \leqslant ba - ba \wedge na = \left(b\left(\frac{1}{n} ba \wedge a\right) - ba \wedge na\right)^+ \leqslant$$

$$\leqslant b\left(\frac{1}{n} ba \wedge a\right) \leqslant \frac{1}{n} b^2 a \quad (n = 1, 2, \ldots).$$

The result follows by observing that

$$ba - ba \wedge na = ba \vee na - na = (ba - na)^+ \quad (n = 1, 2, \dots).$$

The following proposition is crucial for the proof of the main theorem. It is in some respect a local version of [6, Theorem 3.3].

PROPOSITION 2.2. *If* $0 \leqslant F \in (A')_n'$ *and* $0 \leqslant G \in A''$, *then* $G \cdot F \in (A')_n'$.

Proof. By [6, Theorem 4.4.], $(A')_n'$ is an f-algebra with respect to the Arens multiplication. Furthermore, A is embedded as a sub-f-algebra in $(A')_n'$ by identifying a and a'' for all $a \in A$ (for details concerning this embedding, see [12, Section 109]. If $0 \leqslant a \in A$, then $0 \leqslant a'' \in (A')_n'$, so we may apply Lemma 2.1 to the positive elements a'' and F in the f-algebra $(A')_n'$. Consequently,

$$(a'' \cdot F - nF)^+ \leqslant \frac{1}{n} (a'')^2 \cdot F \quad (n = 1, 2, \dots).$$

where $(a'')^2$ stands for $a'' \cdot a''$. Hence,

$$(a'' \cdot F - n^2F)^+ \leqslant \frac{1}{n^2} (a'')^2 \cdot F \quad (n = 1, 2, \dots).$$

Observe now that

$$(a'' \cdot F - n^2F)^+ + a'' \cdot F \wedge n^2F =$$

$$= (a'' \cdot F - n^2F) \vee 0 + a'' \cdot F \wedge n^2F =$$

$$= a'' \cdot F \vee n^2F - n^2F + a'' \cdot F \wedge n^2F =$$

$$= a'' \cdot F + n^2F - n^2F = a'' \cdot F \quad (n = 1, 2, \dots).$$

Suppose now that $f_0 \geqslant f_\tau \downarrow 0$ in A'. We have to verify that $(G \cdot F)(f_\tau) = G(F \cdot f_\tau) \downarrow 0$, in other words, if $d \in \mathbf{R}$ satisfies

$$0 \leqslant d \leqslant (G \cdot F)(f_\tau)$$

for all τ, then we have to show that $d = 0$. It follows from $0 \leqslant F \in (A')_n'$ that $F(f_\tau) \downarrow 0$, so there exists a subsequence $\{f_n\}_{n=1}^\infty$ such that $0 \leqslant F(f_n) \leqslant 1/n^4$ $(n = 1, 2, \dots)$. Therefore, we have for all $0 \leqslant a \in A$ that

$$(a'' \cdot F)(f_n) = (a'' \cdot F - n^2F)^+(f_n) + (a'' \cdot F \wedge n^2F)(f_n) \leqslant$$

$$\leqslant (a'' \cdot F - n^2F)^+(f_0) + n^2F(f_n) \leqslant$$

$$\leqslant \frac{1}{n^2}\cdot(a'')^2\cdot F(f_0)+\frac{1}{n^2}=$$

$$=\frac{1}{n^2}\cdot((a'')^2\cdot F(f_0)+1)\quad (n=1,2,\ldots).$$

This yields

$$\sum_{n=1}^{\infty}(a''\cdot F)(f_n)=\sum_{n=1}^{\infty}a''(F\cdot f_n)=\sum_{n=1}^{\infty}(F\cdot f_n)(a)<\infty$$

