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SUBBROWNIAN OPERATORS

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0. INTRODUCTION

Let H denote a complex Hilbert space and $\mathscr{L}(H)$ the continuous linear operators on H. J. Agler defines an operator $J \in \mathscr{L}(H)$ to be a Brownian shift if, with respect to an orthogonal decomposition of H, J has the form $\begin{pmatrix} V & E \\ 0 & U \end{pmatrix}$ where V is an isometry, U is unitary and $V^*E = 0$. Operators that are obtained by restricting a Brownian shift to an invariant subspace are characterized by the following theorem.

THEOREM A. (Agler [1]). *Let $T \in \mathscr{L}(H)$. T satisfies $I - 2T^*T + T^{*2}T^2 = 0$ if and only if T has an extension to a Brownian shift.*

In this paper, Theorem A is generalized to a larger class of operators. Let H^2 denote the classical Hardy space of analytic functions on the unit disk with square integrable boundary values. Let $S \in \mathscr{L}(H^2)$ denote the unilateral shift. By S_α we mean $S \otimes I$, where I is the identity operator on an α dimensional Hilbert space, $0 \leqslant \alpha \leqslant \infty$. An operator $B \in \mathscr{L}(H)$ is said to be a Brownian operator if H has a decomposition $H = H_1 \oplus H_2$ with respect to which $B = \begin{pmatrix} V & F \\ 0 & S_\alpha^* \oplus U \end{pmatrix}$, where U is unitary, V is an isometry, $F^*V = 0$, $F^*F + (S_\alpha \oplus U^*)(S_\alpha^* \oplus U) = 2I$ and $FF^* + VV^* \geqslant I$.

THEOREM B. *Let $T \in \mathscr{L}(H)$. T has an extension to a Brownian operator if and only if both $I - 2T^*T + T^{*2}T^2 \leqslant 0$ and $\|T\|^2 \leqslant 2$.*

There are several other theorems in the spirit of Theorem B. For example, an operator T satisfying $T^{*3} - 3T^{*2}T + 3T^*T^2 - T^3 = 0$ is known as a 3-symmetric operator. These operators have a canonical model — due to Helton [9] — as multiplication by t on a Sobolev space of functions on the interval $[a, b] = \sigma(T)$. As evidence of the power of this model, Helton connects 3-symmetric operators to classical conjugate point theory and shows, under a technical hypothesis, that 3-symmetric operators extend to Jordan operators of the form $J = A + Q$, where A is selfadjoint (see also [6]). A 2-isometry, an operator satisfying $I - 2T^*T + T^{*2}T^2 = 0$, has a model as multiplication by $e^{i\theta}$ on a Sobolev space of functions analytic in **D**. It is remarkable that a natural disconjugacy theorem exists for first order differential operators (acting on functions analytic in **D**). Agler [1] establishes this theorem and uses it to prove Theorem A.

The proof of Theorem B uses the machinery of complete positivity and replaces conjugate point theory with Weiner-Hopf factorization. This proof likely generalizes to give an extension result for "higher order" Brownian shifts (see Section four).

1. PRELIMINARIES

In this section, we present portions of Agler's hereditary machinery which will be used in the sequel. All of the results may be found in [2]. A polynomial in two non-commuting variables x and y of the form $p(x, y) = \sum p_{nm}x^n y^m$ is a *hereditary polynomial*. The set of hereditary polynomials is denoted $\mathscr{P}$. Often, we will identify $\mathscr{P}$ with $\mathbf{C}[z, w]$ by the formula $p(z, w) = \sum p_{nm}w^m z^n$. Further, a polynomial in the single variable x (or y) can be viewed as a hereditary polynomial in the obvious way. If $\mathscr{A}$ is a unital C^*-algebra if $a \in \mathscr{A}$ and if $p = \sum p_{nm}y^m x^n \in \mathscr{P}$, define

$$p(a, a^*) = \sum p_{nm}a^{*m}a^n.$$

An operator $T \in \mathscr{L}(H)$ is said to *extend* (or have an *extension*) to an operator $J \in \mathscr{L}(K)$ if T is unitarily equivalent to J restricted to an invariant subspace. The following theorem is part of the Arveson Extension Theorem.

THEOREM 1.1 (Agler). *Let $\mathscr{A}$ be a unital C^*-algebra. Let $a \in \mathscr{A}$ and $T \in \mathscr{L}(H)$. There exists a Hilbert space K and a unital representation $\pi: \mathscr{A} \to \mathscr{L}(K)$ such that T extends $\pi(a)$ if and only if for every positive integer M and every $M^2 p^{uv} \in \mathscr{P}$ $(1 \leqslant u, v \leqslant M)$ with $(p^{uv}(a, a^*)) \geqslant 0$ it follows that $(p^{uv}(T, T^*)) \geqslant 0$.*

Fix an operator $T \in \mathscr{L}(H)$ and an open set $G \subseteq \mathbf{C}$ containing $\sigma(T)$. Let $H(G)$, $H(G \times G)$ denote the spaces of functions holomorphic on G and $G \times G$ respectively. The hereditary functional calculus (the map $\mathscr{P} \to \mathscr{L}(H)$ given by $p \to p(T, T^*)$) can be extended to $H(G \times G)$ via the Riesz-Dunford functional calculus by defining, for $f \in H(G \times G)$,

$$f(T, T^*) = -\frac{1}{4\pi} \int_\gamma \int_\gamma f(z, w)\,(w - T^*)^{-1}(z - T)^{-1}\, dz\, dw$$

(where γ is a suitable chosen Jordan arc in G surrounding $\sigma(T)$).

