J. OPERATOR THEORY
23(1990), 339-350

ON THE INTERTWINING OF JOINT ISOMETRIES

AMEER ATHAVALE

This paper may be looked upon as a sequel to [2]. In what follows, $\mathscr{B}(\mathscr{H})$ denotes the set of bounded linear operators on a separable Hilbert space $\mathscr{H}$. An operator tuple $S = (S_1, \ldots, S_m)$ of m communting operators in $\mathscr{B}(\mathscr{H})$ is said to *dilate to* an operator tuple $M = (M_1, \ldots, M_m)$ of m commuting operators in $\mathscr{B}(\mathscr{K})$ if there exists an isometry $V: \mathscr{H} \to \mathscr{K}$ such that $S_1^{n_1} S_2^{n_2} \ldots S_m^{n_m} = V^* M_1^{n_1} M_2^{n_2} \ldots \ldots M_m^{n_m} V$ for all non-negative integers n_i; if moreover Range V is invariant for each M_i, then S is said to *extend to* M. If S dilates to M and M is a tuple of m commuting normals, then S is said to have a *normal dilation* and if S dilates to M and M is a tuple of m commuting unitaries, then S is said to have a *unitary dilation*. If an operator tuple S of m commuting operators in $\mathscr{B}(\mathscr{H})$ extends to a tuple M of m commuting normals in $\mathscr{B}(\mathscr{K})$, then S is said to be *subnormal* (*on* $\mathscr{H}$), with M (*on* $\mathscr{K}$) being a *normal extension* (or *lift*) of S. If S is subnormal, then there is a minimal normal extension M of S unique up to unitary equivalence [19]. A commuting operator tuple S will be called a *toral isometry* (*on* $\mathscr{H}$) if each S_i is an isometry in $\mathscr{B}(\mathscr{H})$; thus S is a total isometry if $I - S_i^* S_i = 0$ for each i, where I denotes the identity operator on $\mathscr{H}$. The multiplication tuple on the Hardy space of the unit polydisk in $\mathbf{C}^m$, hereafter referred to as the Cauchy tuple, is an example of a toral isometry. A commuting operator tuple S will be called a *spherical isometry* (*on* $\mathscr{H}$) if S_i in $\mathscr{B}(\mathscr{H})$ satisfy $I - S_1^* S_1 - \ldots - S_m^* S_m = 0$. The multiplication tuple on the Hardy space of the unit ball in $\mathbf{C}^m$, hereafter referred to as the Szegö tuple, is an example of a spherical isometry. In this paper, we are mainly concerned with the intertwining of a pair of operator tuples at least one of which is a toral or spherical isometry.

A toral isometry S may be looked upon as a subnormal tuple S whose minimal normal extension M has its joint spectrum $\sigma(M)$ contained in the unit torus $\mathbf{T}^m$, the distinguished boundary of the unit polydisk $\mathbf{D}^m$ in $\mathbf{C}^m$. (This is easy to see from Proposition 6.2 in [24], but in Proposition 1 below we present a new proof.) Proposition 2 is a corresponding assertion for a spherical isometry. Before stating the propositions, it will be convenient to stipulate a notational convention: If

$p(z, w) = \sum_{m,n} a_{mn} z^m w^n$ is an analytic polynomial in the complex variables $z = (z_1, \ldots, z_m)$, $w = (w_1, \ldots, w_m)$; then $(p(z, w))(S, S^*)$ is to be interpreted as $(p(z, w))(S, S^*) = \sum_{m,n} a_{mn} S^{*n} S^m$, where $S^* = (S_1^*, \ldots, S_m^*)$.