for all $a\in A^+$. Put $g_k=\sum_{n=1}^{k}F\cdot f_n$ $(k=1,2,\ldots)$. Then $g_k\in(A')^+$ $(k=1,2,\ldots)$ and $\sup_k g_k(a)<\infty$ for all $a\in A^+$. Define $g_0:A^+\to\mathbf{R}$ by $g_0(a)=\sup_n g_n(a)$ for all $a\in A^+$ $\left(\text{i.e., } g_0=\sum_{n=1}^{\infty}F\cdot f_n \text{ on } A^+\right)$. It is easily verified that g_0 is additive on A^+ and thus (by a standard procedure, see e.g. [12, Lemma 83.1]) g_0 extends uniquely to a positive linear functional on A, denoted by g_0 again (actually, $0\leqslant g_n\uparrow g_0$ in A'). It follows from

$$0\leqslant\sum_{n=1}^{k}F\cdot f_n\leqslant g_0\quad (k=1,2,\ldots)$$

and $0\leqslant G\in A''$ that

$$0\leqslant\sum_{n=1}^{k}G(F\cdot f_n)=\sum_{n=1}^{k}(G\cdot F)(f_n)\leqslant G(g_0)\quad (k=1,2,\ldots).$$

But then $0\leqslant d\leqslant(G\cdot F)(f_n)$ $(n=1,2,\ldots)$ gives

$$0\leqslant kd\leqslant G(g_0)\quad (k=1,2,\ldots).$$

Consequently, $d=0$, which completes the proof.

Before we pass on to the next proposition, we first refresh the memory. Recall that for any $F\in A''$ the null ideal of F (or absolute kernel of F) is defined by

$$N_F=\{f\in A':|F|(|f|)=0\}$$

(throughout this paper ideal means order ideal). Obviously, N_F is an ideal in A' and if $F\in(A')'_n$, then N_F is a band. The carrier C_F of any $F\in A''$ is by definition the disjoint complement N_F^d of N_F in A', so C_F is a band as well. Notice that $A'=N_F^{dd}\oplus C_F$ for all $F\in A''$, as A' is Dedekind complete (see [12, Theorem 90.9 (i)]).

Consequently, if $F \in (A')'_n$ then $A' = N_F \oplus C_F$. Moreover, if $0 \leqslant F \in (A')'_n$ and $0 \leqslant G \in A''$, then

$$C_{F \wedge G} = C_F \cap C_G ,$$

as is shown in [12, Theorem 90.7(ii)]. Finally, it is worthwhile to observe that N_F is order dense in A' (i.e., $C_F = N_F^d = \{0\}$) for all $F \in (A')'_s$ (by [8, Theorem 50.4]).

For any $a \in A$, the mapping $\pi_a : A \to A$, defined by $\pi_a(b) = ab = ba$ for all $b \in A$, is an orthomorphism of A, i.e., $\pi_a \in \mathrm{Orth}(A)$. Notice that

$$f \cdot a = \pi'_a(f)$$

for all $a \in A, f \in A'$, where π'_a denotes the adjoint of π_a. Indeed,

$$(f \cdot a)(b) = f(ab) = f(\pi_a(b)) = (\pi'_a f)(b)$$

for all $b \in A$. It is common knowledge that $\pi'_a \in \mathrm{Orth}\,(A')$ ([12, Theorem 142.10 (ii)]), so π'_a leaves invariant all uniformly closed ideals of A' (for the latter result we refer to [10, Theorem 15.2] or [1, Exercise 8.9]). All the preparations have been made now for the next proposition.

PROPOSITION 2.3. *$C_{G \cdot F} \subset C_F$ for all $0 \leqslant F, G \in A''$.*

Proof. The null ideal N_F of F is uniformly closed, so $\pi'_a(N_F) \subset N_F$ for any $a \in A$. Hence, $\pi'_a(f) = f \cdot a \in N_F$ for all $0 \leqslant f \in N_F$, in other words

$$F(f \cdot a) = (F \cdot f)(a) = 0$$

for all $a \in A^+$ and all $0 \leqslant f \in N_F$. This shows that $F \cdot f = 0$ for all $0 \leqslant f \in N_F$ and consequently

$$(G \cdot F)(f) = G(F \cdot f) = G(0) = 0$$

for all $0 \leqslant f \in N_F$. Hence, $N_F \subset N_{G \cdot F}$. By taking disjoint complements we find $C_{G \cdot F} \subset C_F$ and we are done.