LEMMA 1.2. (Lemma 1.16 in [2]). *If* $f_i \in H(G)$ *and* $f \in H(G\times G)$, *and* $g \in H(G\times G)$ *is defined by* $g(z, w) = \overline{f_2(\bar{w})}g(z, w)f_1(z)$, *then* $g(T, T^*) = f_2(T)^* g(T, T^*) f_1(T)$.

LEMMA 1.3. (Lemma 1.17 in [2]). *If* $f, f_n \in H(G\times G)$ *and if* f_n *converges to* f *uniformly on compact subsets of* $G\times G$, *then* $f_n(T, T^*) \to f(T, T^*)$ *in operator norm.*

Lemmas 1.2 and 1.3 will be used freely and without comment in the sequel.

2. THE EXTENSION

In this section we prove Theorem B. For ease of exposition, let $\mathscr{F}$ denote the class of operators T such that $-I + 2T^*T - T^{*2}T^2 \geqslant 0$ and $2 - T^*T \geqslant 0$.

Define $E: H^2 \to H^2 \otimes H^2$ by the formula $Eh = 1 \otimes (h + h(0)(\sqrt{2} - 1))$. Alternatively, $E = 1 \otimes (2 - SS^*)^{1/2}$. For the remainder of this section and the next, K will denote the Hilbert space $(H^2 \otimes H^2) \oplus H^2$ and $J \in \mathscr{L}(K)$ will denote the operator $\begin{bmatrix} V & E \\ 0 & S^* \end{bmatrix}$, where $V = S \otimes I$. Direct computations show

$$2 - J^*J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0$$

$$2J^*J - I - J^{*2}J^2 = \begin{bmatrix} 0 & 0 \\ 0 & I - SS^* \end{bmatrix} \geqslant 0.$$

In particular, $J \in \mathscr{F}$.

To prove Theorem B we intend, for each fixed $T \in \mathscr{F}$, to verify the hypothesis of Theorem 1.1 with $a = J$. Consequently, we will work extensively with linear combinations of the operators $J^{*m}J^n \in \mathscr{L}(K)$. To facilitate computations define operators $G_{n,m}: H^2 \to H^2$ via the following formulas:

$$G_{n,m} = \sum_{k=m}^{n-1} V^{k-m} E S^{*(n-1-k)}$$

$$H_{n,m} = \left(\sum_{k=n}^{m-1} V^{k-n} E S^{*(m-1-k)} \right)^* = G_{m,n}^*$$

$$X_{n,m} = \begin{cases} \displaystyle\sum_{l=0}^{\min(m,n)-2} S^{m-1-l} S^{*n-1-l} + 2S(m-n) & \text{if } m, n \geqslant 1 \\ S(m-n) & \text{if } m = 0 \text{ or } n = 0 \end{cases}$$

where

$$S(k) = \begin{cases} S^k & \text{if } k \geqslant 0 \\ S^{*-k} & \text{if } k < 0. \end{cases}$$

Using the relations $E^*E = 2 - SS^*$ and $E^*V = 0$, it follows that

$$J^{*m}J^n = \begin{bmatrix} V^{*m}V^n & G_{n,m} \\ H_{n,m} & X_{n,m} \end{bmatrix}.$$

For a hereditary polynomial, $p(x, y) = \sum p_{nm}x^n y^m$, define $G(p) = \sum p_{nm}G_{nm}$. Similarly, define $H(p)$, $X(p)$. Then

$$p(J, J^*) = \begin{bmatrix} p(V, V^*) & G(p) \\ H(p) & X(p) \end{bmatrix}.$$

Lemma 2.1. *Let $h \in \mathscr{P}$. If $h(J, J^*) = 0$, then $h = 0$.*

Proof. The relations $X_{n,m} - X_{n-1,m-1} = S^{m-1}S^{*n-1}$ for $n, m \geqslant 2$ and $X_{n,1} = 2S^{*n-1}$, $X_{1,m} = 2S^{m-1}$ for $n, m \geqslant 1$ imply $\{X_{n,m} \mid 1 \leqslant n, m \leqslant N\}$ is a linearly independent set in $\mathscr{L}(H^2)$. Further, as $X_{n,0} = S^{*n}$ and $X_{0,m} = S^m$, we have $X_{n,0} = (1/2)X_{n+1,1}$ and $X_{0,m} = (1/2)X_{1,m+1}$. The assumption $h(J, J^*) = 0$ implies $0 = \sum_{n,m=0}^{N} h_{nm}X_{nm}$. Using the above relations, compute

$$0 = \sum_{m,n=2}^{N} h_{nm}X_{nm} + \sum_{m=0}^{N} \left(\frac{1}{2} h_{0,m} + h_{1,m+1}\right) X_{1,m+1} +$$