PROPOSITION 1. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$. Then statements* (i) *and* (ii) *below are equivalent.*

(i) *S is a toral isometry on $\mathscr{H}$.*

(ii) *S is a subnormal tuple on $\mathscr{H}$ such that the minimal normal extension M of S has its joint spectrum $\sigma(M)$ contained in $\mathbf{T}^m$.*

Proof. That (ii) implies (i) is obvious. So suppose (i) is true. This means $(1 - z_i w_i)(S, S^*) = 0$ for $1 \leqslant i \leqslant m$. For any non-negative integers n_i, this clearly implies $\left[\prod_{i=1}^{m} (1 - z_i w_i)^{n_i}\right](S, S^*) \geqslant 0$. It follows from Proposition 7 in [5] that S is subnormal on $\mathscr{H}$ and there exists an operator valued probability measure ρ_S supported on $\Delta = \{x \in \mathbf{R}^m : 1 - x_1 = 0, \ldots, 1 - x_m = 0\}$ such that $S^{*n} S^n = \int_\Delta x^n \mathrm{d}\rho_S(x)$ for any tuple $n = (n_1, \ldots, n_m)$ of non-negative intergers n_i. Arguing as in the proof of Proposition 8 in [5], it is not difficult to see that the minimal normal extension M of S has its joint spectrum $\sigma(M)$ contained in $\{z \in \mathbf{C}^m : 1 - |z_1|^2 = 0, \ldots, 1 - |z_m|^2 = 0\} = \mathbf{T}^m$. □

PROPOSITION 2. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$. Then statements* (i) *and* (ii) *below are equivalent.*

(i) *S is a spherical isometry on $\mathscr{H}$.*

(ii) *S is a subnormal tuple on $\mathscr{H}$ such that the minimal normal extension M of S has its joint spectrum $\sigma(M)$ contained in the unit sphere $\mathbf{S}^{2m-1}$, the topological boundary of the unit ball $\mathbf{B}^{2m}$ in $\mathbf{C}^m$.*

Proof. That (ii) implies (i) is obvious. So suppose (i) is true. This means $(1 - z_1 w_1 - \ldots - z_m w_m)(S, S^*) = 0$. For any non-negative integers n_i,

$$\left(\prod_{i=1}^{m} (1 - z_i w_i)^{n_i}\right)(S, S^*) =$$

$$= \left(\prod_{i=1}^{m} ((1 - z_1 w_1 - \ldots - z_m w_m) + z_1 w_1 + \ldots + \widehat{z_i w_i} + \ldots + z_m w_m)^{n_i}\right)(S, S^*) \geqslant 0.$$

(Here ^ denotes omission.) It follows from Proposition 7 in [5] that S is subnormal on $\mathscr{H}$ and there exists an operator valued probability measure ρ_S supported on

$$\Delta = \{x \in \mathbf{R}^m : x_1 \geqslant 0, \ldots, x_m \geqslant 0, 1 - x_1 - \ldots - x_m = 0\}$$

such that $S^{*n} S^n = \int_\Delta x^n \mathrm{d}\rho_S(x)$ for any tuple $n = (n_1, \ldots, n_m)$ of non-negative integers

n_i. Arguing as in the proof of Proposition 8 in [5], it is not difficult to see that the minimal normal extension M of S has its joint spectrum $\sigma(M)$ contained in $\{z \in \mathbf{C}^m : 1 - |z_1|^2 - \ldots - |z_m|^2 = 0\} = \mathbf{S}^{2m-1}$. ▨

The role of toral and spherical isometries is vital for the consideration of unitary and normal dilations of operator tuples. To corroborate this claim, we state two results from [1] and [3].

PROPOSITION 3. [1, Corollary]. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$, and let $H^2(\mathbf{D}^m)$ denote the Hardy space of the unit polydisk in $\mathbf{C}^m$. Then statements* (i) *and* (ii) *below are equivalent.*

(i) $\left(\prod_{i=1}^{m} (1 - z_i w_i)^{n_i}\right)(S, S^*) \geqslant 0$ *for all integers n_i statisfying* $0 \leqslant n_i \leqslant 1$.

(ii) *There exist a Hilbert space $\mathscr{H}'$ and a unital $*$-representation $\pi : \mathscr{B}(H^2(\mathbf{D}^m)) \to \mathscr{B}(\mathscr{H}')$ such that S extends to $\pi(M_z^*) = (\pi(M_{z_1}^*), \ldots, \pi(M_{z_m}^*))$, where $M_z = (M_{z_1}, \ldots, M_{z_m})$ denotes the Cauchy tuple.*

PROPOSITION 4. [3, Theorem 4.2]. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$ such that the Taylor spectrum $\sigma(S)$ of S is contained in the closed unit ball $\overline{\mathbf{B}}^{2m}$. Let also $H^2(\mathbf{B}^{2m})$ denote the Hardy space of the unit ball in $\mathbf{C}^m$. Then statements* (i) *and* (ii) *below are equivalent.*

(i) $((1 - z_1 w_1 - \ldots - z_m w_m)^k)(S, S^*) \geqslant 0$ *for* $1 \leqslant k \leqslant m$.