It is shown in [6, Lemma 4.2] that for any $F \in (A')'_n$ the mapping $\nu_F : A' \to A'$ defined by

$$\nu_F(f) = F \cdot f$$

for all $f \in A'$, satifies $\nu_F \in \mathrm{Orth}(A')$. This result be needed in the proof of the next proposition.

PROPOSITION 2.4. *$C_{G \cdot F} \subset C_G$ for all $0 \leqslant F \in (A')'_n$, $0 \leqslant G \in A''$.*

Proof. The orthomorphism $\nu_F \in \mathrm{Orth}(A')$ leaves invariant all uniformly closed ideals of A', so in particular $\nu_F(N_G) \subset N_G$. Hence,

$$\nu_F(f) = F \cdot f \in N_G$$

for all $0 \leqslant f \in N_G$, i.e.,

$$G(F \cdot f) = (G \cdot F)(f) = 0$$

for all $0 \leqslant f \in N_G$. This shows that $N_G \subset N_{G \cdot F}$ and thus $C_{G \cdot F} \subset C_G$, which is the desired result.

PROPOSITION 2.5. *$N^{\mathrm{dd}}_{G \cdot F} = A'$ for all $0 \leqslant F \in (A')'_{\mathrm{n}}$, $0 \leqslant G \in (A')'_{\mathrm{s}}$.*

Proof. Combining Propositions 2.3 and 2.4 we find

$$C_{G \cdot F} \subset C_F \cap C_G.$$

As observed above (we emphasize that we use here, as in the proof of Proposition 2.4, that $0 \leqslant F \in (A')'_{\mathrm{n}}$),

$$C_F \cap C_G = C_{F \wedge G}.$$

It follows from $F \wedge G = 0$ that $C_F \cap C_G = C_0 = \{0\}$. Hence, $C_{G \cdot F} = \{0\}$ implying that

$$C^{\mathrm{d}}_{G \cdot F} = N^{\mathrm{dd}}_{G \cdot F} = A'.$$

The next proposition is an essential ingredient in the proof of the main result.

PROPOSITION 2.6. *If $F \in (A')'_{\mathrm{n}}$ and $G \in (A')'_{\mathrm{s}}$, then $G \cdot F = 0$.*

Proof. We may assume without lose of generality that $F \geqslant 0$ and $G \geqslant 0$. By Proposition 2.2., $G \cdot F \in (A')'_{\mathrm{n}}$ and thus $N_{G \cdot F}$ is a band in A', i.e., $N_{G \cdot F} = N^{\mathrm{dd}}_{G \cdot F}$. By Proposition 2.5, $N^{\mathrm{dd}}_{G \cdot F} = A'$. These two equalities yield $N_{G \cdot F} = A'$, so $G \cdot F = 0$.

The Archimedean f-algebra A is commutative, so by property (iii) of the introduction

$$a'' \cdot f = f \cdot a$$

for all $a \in A$, $f \in A'$. An immediate consequence is that $G \cdot a'' = a'' \cdot G$ for all $a \in A$, $G \in A''$. Indeed,

$$(G \cdot a'')(f) = G(a'' \cdot f) = G(f \cdot a) = (G \cdot f)(a) =$$

$$= a''(G \cdot f) = (a'' \cdot G)(f)$$

for all $f \in A'$.

PROPOSITION 2.7. *$F \cdot G = 0$ for all $F \in A''$, $G \in (A')'_{\mathrm{s}}$.*

Proof. Since $a'' \in (A')'_{\mathrm{n}}$ for all $a \in A$, it follows from the above observation and Proposition 2.6 that

$$G \cdot a'' = a'' \cdot G = 0$$

for all $a \in A$. Hence,

$$(a'' \cdot G)(f) = (G \cdot f)(a) = 0$$

for all $a \in A$, $f \in A'$, showing that $G \cdot f = 0$ for all $f \in A'$. Consequently,

$$(F \cdot G)(f) = F(G \cdot f) = F(0) = 0$$

for all $f \in A'$, i.e., $F \cdot G = 0$ and we are through.

We have gathered now all prerequisites for the proof of the main theorem, which generalizes both [6, Theorem 4.4] and [11, Theorem 2.2]. It gives a positive answer to the first conjecture made in [5].