$$+ \sum_{n=1}^{N} \left(\frac{1}{2} h_{n,0} + h_{n+1,1}\right) X_{n+1,1}.$$

The linear independence of $\{X_{n,m} \mid 1 \leqslant n, m \leqslant N\}$ yields

$$h_{nm} = 0 \quad \text{for } n, m \geqslant 2$$

$$\frac{1}{2} h_{0,m} + h_{1,m+1} = 0 \quad \text{for all } m$$

$$\frac{1}{2} h_{n,0} + h_{n+1,1} = 0 \quad \text{for all } n.$$

However $h(J, J^*) = 0$ also implies $h(V, V^*) = 0$. Since

$$h(V, V^*) = \sum_{m=0}^{N} (h_{0,m} + h_{1,m+1})V^{*m} + \sum_{n=1}^{N} (h_{n,0} + h_{n+1,1})V^n,$$

it follows that

$$h_{0,m} + h_{1,m+1} = 0$$

$$h_{n,0} + h_{n+1,1} = 0.$$

Consequently $h_{nm} = 0$ for all n, m. ▨

Define the *order* of a hereditary polynomial $h \in \mathscr{P}$, denoted $O(h)$, to be the smallest N such that $h_{nm} = 0$ if $m \geqslant N$ or $n \geqslant N$. Let M_N denote the $N\times N$ matrices with complex entries.

LEMMA 2.2. *Fix N. Let $h^k \in \mathscr{P}$. If $O(h^k) \leqslant N$ for all k and if $h^k(J, J^*) \to 0$ in operator norm, then $h^k_{nm} \to 0$ for every n, m.*

Proof. Lemma 2.1 says that $\{J^{*m}J^n \mid 0 \leqslant m,\ n \leqslant N\}$ is a linearly independent set of operators in $\mathscr{L}(K)$. Thus the formula $\|(h_{nm})_{m,n=0}^N\| = \left\| \sum_{m,n=0}^{N} h_{nm}J^{*m}J^n \right\|$ defines a norm on M_{N+1}, which is boundedly equivalent with any other norm on M_{N+1}. Hence, $h^k(J, J^*) \to 0$ implies $h^k_{nm} \to 0$ for each m, n. ▨

If $Q(z) = \sum_{a=0}^{N} Q_a z^a$ is an $M\times M$ matrix valued analytic polynomial we may view $Q(z)$ as an $M\times M$ matrix of hereditary polynomials (q^{uv}) by letting $q^{uv}(x) = \sum_{a=0}^{N} q_a^{uv} x^a$ where $Q_a = (q_a^{uv})_{u,v=1}^M$. For an $M\times M$ matrix of hereditary polynomials $P = (p^{uv})$ let $G(P) = (G(p^{uv}))_{u,v=1}^M$ For any operator A, define $P(A) = (p^{uv}(A))$.

PROPOSITION 2.3. *Fix N, M. Let p^{uv} $(1 \leqslant u, v \leqslant M)$ be M^2 hereditary polynomials with $O(p^{uv}) \leqslant N$. Define $P(z, w) = (p^{uv}(z, w))$ and $R(e^{i\theta}) = P(e^{i\theta}, e^{-i\theta})$. If there exists an analytic $M\times M$ matrix valued polynomial $Q(z) = \sum_{a=0}^{N} Q_a z^a$ such that $Q(e^{i\theta})^*Q(e^{i\theta}) = R(e^{i\theta})$ then $Q(V)^*G(Q) = G(P)$.*

Proof. We must show that

$$\sum_{w=1}^{M} q^{wu}(V)^*G(q^{wv}) = G(p^{uv}) \quad (1 \leqslant u, v \leqslant M).$$

It suffices to compute both sides and compare. $q^{wu}(V)^* = \sum_{a=0}^{N} \overline{q_a^{wu}} V^{*a}$ and

$$G(q^{wv}) = \sum_{k=1}^{N} q_k^{wv} \sum_{s=0}^{k-1} V^s E S^{*(k-1-s)} = \sum_{s=0}^{N-1} V^s \sum_{k=s+1}^{N} q_k^{wv} E S^{*(k-1-s)}.$$

Using the relation $V^*E = 0$, changing the order of summation and making the substitutions $b = s - a$, $c = k - a$ gives

$$\sum_{w=1}^{M} q^{wu}(V)^* G(q^{wv}) = \sum_{w=1}^{M} \sum_{s=0}^{N-1} \sum_{a=0}^{s} \sum_{k=s+1}^{N} V^{s-a} \overline{q_a^{wu}} q_k^{wv} E S^{*(k-1-s)} =$$

$$= \sum_{w=1}^{M} \sum_{b=0}^{N-1} \sum_{a=0}^{N-b} \sum_{k=a+b+1}^{N} V^b \overline{q_a^{wu}} q_k^{wv} E S^{*(k-1-a-b)} =$$

$$= \sum_{w=1}^{M} \sum_{b=0}^{N-1} \sum_{a=0}^{N-b-1} \sum_{c=b+1}^{N-a} V^b \overline{q_a^{wu}} q_{a+c}^{wv} E S^{*(c-1-b)} =$$

$$= \sum_{b=0}^{N-1} \sum_{c=b+1}^{N} V^b \left(\sum_{w=1}^{M} \sum_{a=0}^{N-c} \overline{q_a^{wu}} q_{a+c}^{wv} \right) E S^{*(c-1-b)}.$$