(ii) *There exist a Hilbert space $\mathscr{H}'$ and a unital $*$-representation $\pi : \mathscr{B}(H^2(\mathbf{B}^{2m})) \to \mathscr{B}(\mathscr{H}')$ such that S extends to $\pi(M_z^*)$, where M_z denotes the Szegö tuple.*

REMARK 1. Since π is a unital $*$-representation, $\pi(M_z)$ is a toral isometry in Proposition 3 and $\pi(M_z)$ is a spherical isometry in Proposition 4. Thus conditions (i) in Proposition 3 guarantee a unitary dilation of S and conditions (i) in Proposition 4 guarantee a normal dilation M of S with $\sigma(M)$ contained in $\mathbf{S}^{2m-1}$. Indeed, the result in Proposition 4 is the spherical analog of the Sz.-Nagy–Brehmer or regular unitary dilation for operator tuples [6], [16], [24]. We will return to Propositions 3 and 4 in the context of the intertwining of operator tuples.

DEFINITION. Let $S = (S_1, \ldots, S_m)$ be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$ and T a tuple of m commuting operators in $\mathscr{B}(\mathscr{K})$. If a bounded linear map $X: \mathscr{H} \to \mathscr{K}$ is such that $XS_i = T_i X$ for each i, then X is said to be an *intertwining operator* (*for S and T*) and we denote this fact by $XS = TX$. If $X: \mathscr{H} \to \mathscr{K}$ and $Y: \mathscr{K} \to \mathscr{H}$ are two intertwining operators such that $XS = TX$ and $YT = SY$, and both X and Y are injective and have dense ranges, then S is said to be *quasisimilar to T*. The operator tuple S is said to be *similar to T* or *unitarily equivalent to T* according as one can find an invertible or a unitary intertwining operator for S and T.

Our occasional laxity of language in the use of above definitions can be justified by noting that "quasisimilar to," "similar to" and "unitarily equivalent to" are de facto equivalence relations. Generalizing some earlier work of Clary [7], Hastings [17] provided a function-theoretic characterization of subnormal tuples quasisimilar to the Cauchy tuple. A function-theoretic characterization of subnormal tuples quasisimilar to the Szegö tuple was provided in [2]. Using the ideas present in the proof of Proposition 4.1 in [8] and easily modifying the proofs of Lemma 2 and Theorem 2 in [17], one can prove Proposition 5 below for the case of a toral isometry. Using the ideas present in the proof of Proposition 4.1 in [8] and easily modifying the proofs of Proposition 3 and Theorem 1 in [2], one can prove Proposition 5 for the case of a spherical isometry. At this stage, it is emphasized that excepting Proposition 5, in none of the propositions in this paper we require the subnormal tuples to be cyclic; that is, unitarily equivalent to the multiplication tuple M_z^μ on the closure $H^2(\mu)$ of the set of analytic polynomials in $L^2(\mu)$, μ being a compactly supported positive measure on $\mathbf{C}^m$.

PROPOSITION 5. *Let S be a cyclic toral (sperical) isometry on $\mathscr{H}$ and let T be a cyclic subnormal tuple on $\mathscr{K}$. Then S is similar to T if and only if there are compactly supported measures μ and ν with μ supported on $\mathbf{T}^m$ $(\mathbf{S}^{2m-1})$ such that S is unitarily equivalent to M_z^μ, T is unitarily equivalent to N_z^ν and there are positive constants c and d satisfying*

(i) $\int |p|^2 d\mu \leqslant c \int |p|^2 d(\nu|\mathbf{T}^m) \quad \left(\int |p|^2 d\mu \leqslant c \int |p|^2 d(\nu|\mathbf{S}^{2m-1})\right),$

(ii) $\int |p|^2 d\nu \leqslant d \int |p|^2 d\mu$

for every m-variable analytic polynomial p.