THEOREM 2.8. *If A is an f-algebra with point separating order dual A', then the order bidual A'' is a Dedekind complete (hence Archimedean and commutative) f-algebra with respect to the Arens multiplication. All such f-algebras are therefore Arens regular.*

Proof. It was shown already in [6, Theorem 4.1] that A'' is a (Dedekind complete) lattice ordered algebra with respect to the Arens multiplication. It remains therefore to verify that if $F, G, H \in (A'')^+$ and $F \wedge G = 0$, then

$$F \cdot H \wedge G = H \cdot F \wedge G = 0.$$

Decompose F, G and H according to the order direct sum decomposition $A'' = (A')'_n \oplus (A')'_s$, so

$$F = F_1 + F_2, \quad G = G_1 + G_2, \quad H = H_1 + H_2$$

with $0 \leqslant F_1, G_1, H_1 \in (A')'_n$ and $0 \leqslant F_2, G_2, H_2 \in (A')'_s$. It follows from Propositions 2.6 and 2.7 that

$$F \cdot H = F_1 \cdot H_1 + F_1 \cdot H_2 + F_2 \cdot H_1 + F_2 \cdot H_2 = F_1 \cdot H_1.$$

Hence,

$$F_1 \cdot H \wedge G = F_1 \cdot H_1 \wedge (G_1 + G_2) =$$

$$= F_1 \cdot H_1 \wedge G_1 + F_1 \cdot H_1 \wedge G_2$$

(the latter equality follows from $G_1 \wedge G_2 = 0$). But $0 \leqslant F_1 \cdot H_1 \in (A')'_n$, as $(A')'_n$ is an algebra and thus $0 \leqslant G_2 \in (A')'_s$ yields $F_1 \cdot H_1 \wedge G_2 = 0$. Moreover, it follows from $0 \leqslant F_1 \leqslant F$, $0 \leqslant G_1 \leqslant G$ and $F \wedge G = 0$ that $F_1 \wedge G_1 = 0$ as well. Consequently, $F_1 \cdot H_1 \wedge G_1 = 0$ because $0 \leqslant F_1, G_1, H_1 \in (A')'_n$ and $(A')'_n$ is an f-algebra by [6, Theorem 4.4]. We have shown therefore that $F \cdot H \wedge G = 0$. Similarly, it is proved that $H \cdot F \wedge G = 0$ and the proof is complete.

The next corollary shows that the second conjecture of [5] is also true.

COROLLARY 2.9. *$F \cdot G = G \cdot F = 0$ for all $F \in A''$, $G \in (A')'_s$,* i.e., *right and left Arens multiplication with singular functionals are trivial.*

Proof. This is a direct consequence of Proposition 2.7 and the fact that the Archimedean f-algebra A'' is commutative.

Compare the next corollary with Proposition 2.2.

COROLLARY 2.10. *$F \cdot G \in (A')'_n$ for all $F, G \in A''$.*

Proof. Decompose $F = F_1 + F_2$, $G = G_1 + G_2$ according to the order direct sum $A'' = (A')'_n \oplus (A')'_s$. Then

$$F \cdot G = F_1 \cdot G_1 + F_1 \cdot G_2 + F_2 \cdot G_1 + F_2 \cdot G_2 = F_1 \cdot G_1$$

by Corollary 2.9. Hence, $F \cdot G = F_1 \cdot G_1 \in (A')'_n$.

Fix $F \in A''$ for the moment and consider the mapping $\pi_F \colon A'' \to A''$ defined by

$$\pi_F(G) = F \cdot G = G \cdot F$$

for all $G \in A''$.

COROLLARY 2.11. *$\pi_F \in \mathrm{Orth}(A'')$ for all $F \in A''$.*

Proof. Obviously, $\pi_F = \pi_{F^+} - \pi_{F^-}$, so we may confine ourselves to considering the case $F \geqslant 0$. We have to show that $G \wedge H = 0$ in A'' implies $\pi_F(G) \wedge \wedge H = 0$. This, however, is immediate from the fact that A'' is an f-algebra with respect to the Arens multiplication.