Now since $Q^*(e^{i\theta})Q(e^{i\theta}) = R(e^{i\theta})$,

$$\sum_{w=1}^{M} \sum_{a=0}^{N-c} \overline{q_a^{uw}} q_{a+c}^{wv} = r_c^{uv}$$

where $R(e^{i\theta}) = \sum_{-N}^{N} R_c e^{ic\theta}$ and $R_c = (r_c^{uv})$. Thus

$$\sum_{w=1}^{M} q^{wu}(V)^* G(q^{wv}) = \sum_{b=0}^{N-1} \sum_{c=b+1}^{N} V^b r_c^{uv} E S^{*(c-1-b)}. \tag{2.4}$$

The relation $E^*V = 0$ implies $G_{nm} = 0$ if $n - m < 1$. Therefore,

$$G(p^{uv}) = \sum_{k=1}^{N} \left(\sum_{n-m=k} p_{nm}^{uv} \right) \sum_{l=0}^{k-1} V^l E S^{*(k-l-1)} =$$

$$= \sum_{l=0}^{N-1} \sum_{k=l+1}^{N} V^l \left(\sum_{n-m=k} p_{nm}^{uv} \right) E S^{*(k-l-1)} =$$

$$= \sum_{l=0}^{N-1} \sum_{k=l+1}^{N} V^l r_k^{uv} E S^{*(k-l-1)}$$

where, in the last equality, one uses $R(e^{i\theta}) = P(e^{i\theta}, e^{-i\theta})$. Comparing the above equality with (2.4) proves the proposition. ▨

The next theorem is a special case of Theorem 3.2 in [12].

THEOREM 2.5. *Let F_n be $M\times M$ matrices for $-N \leqslant n \leqslant N$. Define* $F(e^{i\theta}) = \sum_{n=-N}^{N} F_n e^{in\theta}$. *If $F(e^{i\theta}) > 0$, then there exist Q_n, $M\times M$ matrices $(0 \leqslant n \leqslant N)$, such that $Q(z) = \sum_{n=0}^{N} Q_n z^n$ is invertible for each $|z| < 1$ and $Q(e^{i\theta})^* Q(e^{i\theta}) = F(e^{i\theta})$.*

In applications $F(e^{i\theta}) > 0$; in which case there is a matrix valued function $Q^{-1}(z)$ analytic in a neighborhood of $\overline{\mathbf{D}}$ such that $Q^{-1}(z)Q(z) = I$. In particular, for any isometry W, $Q(W)^*Q(W) = P(W)$ and $Q^{-1}(W) = Q(W)^{-1}$.

The next lemma is specialized from and contained in the proof of Theorem 2.3 in [2] (see (2.13)).

THEOREM 2.6. (Agler). *Let $g^{uv} \in \mathscr{P}$ $(1 \leqslant u, v \leqslant M)$ and fix $0 < t < 1$. If $(g^{uv}(S^*, S)) \geqslant 0$, then there exists functions $f_l^{\alpha}(z)$ $(1 \leqslant \alpha \leqslant M, l \geqslant 1)$ defined and analytic on $t\mathbf{D}$ such that*

$$g^{uv}(z, w) = \sum_{l=1}^{\infty} \overline{f_l^u(\overline{w})}\,(1 - zw) f_l^v(z)$$

with the series converging uniformly on compact subsets of $t\mathbf{D}\times t\mathbf{D}$.

For each positive integer M, let I_M denote the $M\times M$ identity matrix. If $A \in \mathscr{L}(H)$, $I_M \otimes A$ is the $M\times M$ operator matrix with diagonal entries A and off diagonal entries 0. For a C^*-algebra $\mathscr{A}$, let $M_n(\mathscr{A})$ denote the C^*-algebra of $n\times n$ matrices with entries from $\mathscr{A}$. The natural $*$-isomorphism $M_n(M_m(\mathscr{A})) \to M_m(M_n(\mathscr{A}))$ is known as the canonical shuffle (see [10]). For example, the canonical shuffle $M_2(M_2(\mathscr{A})) \to M_2(M_2(\mathscr{A}))$ is given by the formula

$$\begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} & \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \end{pmatrix} \to \begin{pmatrix} \begin{pmatrix} a_{11} & b_{11} \\ c_{11} & d_{11} \end{pmatrix} & \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix} \\ \begin{pmatrix} a_{21} & b_{21} \\ c_{21} & d_{21} \end{pmatrix} & \begin{pmatrix} a_{22} & b_{22} \\ c_{22} & d_{22} \end{pmatrix} \end{pmatrix}.$$