Crucial for the proof of Theorem 1 in [2] (as well as a part of Proposition 5 above) is Proposition 6 below which is in fact a particularized version of an approximation theorem related to the solution of the inner function problem on the unit ball [21, Theorem 3.5]. Proposition 6 will be appealed to frequently in the sequel.

PROPOSITION 6. *Suppose that*

(a) *$f\colon \bar{\mathbf{B}}^{2m} \to (0, \infty)$ is continuous, and*

(b) *θ is a positive measure on $\mathbf{S}^{2m-1}$.*

Then there exists a sequence $\{p_j\}$ of m-variable analytic polynomials such that

(1) $|p_j| < f$ *on* $\bar{\mathbf{B}}^{2m}$,

(2) $\lim_{j\to\infty} p_j(z) = 0$ *uniformly on every compact subset of* $\mathbf{B}^{2m}$, *and*

(3) $\lim_{j\to\infty} |p_j(\zeta)| = f(\zeta)$ ζ-a.e. $[\theta]$.

EXAMPLE 1. In the case of the polydisk algebra, the Poisson-Szegö kernel

$$P(w, \xi) = \frac{1}{(2\pi)^m} \prod_{i=1}^{m} \frac{1 - |w_i|^2}{|1 - w_i\xi_i|^2} \quad (w \in \mathbf{D}^m,\ \xi \in \mathbf{T}^m)$$

gives rise to the representing measures $d\mu_w = P(w, \xi)d\theta_1 \dots d\theta_m$ for the points w in $\mathbf{D}^m$, where $d\theta_i$ denotes the arc-length measure on $\mathbf{T}$. In the case of the ball algebra, the Poisson-Szegö kernel

$$Q(w, \xi) = \frac{(m-1)!}{(2\pi)^m}\left[\frac{1 - |w_1|^2 - \dots - |w_m|^2}{|1 - w_1\xi_1 - \dots - w_m\xi_m|^{2m}}\right] \quad (w \in \mathbf{B}^{2m},\ \xi \in \mathbf{S}^{2m-1})$$

gives rise to the representing measures $d\nu_w = Q(w, \xi)d\sigma$ for the points w in $\mathbf{B}^{2m}$, where $d\sigma$ denotes the surface area measure. Note that μ_w $(w \in \mathbf{D}^m)$ are all mutually boundedly absolutely continuous and it follows from Proposition 5 that $M_z^{\mu_w}$ all lie in the similarity orbit of the Cauchy tuple. Similarly, $M_z^{\nu_w}$ $(w \in \mathbf{B}^{2m})$ all lie in the similarity orbit of the Szegö tuple. As is pointed out in [22], if ξ is any point on $\mathbf{S}^{2m-1}$, then since $f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi e^{i\theta})d\theta$ for any f in the ball algebra, there is a representing measure $n(\xi)$ for the origin 0 supported on $\{\xi e^{i\theta}: -\pi \leq \theta \leq \pi\}$. If α and β are two points on $\mathbf{S}^{2m-1}$ such that $|\alpha_j| \neq |\beta_j|$ for any j, then support $n(\alpha) \cap$ $\cap$ support $n(\beta) = 0$ and it follows from the considerations in [2] that $M_z^{n(\alpha)}$ cannot be quasisimilar to $M_z^{n(\beta)}$.

In the one-dimensional case, Hoover [18] showed that quasisimilar toral isometries are unitarily equivalent. It was pointed out by Hastings in [17] that for $m \geq 2$, quasisimilar toral isometries need not be unitarily equivalent. In conjunction with Proposition 5, Hastings's example actually shows that similar toral isometries need not be unitarily equivalent. As is shown by Example 2 below, similar spherical isometries need not be unitarily equivalent either.