Let us consider now the case that A is a Banach-f-algebra, that is, A is an f-algebra equipped with a Riesz (or lattice) norm such that $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in A^+$. It follows from $\|a\| = \| |a| \|$ for all $a \in A$, that $\|ab\| \leqslant \|a\| \cdot \|b\|$ for all $a, b \in A$, so A is certainly a Banach algebra. Moreover, $A' = A^*$, the norm dual of A, as A is a Banach lattice ([12, Theorem 102.3(iii)]). Hence, the order dual of A separates the points of A by the Hahn-Banach theorem, so A is Archimedean.

The following inequalities are used by E. Scheffold in [11] to show that A'' is a Banach-f-algebra with respect to the Arens multiplication for all Banach-f-algebras A. We obtain them as a consequence of Corollary 2.11.

COROLLARY 2.12. *Let A be a Banach-f-algebra. Then*
(i) $0 \leqslant F \cdot G \leqslant r(F)G \leqslant \|F\|G$
(ii) $0 \leqslant G \cdot F \leqslant r(F)G \leqslant \|F\|G$
for all $0 \leqslant F, G \in A''$ (where $r(F)$ denotes the spectral radius of F and $\|F\|$ the operator norm).

Proof. Since $A'' = A^{**}$ is a Banach lattice (see e.g. [12, Theorem 102.3(iii)], we have $\mathrm{Orth}(A'') = Z(A'')$, the center of A'' ([12, Corollary 144.3(ii)]). By Corollary 2.11, $\pi_F \in Z(A'')$, so there exists $\lambda > 0$ such that $0 \leqslant \pi_F \leqslant \lambda I$. In fact,

$$r(\pi_F) = \|\pi_F\| = \inf\{\lambda > 0 : 0 \leqslant \pi_F \leqslant \lambda I\},$$

a result which can be found in e.g. [12, Section 144]. This shows that $0 \leqslant \pi_F \leqslant r(\pi_F)I$. It is well-known that A'' is a Banach algebra with respect to the Arens multiplication ([4, Section 9.13]), so

$$\pi_F^n(G) = F^n \cdot G \quad (n = 1, 2, \ldots)$$

implies

$$\|\pi_F^n(G)\| \leqslant \|F^n\| \cdot \|G\| \quad (n = 1, 2, \ldots)$$

for all $G \in A''$. It follows that $\|\pi_F^n\| \leqslant \|F^n\|$ $(n = 1, 2, \ldots)$ and hence

$$r(\pi_F) = \lim_{n \to \infty} \|\pi_F^n\|^{\frac{1}{n}} \leqslant \lim_{n \to \infty} \|F^n\|^{\frac{1}{n}} = r(F).$$

Consequently, $0 \leqslant \pi_F \leqslant r(F)I$. This shows that

$$0 \leqslant \pi_F(G) = F \cdot G = G \cdot F \leqslant r(F)G$$

for all $0 \leqslant G \in A''$. It remains to observe that $r(F) \leqslant \|F\|$ and we are done.

Let, for the moment, A be an arbitrary Archimedean f-algebra. Denote by $N(A)$ the set of all nilpotent elements in A, i.e.,

$$N(A) = \{a \in A : a^k = 0 \text{ for some } k \in \mathrm{N}\}$$

(where k is a priori depending on a). It is shown in [10, Proposition 10.2] and partly in [12, Theorem 142.5] that $N(A)$ is a band in A satisfying

$$N(A) = \{a \in A : a^2 = 0\} =$$

$$= \{a \in A : ab = ba = 0 \text{ for all } b \in A\},$$

so $N(A)$ is a zero-sub-f-algebra of A (by which we mean that the multiplication in $N(A)$ is trivial). Moreover, it is shown that $ab \in N(A)^{\mathrm{d}}$ for all $a, b \in A$ and hence $N(A)^{\mathrm{d}}$ is a *semiprime* sub-f-algebra of A (recall that an f-algebra is said to be semiprime whenever the only nilpotent is 0). Finally, if A has the principal projection property (which is certainly true if A is Dedekind complete), then $N(A)$ is a projection band, i.e.,

$$A = N(A) \oplus N(A)^{\mathrm{d}}.$$

It is staightforward from the last description of $N(A)$ above that any Archimedean f-algebra A with unit element is semiprime ($N(A) = \{0\}$). So far the generalities concerning the band of nilpotents.