LEMMA 2.7. *Fix N, M. Let p^{uv} $(1 \leqslant u, v \leqslant M)$ be hereditary polynomials with $O(p^{uv}) \leqslant N$. If $(p^{uv}(J, J^*)) \geqslant \varepsilon > 0$, then there exists $Q(z) = \sum_{k=0}^{N} Q_k z^k$ inver-*

tible in a neighborhood of $\overline{\mathbf{D}}$ *and there exists* $g^{uv} \in \mathscr{P}$ $(1 \leq u, v \leq M)$ *with* $O(g^{uv}) \leq N$ *and* $(g^{uv}(S^*, S))_{u,v=1}^M \geq 0$ *such that*

$$P(J, J^*) = Q(J)^*[I_M \otimes (2 - J^*J)]Q(J) + \left(\begin{bmatrix} 0 & 0 \\ 0 & g^{uv}(S^*, S)\end{bmatrix}\right)_{u,v=1}^M.$$

Alternately,

$$p^{uv}(J, J^*) = \sum_{a=1}^{M} q^{au}(J)^*(2 - J^*J)q^{av}(J, J^*) + \begin{bmatrix} 0 & 0 \\ 0 & g^{uv}(S^*, S)\end{bmatrix}.$$

Proof. $P(J, J^*) = \left(\begin{bmatrix} P^{uv}(V, V^*) & G(p^{uv}) \\ H(p^{uv}) & X(p^{uv})\end{bmatrix}\right)_{u,v=1}^M$. Applying the canonical shuffle, we find $P(J, J^*)$ is unitarily equivalent to the operator

$$(*) \qquad \begin{bmatrix} P(V, V^*) & G(P) \\ H(P) & X(P)\end{bmatrix} \geq \varepsilon > 0.$$

In particular $P(V, V^*) \geq \varepsilon$. Hence $R(e^{i\theta}) =: P(e^{i\theta}, e^{-i\theta}) > 0$. By Theorem 2.5, there exists $Q(z) = \sum_{a=0}^{N} Q_a z^a$ such that $Q(z)$ is invertible in a neighborhood of $\overline{\mathbf{D}}$ and $Q(e^{i\theta})^*Q(e^{i\theta}) = R(e^{i\theta})$. Using the relation $2 - J^*J = \begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}$ (and the canonical shuffle) it follows that $Q(J)^*[I_M \otimes (2 - J^*J)]Q(J)$ is unitarily equivalent to

$$\begin{bmatrix} Q(V)^*Q(V) & Q(V)^*G(Q) \\ G(Q)^*Q(V) & G(Q)^*G(Q)\end{bmatrix}.$$

By Proposition 2.4, it follows that $P(J, J^*) - Q(J)^*[I_M \otimes (2 - J^*J)]Q(J)$ is unitarily equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & X(P) - G(Q)^*G(Q)\end{bmatrix}.$$

Since $X(P) - G(Q)^*G(Q)$ is a (matrix valued) linear combination of $\{S^m S^{*n} \mid 0 \leq m, n \leq N\}$ there exists $g^{uv} \in \mathscr{P}$ $(1 \leq u, v \leq M)$ such that $(g^{uv}(S^*, S)) = X(P) - G(Q)^*G(Q)$. Unshuffling, we find

$$P(J, J^*) = Q(J)^*[I_M \otimes (2 - J^*J)]Q(J) + \left(\begin{bmatrix} 0 & 0 \\ 0 & g^{uv}(S^*, S)\end{bmatrix}\right)_{u,v=1}^M.$$

The fact that Q^{-1} exists and is analytic in a neighborhood of $\overline{\mathbf{D}}$ implies $Q(V)^{-1} = Q^{-1}(V)$. From (*) it follows

$$0 \leqslant \begin{bmatrix} Q(V)^{*-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Q(V)^*Q(V) & G(P) \\ G(P)^* & X(P) \end{bmatrix} \begin{bmatrix} Q(V)^{-1} & 0 \\ 0 & I \end{bmatrix} =$$

$$= \begin{bmatrix} I & Q(V)^{*-1}G(P) \\ G(P)^*Q(V)^{-1} & X(P) \end{bmatrix}.$$

Thus, using Proposition 2.4 and the Cholesky algorithm obtains

$$(g^{uv}(S^*, S)) = X(P) - G(Q)^*G(Q) \geqslant 0.$$

This completes the proof. ▨

The following lemma can be found in [11].

Lemma 2.8. *If $T \in \mathscr{F}$, then $T^*T - I \geqslant 0$.*

Theorem 2.9. *Fix M, N. Let $p^{uv} \in \mathscr{P}$ $(1 \leqslant u, v \leqslant M)$ with $O(p^{uv}) \leqslant N$. If $P(J, J^*) = (p^{uv}(J, J^*)) \geqslant 0$, then $P(T, T^*) \geqslant 0$ for every $T \in \mathscr{F}$.*