EXAMPLE 2. Let σ be the normalized surface area measure on $\mathbf{S}^3$, so that $H^2(\sigma)$ is the Hardy space $H^2(\mathbf{B}^4)$ of $\mathbf{B}^4$. Let μ on $\mathbf{S}^3$ be defined by $d\mu = (1 + |\xi_1|)d\sigma(\zeta)$. It is easy to see from Proposition 4 that M_z^σ is in fact similar to M_z^μ. However, M_z^σ is not unitarily equivalent to M_z^μ. For, suppose there exists an intertwining unitary operator $U: H^2(\sigma) \to H^2(\mu)$. If 1 denotes the constant function 1 in $H^2(\sigma)$, then letting $\varphi = U1$, we must have

$$\int |p(\zeta)|^2|\varphi(\zeta)|^2(1 + |\zeta_1|)d\sigma(\zeta) = \int |p(\zeta)|^2 d\sigma(\zeta)$$

for any m-variable analytic polynomial p. Now let θ on $\mathbf{S}^3$ be defined by

$$d\theta(\zeta) = |\varphi(\zeta)|^2(1 + |\zeta_1|)d\sigma(\zeta) + d\sigma(\zeta).$$

If $f(\zeta)$ is any positive continuous function on $\mathbf{S}^3$, then extend it to a positive continuous function on $\mathbf{B}^4$, still denoted f, and appeal to Proposition 6 to conclude that

$$\int f(\zeta)|\varphi(\zeta)|^2(1 + |\zeta_1|)d\sigma(\zeta) = \int f(\zeta)d\sigma(\zeta).$$

Thus $|\varphi(\zeta)|^2(1 + |\zeta_1|) = 1$ a.e. $[\sigma]$. Note that 1 is in $\ker(M_{z_1}^u)^* \cap \ker(M_{z_2}^u)^*$, hence φ must be in $\ker(M_{z_1}^u)^* \cap \ker(M_{z_2}^u)^*$. It is easy to check that $\int \zeta_1^k \zeta_2^l(1 + |\zeta_1|)d\sigma(\zeta) = 0$ for $(k, l) \neq (0, 0)$. Thus $\ker(M_{z_1}^u)^* \cap \ker(M_{z_2}^u)^*$ is spanned by the constant function 1. Since $|\varphi(\zeta)|^2(1 + |\zeta_1|) = 1$ a.e. $[\sigma]$, this forces $|\zeta_1|$ to be constant a.e. $[\sigma]$, which is obviously not true.

Despite the negative results referred to above, it may be possible to deduce unitary equivalence from quasisimilarity in some special cases as shown by the next proposition. Recall that two operators S_1 and S_2 in $\mathscr{B}(\mathscr{H})$ are said to be *doubly commuting* if $S_1S_2 = S_2S_1$ and $S_1^*S_2 = S_2S_1^*$. For the definition of a shift operator, refer to [24].

PROPOSITION 7. *Let $S = (S_1, S_2)$ and $T = (T_1, T_2)$ be toral isometries on $\mathscr{H}$ and $\mathscr{K}$ respectively, such that S_1 and S_2 are doubly commuting shifts as are T_1 and T_2. If S is quasisimilar to T, then S is unitarily equivalent to T.*

Proof. Let S and T be toral isometries on $\mathscr{H}$ and $\mathscr{K}$ such that S_1 and S_2 are doubly commuting shifts as are T_1 and T_2. Let further $X: \mathscr{H} \to \mathscr{K}$ and $Y: \mathscr{K} \to \mathscr{H}$ be the intertwining operators for S and T such that X and Y are injective and have dense ranges. From Theorem 1 in [23], one can write $\mathscr{H} = \bigoplus_{p,q\geq 0} S_1^pS_2^qA$ and $\mathscr{K} = \bigoplus_{p,q\geq 0} T_1^pT_2^qB$, where $A = \ker S_1^* \cap \ker S_2^*$, $B = \ker T_1^* \cap \ker T_2^*$, and where $\oplus$ denotes the orthogonal direct sum. Since $Y^*S_i^* = T_i^*Y^*$, it follows that $Y^*A \subset B$. Since Y^* is one-one, $\dim A \leqslant \dim B$. Similarly, $X^*T_i^* = S_i^*X^*$ forces $\dim B \leqslant \dim A$. Thus there exists a unitary from A onto B. Using the representations for $\mathscr{H}$ and $\mathscr{K}$, it is easy to deduce that S is unitarily equivalent to T. ∎

QUESTION 1. Are quasisimilar toral (spherical) isometries necessarily similar?