We now return to the case considered throughout this paper, viz. A is an f-algebra with point separating order dual A'. We shall study the band $N(A'')$ of all nilpotents in the f-algebra A'' in more detail. Note first that $N(A'')$ is a projection band in A'', because A'' is Dedekind complete. By Corollary 2.9,

$$G^2 = G \cdot G = 0$$

for all $G \in (A')'_s$, which shows that $(A')'_s \subset N(A'')$. In the next proposition we shall derive a necessary and sufficient condition for equality.

Proposition 2.13. *The following statements are equivalent.*
(i) $(A')'_s = N(A'')$
(ii) $(A')'_n$ *is semiprime.*

Proof. (i) $\Rightarrow$ (ii). It follows from the hypothesis that

$$(A')'_n = \{(A')'_s\}^d = N(A'')^d,$$

so $(A')'_n$ is semiprime, according to the remarks above.

(ii) $\Rightarrow$ (i). Choose $0 \leqslant F \in (A')'_n$ and decompose $F = F_1 + F_2$ with $0 \leqslant F_1 \in N(A'')$ and $0 \leqslant F_2 \in N(A'')^d$. It follows from $0 \leqslant F_1 \leqslant F$ that $F_1 \in (A')_n$ as well. On the other hand, $F_1 \in N(A'')$ yields $F_1^2 = 0$. By hypothesis, $(A')'_n$ is semiprime, so $F_1 = 0$. This proves that $F = F_2 \in N(A'')^d$. We have shown therefore that $(A')'_n \subset N(A'')^d$, so $(A')'_s \supset N(A'')$. Combined with the inclusion $(A')'_s \subset N(A'')$ (which always holds), we find $(A')'_s = N(A'')$, as desired.

Necessary and sufficient conditions that $(A')'_n$ be semiprime are derived in [6, Sections 6 and 7].

Remark 2.14. We claim that $A'' = (A')'_n$ if A'' is semiprime with respect to the Arens multiplication. Indeed, if $G \in (A')'_s$, then $G^2 = 0$, so $G = 0$, as the only nilpotent in A'' is 0. Hence, $(A')'_s = \{0\}$, i.e., $A'' = (A')'_n$. This result is shown by E. Scheffold for the class of Banach-f-algebras in [11, Theorem 2.11]. Hence, if A'' has a unit element (which implies that A'' is semiprime), then $A'' = (A')'_n$ as well. In particular, if A has a unit element e (so e'' is the unit element of A'', according to property (ii) in the introduction), then $A'' = (A')'_n$. The last result is shown in [6, Corollary 3.4], so the initial remark above is a considerable improvement upon this result.

We end this section with some observations concerning the mapping ν_F as introduced in the paragraph preceding Proposition 2.4. It was noticed there that the mapping $\nu_F : A' \to A'$ defined by $\nu_F(f) = F \cdot f$ for all $f \in A'$ is an orthomorphism of A' whenever $F \in (A')'_n$. Of course, this mapping ν_F can be defined for all $F \in A''$ and ν_F is linear and order bounded (since $\nu_F = \nu_{F^+} - \nu_{F^-}$, the difference of two positive mappings). It turns out a posteriori that $\nu_F \in \mathrm{Orth}(A')$ for all $F \in A''$.

LEMMA 2.15. $\nu_F \in \mathrm{Orth}(A')$ *for all* $F \in A''$.

Proof. It is shown in the proof of Proposition 2.7 that $G \cdot f = 0$ for all $f \in A'$, $G \in (A')'_s$. Take $F \in A''$ arbitrary and decompose $F = F_1 + F_2$ with $F_1 \in \in (A')'_n$, $F_2 \in (A')'_s$. By [6, Lemma 4.2], $\nu_{F_1} \in \mathrm{Orth}(A')$. Moreover,

$$\nu_F(f) = (F_1 + F_2) \cdot f = F_1 \cdot f + F_2 \cdot f = F_1 \cdot f = \nu_{F_1}(f)$$

for all $f \in A'$, where we use that $F_2 \in (A')'_s$. Hence, $\nu_F = \nu_{F_1} \in \mathrm{Orth}(A')$.