Proof. We begin with an $M \times M$ matrix of hereditary polynomials $P = (p^{uv})$ such that $O(p^{uv}) \leqslant N$ and $P(J, J^*) \geqslant 0$. By a standard approximation technique $\left(\text{replacing } P \text{ by } P + \begin{bmatrix} \varepsilon & & 0 \\ & \varepsilon & \\ 0 & & \ddots \end{bmatrix} \text{ and letting } \varepsilon \text{ tend to } 0\right)$ we may assume that $P(J, J^*) \geqslant \varepsilon > 0$. Let g^{uv}, Q be as in Lemma 2.7. Fix $0 < r < 1$ and let $t = (1 + r)/2$. Since $(g^{uv}(S^*, S)) \geqslant 0$, Theorem 2.6 produces functions $f_l^u(z)$ defined and analytic on $t\mathbf{D}$ such that

$$g^{uv}(z, w) = \sum_{l=1}^{\infty} \overline{f_l^v(\overline{w})}\,(1 - zw) f_l^u(z)$$

where the series converges uniformly on compact subsets of $t\mathbf{D} \times t\mathbf{D}$. Define $g_r^{uv}(z, w) = g^{uv}(rz, rw)$ and $f_{lr}^u(z) = f_l^u(rz)$. Then

$$g_r^{uv}(z, w) = \sum_{l=1}^{\infty} \overline{f_{lr}^v(\overline{w})}\,(1 - r^2zw) f_{lr}^u(z) \tag{2.10}$$

with f_{lr}^u defined and analytic on $(t/r)\mathbf{D}$ and where the series converges uniformly on compact subsets of $(t/r)\mathbf{D} \times (t/r)\mathbf{D}$.

Let

$$s^{uv}(z, w) = \sum_{a=1}^{M} \overline{q^{au}(\overline{w})}\,(2 - zw)q^{av}(z)$$

$((s^{uv}(z, w) = Q(\overline{w})^*(I_M \otimes (2 - zw)Q(z))$. Define (for each $0 < r < 1$) hereditary polynomials

(2.11) $$h_r^{uv}(z, w) = s^{uv}(z, w) - g_r^{uv}(z, w) + wg_r^{uv}(z, w)z,$$

$O(h_r^{uv}) \leq N + 1$, since $O(g_r^{uv}) \leq N$. Define auxiliary hereditary polynomials $b_1(x, y) = xy - 1$, $b_2(x, y) = -1 + 2xy - x^2y^2$ and $k_r(x, y) = r^2 b_2(x, y) + (1 - r^2)b_1(x, y))$. Then, using (2.10) and rearranging, (2.11) becomes

(2.12) $$h_r^{uv}(z, w) = s^{uv}(z, w) + \sum_{j=1}^{\infty} \overline{f_{jr}^u(\overline{w})} k_r(z, w) f_{jr}^v(z)$$

(with the series converging uniformly on compact subsets of $(t/r)\mathbf{D} \times (t/r)\mathbf{D}$). Compute

$$k_r(J, J^*) = \begin{bmatrix} 0 & 0 \\ 0 & I - r^2SS^* \end{bmatrix}.$$

If $\varphi(z) = \sum \varphi_n z^n$ is any analytic polynomial, then

$$k_r(J, J^*)\varphi(J) = \begin{bmatrix} 0 & 0 \\ 0 & I - r^2SS^* \end{bmatrix} \begin{bmatrix} \varphi(V) & G(\varphi) \\ 0 & \varphi(S^*) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & (I - r^2SS^*)\varphi(S^*) \end{bmatrix}.$$

Thus, if $\varphi, \psi \in H[(t/r)\mathbf{D}]$, by choosing sequences of polynomial $\psi_n \to \psi$, $\varphi_n \to \varphi$ converging uniformly on $t'\overline{\mathbf{D}}$ where $1 < t' < t/r$ and applying Lemma 1.3, it follows that

$$\varphi(J^*)k_r(J, J^*)\psi(J) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi(S^*)^*(I - r^2SS^*)\psi(S^*) \end{bmatrix}.$$

The above formula and (2.12) yield

$$h_r^{uv}(J, J^*) = s^{uv}(J, J^*) + \sum_{j=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & f_{jr}^u(S^*)^*(I - r^2SS^*)f_{jr}^v(S^*) \end{bmatrix}.$$

Using the definition of $s(z, w)$ and (2.10) gives

$$h_r^{uv}(J, J^*) = \sum_{a=1}^{M} q^{au}(J)^*(2 - J^*J)q^{av}(J) + \begin{bmatrix} 0 & 0 \\ 0 & g_r^{uv}(S^*, S) \end{bmatrix}.$$

Since $g_r^{uv}(S^*, S) \to g^{uv}(S^*, S)$ in operator norm, it follows from Lemma 2.7 that $h_r^{uv}(J, J^*) \to p^{uv}(J, J^*)$ in operator norm. Consequently, by Lemmas 2.1 and 2.2

$$h_r^{uv}(T, T^*) \to p^{uv}(T, T^*)$$

for any $T \in \mathscr{F}$. As $T \in \mathscr{F}$ implies $\sigma(T) \subseteq \overline{\mathbf{D}}$ and $t/r > 1$ from (2.12),

$$h_r^{uv}(T, T^*) = s^{uv}(T, T^*) + \sum_{l=1}^{\infty} f_{lr}^{u}(T)^* k_r(T, T^*) f_{lr}^{v}(T),$$

where the series converges in operator norm. Since, for $T \in \mathscr{F}$, $2 - T^*T \geqslant 0$, $(s^{uv}(T, T^*)) \geqslant 0$. Since, for $T \in \mathscr{F}$, $b_1(T, T^*)$ and $b_2(T, T^*)$ are positive semidefinite ($b_1(T, T^*) \geqslant 0$ by Lemma 2.8), $k_r(T, T^*) \geqslant 0$. Thus, $(h_r^{uv}(T, T^*)) \geqslant 0$. Whence,

$$(p^{uv}(T, T^*)) \geqslant 0.$$

▨

Proof of Theorem B. Fix $T \in \mathscr{F}$. Let $p^{uv} \in \mathscr{P}$ $(1 \leqslant u, v \leqslant M)$. If $(p^{uv}(J, J^*)) \geqslant 0$, then $(p^{uv}(T, T^*)) \geqslant 0$ by Theorem 2.9. Thus, by Theorem 2.1 there exists a Hilbert space K' and a unital representation $\pi : \mathscr{L}(K) \to \mathscr{L}(K')$ such that T extends to $\pi(J)$. Consequently, the proof of Theorem B will be complete once we establish:

LEMMA 2.13. *If $\pi : \mathscr{L}(K) \to \mathscr{L}(K)$ is a unital representation, then $\pi(J)$ is a Brownian operator.*

Proof. Let $K_1 \oplus K_2 = K$ be the orthogonal decomposition of K with respect to which $J = \begin{pmatrix} V & E \\ 0 & S^* \end{pmatrix}$. Let P_i be the orthogonal projection of K onto K_i. Let $P_i' = \pi(P_i)$ and let $P_i'K' = K_i'$. Since $P_1 + P_2 = I$ and π is unital $K' = K_1' \oplus K_2'$ Moreover, since $V : K_1 \to K_1$ is unitary so is $V' =: P_1'\pi(B)P_1' : K_1' \to K_1'$. Further, by the Wold decomposition in [2], the operator $P_2'\pi(J)P_2' : K_2' \to K_2'$ is unitarily equivalent to $S_\beta^* \oplus U'$ for some $0 \leqslant \beta \leqslant \infty$ and some unitary operator U'. Let $F' : K_2' \to K_1'$ be $P_2'\pi(B)P_1'$. Then, since $E^*E + I = 2I - SS^*$, $2I = F'^*F' + (S_\beta \oplus \otimes U'^*)(S_\beta^* \oplus U')$. Further, since

$$EE^* + VV^* \geqslant I$$

$$I_{K_1'} \leqslant \pi(P_2BP_1B^*P_2 + P_1VP_1V^*P_1) = F'F'^* + V'V'^*.$$

Consequently $\pi(J) = \begin{pmatrix} V' & F' \\ 0 & S_\beta^* \oplus U' \end{pmatrix}$ is a Brownian operator. ∎

3. OPTIMAL EXTENSIONS

In this section, we show that Theorem B is the "best possible" extension theorem for the class $\mathscr{F}$. As before, $\mathscr{F}$ is the class of operators T such that $I - 2T^*T + T^{*2}T^2 \leqslant 0$ and $\|T\|^2 \leqslant 2$. For convenience, let $\partial\mathscr{F}$ denote the Brownian operators. Of course, $\partial\mathscr{F} \subseteq \mathscr{F}$. To describe the sense in which Theorem B is "best", we recall some ideas from J. Agler's approach to model theory [4]. A subcollection $\mathscr{B} \subseteq \mathscr{F}$ is a boundary for $\mathscr{F}$ if

(i) given $B \in \mathscr{B}$ and π a unital representation, $\pi(B) \in \mathscr{B}$;

(ii) given $B \in \mathscr{L}(K) \cap \mathscr{B}$ and $H \subseteq K$ a reducing subspace for B, $B|H \in \mathscr{B}$;

(iii) given $B_i \in \mathscr{B}$ for $i \in I$ (some index set), $\bigoplus_{i \in I} B_i \in \mathscr{B}$;

(iv) for every $T \in \mathscr{F}$ there exists a $B \in \mathscr{B}$ such that T has an extension to B.

In particular, $\partial\mathscr{F}$ is a boundary for $\mathscr{F}$. Given $T \in \mathscr{F}$ an extension of T always exists by simply choosing any $T_0 \in \mathscr{F}$ and noting that T extends to $T' = T \oplus T_0$. An operator $T \in \mathscr{F}$ is said to be *extremal* if whenever T extends to $T' \in \mathscr{F}$ there exist $T_0 \in \mathscr{F}$ such that $T' = T \oplus T_0$. Let $\mathscr{E}$ denote the extremal operators in $\mathscr{F}$. If $\mathscr{B}$ is any boundary for $\mathscr{F}$, then $\mathscr{E} \subseteq \mathscr{B}$.

THEOREM 3.1. $\mathscr{E} = \partial\mathscr{F}$.