Recall that if A is a function algebra on a compact set X in $\mathbf{C}^m$, then a representation u of A on $\mathscr{H}$ is an algebra homomorphism $u: A \to \mathscr{B}(\mathscr{H})$ satisfying $u(1) = I$ and $\|u(\varphi)\| \leqslant \|\varphi\|$ for any φ in A, where $\|\varphi\|$ denotes the supremum norm of φ. It was noted in [20] that if X is an intertwining operator for two toral isometries S and T, then X has a norm-preserving lift $\tilde{X}$ intertwining the minimal normal (in this case unitary) extensions M and N of S and T. The following proposition is the analog of this result for spherical isometries.

PROPOSITION 8. *Let S and T be two spherical isometries on $\mathscr{H}$ and $\mathscr{K}$, respectively, and let M on $\tilde{\mathscr{H}}$ and N on $\tilde{\mathscr{K}}$ be the minimal normal extensions of S and T. If $X: \mathscr{H} \to \mathscr{K}$ satisfies $XS = TX$, then X extends to an operator $\tilde{X}: \tilde{\mathscr{H}} \to \tilde{\mathscr{K}}$ such that $\tilde{X}M = N\tilde{X}$ and $\|\tilde{X}\| = \|X\|$.*

Proof. Let X be the intertwining operator for the spherical isometries S and T. In view of Proposition 2, the minimal normal extensions M and N of S and T satisfy

$$p(M)^*p(M) = \int |p(\zeta)|^2 \mathrm{d}\rho_M(\zeta) \quad \text{and} \quad p(N)^*p(N) = \int |p(\zeta)|^2 \mathrm{d}\rho_N(\zeta),$$

where ρ_M and ρ_N are the spectral measures on $\mathbf{S}^{2m-1}$ associated with M and N, and p is any m-variable analytic polynomial. Letting $\rho_S(\cdot) = P_{\mathscr{H}}\rho_M(\cdot)/\mathscr{H}$ and $\rho_T = P_{\mathscr{K}}\rho_N(\cdot)/\mathscr{K}$, where $P_{\mathscr{H}}, P_{\mathscr{K}}$ denote appropriate projections, we see that for any vector u in $\mathscr{H}$ and v in $\mathscr{K}$, $\|p(S)u\|^2 = \int |p(\zeta)|^2 \mathrm{d}\langle \rho_S(\zeta)u, u\rangle_{\mathscr{H}}$ and $\|p(T)v\|^2 = \int |p(\zeta)|^2 \mathrm{d}\langle \rho_T(\zeta)v, v\rangle_{\mathscr{K}}$. If w is any vector in $\mathscr{H}$, then letting $u = Xw$, one has

$$\int |p(\zeta)|^2 \mathrm{d}\langle \rho_T(\zeta)Xw, Xw\rangle_{\mathscr{K}} = \|p(T)Xw\|^2 = \|Xp(S)w\|^2 \leqslant \|X\|^2\|p(S)w\|^2 =$$

$$= \|X\|^2 \int |p(\zeta)|^2 \mathrm{d}\langle \rho_S(\zeta)w, w\rangle_{\mathscr{H}}.$$

Now define a measure θ on $\mathbf{S}^{2m-1}$ by $\theta(\cdot) = \langle \rho_T(\cdot)Xw, Xw\rangle_{\mathscr{K}} + \langle \rho_S(\cdot)w, w\rangle_{\mathscr{H}}$. Appealing to Proposition 4, it follows that

$$\int f(\zeta)\mathrm{d}\langle \rho_T(\zeta)Xw, Xw\rangle_{\mathscr{K}} \leqslant \|X\|^2 \int f(\zeta)\mathrm{d}\langle \rho_S(\zeta)w, w\rangle_{\mathscr{H}}$$

for any positive continuous function f on $\mathbf{S}^{2m-1}$. This means $\langle \rho_T(\cdot)Xw, Xw\rangle_{\mathscr{K}} \leqslant \|X\|^2\langle \rho_S(\cdot)w, w\rangle_{\mathscr{H}}$ for any w in $\mathscr{H}$. It follows now from Lemma 4.1 in [20] that X has a norm-preserving lift $\tilde{X}$ satisfying $\tilde{X}M = N\tilde{X}$. ▨