Consider now the mapping $\nu \colon A'' \to \mathrm{Orth}(A')$, defined by $\nu(F) = \nu_F$ for all $F \in A''$. Clearly, ν is linear and positive. Using the formulas

$$f \cdot ab = (f \cdot a) \cdot b \quad (a, b \in A; f \in A')$$

$$(F \cdot f) \cdot a = F \cdot (f \cdot a) \quad (a \in A, f \in A', F \in A''),$$

a straightforward calculation shows that

$$(F \cdot G) \cdot f = F \cdot (G \cdot f) \quad (f \in A';\ F, G \in A'').$$

Hence, $\nu_{F\,G} = \nu_F \nu_G$, i.e., ν is an algebra homomorphism. It is easily deduced now that ν is a Riesz homomorphism as well. To this end, it suffices to show that $F \wedge G = 0$ in A'' implies $\nu(F) \wedge \nu(G) = 0$. Since A'' is an f-algebra, we have $F \cdot G = 0$ and thus

$$\nu(F \cdot G) = \nu(F) \cdot \nu(G) = 0.$$

But $\mathrm{Orth}(A')$ is an Archimedean f-algebra with unit element, so $\mathrm{Orth}(A')$ is semiprime. Consequently, $\nu(F) \wedge \nu(G) = 0$ (see e.g. [12, Theorem 142.3(i)]), which proves the claim.

We are now in a position to formulate the next theorem, the proof of which is a complete imitation of the corresponding proof for $(A')'_n$ [6, Theorem 5.2] and is therefore omitted (observe, by the way, that if A'' has a unit element, then $A'' = (A')'_n$).

THEOREM 2.16. *The following statements are equivalent.*

(i) ν *is onto,*

(ii) A'' *has a unit element with respect to the Arens multiplication,*

(iii) ν *is bijective (so* $\mathrm{Orth}(A')$ *and* A'' *are f-algebra isomorphic).*

The mapping $\rho: A'' \to \mathrm{Orth}(A'')$ defined by $\rho(F) = \pi_F$ for all $F \in A''$ (for the meaning of π_F, see the paragraph preceding Corollary 2.11) is an algebra and Riesz homomorphism and $\rho(A'')$ is both a Riesz subspace and an algebra ideal in $\mathrm{Orth}(A'')$. Furthermore, ρ is injective if and only if A'' is semiprime and necessary and sufficient for A'' to have a unit element is that ρ be bijective (so A'' and $\mathrm{Orth}(A'')$ are f-algebra isomorphic). For details on this mapping ρ we refer to e.g. [7, Section 2] and [10, Section 12]. It follows therefore from Theorem 2.16 that $\mathrm{Orth}(A')$, A'' and $\mathrm{Orth}(A'')$ are isomorphic as f-algebras, whenever A'' has a unit element.

In precisely the same way as is shown in Remark 5.3 of [6], it is deduced that in general the kernel ker ν of ν satisfies $\ker \nu = N(A'')$, the band of all nilpotents in A''. Hence, ν is injective if and only if A'' is semiprime with respect to the Arens multiplication (and hence $A'' = (A')'_n$ according to Remark 2.14). We state this result as a theorem.

THEOREM 2.17. *The following statements are equivalent.*

(i) ν *is* 1-1.

(ii) A'' *is semiprime with respect to the Arens multiplication.*

One final remark. In the case that A'' has a unit element E, the f-algebra $\mathrm{Orth}(A)$ can be embedded injectively as a sub-f-algebra in A''. More precisely, the mapping $\Phi: \mathrm{Orth}(A) \to A''$ defined by $\Phi(\pi) = \pi''(E)$ for all $\pi \in \mathrm{Orth}(A)$ (here π'' denotes the second adjoint of π, so $\pi'' \in \mathrm{Orth}(A'')$) is an injective algebra and Riesz homomorphism. For details of the proof we refer the reader to [6, Theorem 5.4].

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