Clearly $\mathscr{E} \subseteq \partial\mathscr{F}$. Thus, it suffices to show that $\partial\mathscr{F} \subseteq \mathscr{E}$. For simplicity, we will show only that the operator $B = \begin{pmatrix} V & F \\ 0 & S^* \end{pmatrix}$ acting on $(H^2 \otimes H^2) \oplus H^2 =: K$ is in $\mathscr{E}$. To this end, suppose B extends to a $T \in \mathscr{F} \cap \mathscr{L}(K')$ where $K' \supseteq K$. With respect to the decomposition of K' as $(H^2 \otimes H^2) \oplus H^2 \oplus (K)^\perp$ the operator T has the form

$$\begin{pmatrix} V & F & X \\ 0 & S^* & Y \\ 0 & 0 & Z \end{pmatrix}$$

for some operators X, Y, Z. Compute

$$T^*T = \begin{pmatrix} I & 0 & V^*X \\ 0 & 2I & F^*X + SY \\ * & * & * \end{pmatrix}.$$

Using the relations $2 \geq T^*T \geq 1$, it follows that $V^*X = 0$ and $F^*X + SY = 0$. Compute

$$T^{*2}T^2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 2I + Q & F^*X + 2SY \\ * & * & * \end{pmatrix}$$

where Q is the projection onto the range of S. Since $2T^*T - I - T^{*2}T^2 \geq 0$, it follows that the range of F^*X is contained in the range of P, where $P = I - Q$. But using the relation $F^*X + SY = 0$, we find that $PSY = SY = 0$. Since S is injective $Y = 0$. Since $FF^* + VV^* \geq I$, it follows that $X = 0$. Hence $T = B \oplus Z$ and thus $B \in \mathcal{E}$ as desired. ▨

4. FURTHER RESULTS AND PROBLEMS

In a routine way, Theorem A can be recovered as a Corollary of Theorem B. Moreover, certainly one could establish a version of Theorem A for "higher order" Brownian shifts. For example, an operator $T \in \mathcal{L}(H)$ is said to be a 3-isometry if

$$(*) \qquad I - 3T^*T + 3T^{*2}T^2 - T^{*3}T^3 = 0.$$

Given $T \in \mathcal{L}(H)$, let $B_0(T) = 2 - T^*T$, $B_1(T) = -2 + 3T^*T - T^{*2}T^2$, and $B_2(T) = I - 2T^*T + T^{*2}T^2$. If T is a 3-isometry, then, by iterating $(*)$, we obtain

$$T^{*n}T^n = I + nB_1(T) + \frac{n(n+1)}{2} B_2(T).$$

In particular, $B_2(T) \geq 0$. Let $\mathcal{G}$ be the class of operators satisfying

(i) T is a 3-isometry
(ii) $B_0(T) \geq 0$
(iii) $B_1(T) \geq 0$.

Let $U \in \mathcal{L}(H)$ be a unitary operator and let $V_i \in \mathcal{L}(H_i)$ for $i = 1,2$ be isometries. Let $E_1 : H \to H_1$ and $E_2 : H_1 \to H_2$ be isometries such that $V_i^* E_i = 0$. Let $\partial\mathcal{G}$ denote the collection of operators of the form

$$\begin{pmatrix} V_2 & E_2 & 0 \\ 0 & V_1 & E_1 \\ 0 & 0 & U \end{pmatrix}.$$

A straightforward computation shows $\partial\mathcal{G} \subseteq \mathcal{G}$.

CONJECTURE 4.1. *If $T \in \mathscr{G}$, then T has an extension to a $J \in \partial\mathscr{G}$.*

In another direction, one could attempt to generalize Theorem A to tuples of operators. For each $i = 1, 2, \ldots, n$, let $J_i = \begin{pmatrix} V_i & E_i \\ 0 & U_i \end{pmatrix}$ be Brownian shifts; i.e., U_i is unitary, each V_i is an isometry and $V_i^* E_i = 0$. The tuple $\mathbf{J} = (J_1, J_2, \ldots, J_n)$ will be called a tuple of Brownian shifts if $E_j U_i = V_i E_j$ for each $i \neq j$ and if $\sum_{i=1}^{n} E_i^* E_i = I$. Given a tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ with $T_i \in \mathscr{L}(H)$ and a multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}^n$, define $\mathbf{T}^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} \ldots T_n^{\alpha_n}$. The tuple $\mathbf{T}$ is said to have an extension to a tuple $\mathbf{J} = (J_1, J_2, \ldots, J_n)$ where $J_i \in \mathscr{L}(K)$ if $H \subseteq K$, $J_i H \subseteq H$ and $T_i = J_i|H$ for each i.

If a tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ satisfies both $\mathbf{T}^{*\alpha}\mathbf{T}^\alpha = \sum_{i=1}^{n} \alpha_i (T_i^* T_i - I) + I$ for every $\alpha \in \mathbf{N}^n$ and $\sum_{i=1}^{n} (T_i^* T_i - I) \leqslant I$, then does $\mathbf{T}$ have an extension to an n-tuple of Brownian shifts?

An attempt to prove this by mimicking the proof of Theorem A given (implicitly) in Section two fails since the analogue of Theorem 2.5 is false in several variables.

REMARK 4.2. In an earlier version of this paper it was asserted that to show, for $T \in \mathscr{F}$ and $J = \begin{bmatrix} V & E \\ 0 & S^* \end{bmatrix}$, $(p^{uc}(J, J^*)) \geqslant 0$ implies $(p^{uc}(T, T^*)) \geqslant 0$ it was enough to verify the above statement for hereditary polynomials p^{uc} of the special form $p^{uc}(x, y) = \sum_{n=0}^{N} p_n^{uc} x^n y^n$. While this reduction may be valid, the proof given tacitly assumed T to be invertible, a condition violated by the generic $T \in \mathscr{F}$.

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