REMARK 2. It was observed in [20] that every toral isometry gives rise to a representation of the polydisk algebra as restricted to the distinguished boundary $\mathbf{T}^m$ of $\mathbf{D}^m$. Given any spherical isometry S, Proposition 2 allows us to obtain a representation u of the ball algebra through the correspondence $u(z_i) = S_i$. The results in [20] related to the intertwining of toral isometries depended crucially on the fact that the polydisk algebra as restricted to $\mathbf{T}^m$ is an approximating in modulus algebra; that is, a positive continuous function on $\mathbf{T}^m$ can be approximated uniformly by the moduli of polynomials on $\mathbf{T}^m$. For the questions related to the intetwining of spherical isometries, our savior is Proposition 6. Indeed, it is left to the reader to verify that the assertions of Theorem 5.2, 5.3 and 6.1 in [20] remain valid for the representations of the ball algebra as well. In particular, if the intertwining operator X for two spherical isometries has dense range, then so has the lift $\tilde{X}$. In view of the negative results on the unitary equivalence of quasisimilar toral or spherical isometries, such observations are useful as is borne out by Proposition 9 below. Note also that letting S to be the same as T amounts to considering commutant lifting theorems for toral and spherical isometries. In particular, Proposition 8 says that an operator X in the commutant of a spherical isometry has a norm-preserving lift to an operator $\tilde{X}$ in the commutant of the minimal normal extension M of S. As far as the author knows, it is an open question whether the ball algebra as restricted to $\mathbf{S}^{2m-1}$ $(m > 1)$ is an approximating in modulus algebra or not.

Lemma 1. *Let M and N be tuples of commuting normals in $\mathscr{B}(\mathscr{H})$ and $\mathscr{B}(\mathscr{K})$ respectively. If $X: \mathscr{H} \to \mathscr{K}$ and $Y: \mathscr{K} \to \mathscr{H}$ have dense ranges, and $XM = NX$ and $YN = MY$, then M is unitarily equivalent to N.*

Proof. Let U and V be the partial isometries in the polar decompositions of X and Y. It follows from Lemma 4.1 of [13] that $(\ker X)^{\perp}$ reduces each M_i, $(\ker Y)^{\perp}$ reduces each N_i, $U(\ker X)^{\perp}$ is a unitary intertwining $M_i(\ker X)^{\perp}$ and N_i, and $V(\ker Y)^{\perp}$ is a unitary intertwining $N_i(\ker Y)^{\perp}$ and M_i. Using Theorem 1.3 of [15] (which generalizes trivially to operator tuples), one concludes that M is unitarily equivalent to N. ∎

Proposition 9. *Let S and T be toral (spherical) isometries on $\mathscr{H}$ and $\mathscr{K}$ respectively. If S is quasisimilar to T, then the minimal normal extension M of S is unitarily equivalent to the minimal normal extension N of T.*

Proof. In the case of toral isometries, appeal to Lemmas 5.1 and 5.2 in [20] and Lemma 1 above. In the case of spherical isometries, appeal to Proposition 8, Remark 2 and Lemma 1. ∎

Corollary 1. *Let S and T be two toral (spherical) isometries. If S is quasisimilar to T, then $\hat{\sigma}(S) = \hat{\sigma}(T)$, where $\hat{\sigma}(S)$ denotes the joint spectrum of S with respect to the polynomial algebra generated by I and $S_1, \ldots, S_m$.*

Proof. This follows straight from Proposition 8 above and Theorem 4 in [17]. ∎

Remark 3. The conclusion of Proposition 8 may not hold if only one of S and T is a toral (spherical) isometry. For, consider

$$d\mu = (1/2\pi)^m d\theta_1 d\theta_2 \ldots d\theta_m + \delta(0) \quad (d\mu = ((m-1)!/2\pi^m)d\sigma + \delta(0)),$$

where $d\theta_i$ denotes the arc-length measure on $\mathbf{T}$ ($d\sigma$, the surface area measure on $\mathbf{S}^{2m-1}$) and $\delta(0)$, the point mass at the origin 0. Since $(d\mu - \delta(0))$ evaluates a polynomial at the origin, it is easy to see from Proposition 4 that M_z^μ is similar to the Cauchy (Szegö) tuple. However, the spectrum of the minimal normal extension of M_z^μ is $\mathbf{T}^m \cup \{0\}$ $(\mathbf{S}^{2m-1} \cup \{0\})$.

Taking a cue from the final remarks in [20], we now give a couple of concrete results on intertwining a pair of operator tuples, one of which is not necessarily a toral or spherical isometry. The author will refrain from explicating the sense of minimality for unitary and normal dilations (see [20]); also the proof of Proposition 10 will be skipped since it is essentially the same as the proof of Proposition 11.

PROPOSITION 10. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$ satisfying conditions* (i) *of Proposition* 3, *and let $\tilde{S}$ be a minimal unitary dilation of S guaranteed by Proposition* 3. *If T is a toral isometry on $\mathscr{K}$ and $X: \mathscr{H} \to \mathscr{K}$ is an intertwining operator for S and T, then X has a norm-preserving lift $\tilde{X}$ intertwining $\tilde{S}$ and N, the minimal normal (in fact, unitary) extension of T.*

PROPOSITION 11. *Let S be a tuple of m commuting operators in $\mathscr{B}(\mathscr{H})$ with the Taylor spectrum $\sigma(S)$ of S in $\bar{\mathbf{B}}^{2m}$, and satisfying conditions* (i) *of Proposition* 4. *Let $\tilde{S}$ with $\sigma(\tilde{S})$ in $\mathbf{S}^{2m-1}$ be a minimal normal dilation of S guaranteed by Proposition* 4. *If T is a spherical isometry on $\mathscr{K}$ and $X: \mathscr{H} \to \mathscr{K}$ is an intertwining operator for S and T, then X has a norm-preserving lift $\tilde{X}$ intertwining $\tilde{S}$ and N, the minimal normal extension of T.*

Proof. For S and T as above, let $\rho_{\tilde{S}}$ and ρ_N be the spectral measures on $\mathbf{S}^{2m-1}$ associated with $\tilde{S}$ and N. Let ρ_T be the compression of $\rho_N(\cdot)$ to $\mathscr{K}$. Then for any w in $\mathscr{H}$,

$$\int |p(\zeta)|^2 d\langle \rho_T(\zeta)Xw, Xw\rangle = \|p(T)Xw\|^2 = \|Xp(S)w\|^2 \leqslant$$

$$\leqslant \|X\|^2\|p(S)w\|^2 \leqslant \|X\|^2\|p(\tilde{S})w\|^2 = \|X\|^2 \int |p(\zeta)|^2 d\langle \rho_{\tilde{S}}(\zeta)w, w\rangle.$$

The result follows by a now-familiar argument and Lemma 4.1 in [20]. ▨

For the technical jargon employed in the remainder of this paper, a basic reference is [10]. In [4], the author discussed a notion of the dual of a subnormal tuple S on $\mathscr{H}$. If S on $\mathscr{H}$ has a minimal normal extension M on $\tilde{\mathscr{H}}$, then the dual $S' = (S_1', \ldots, S_m')$ of S is defined by setting $S_i' = N_i^*|\mathscr{H}^\perp$, where $\mathscr{H}^\perp = \tilde{\mathscr{H}} \ominus \mathscr{H}$. The main emphasis of [4] was to discuss the interdependence of the Fredholm properties of S, S' and M. In view of Propositions 1 and 2, it follows that if S is a toral (spherical) isometry, then so is its dual S'! Recall that an operator tuple S is said to be essentially normal if $S_j^* S_i - S_i S_j^*$ is compact for each i and j. For a Fredholm tuple S, Index S denotes the Fredholm index of S.

PROPOSITION 12. *If S is an essentially normal spherical isometry, then both S and S' are Fredholm and* Index $S = (-1)^{m+1}$Index S'; *in particular, for m even, S cannot be similar to S' if* Index $S \neq 0$.

Proof. If S is an essentially normal spherical isometry, then it follows from Corollary 2 in [4] and Proposition 2 that the Taylor essential spectrum $\sigma_e(S)$ of S is contained in $\mathbf{S}^{2m-1}$. Thus S is Fredholm and by Proposition 1 in [4], so is S'. Now appeal to Proposition 3 of [4] to conclude that Index $S' = (-1)^{m+1}$Index S.

From an easy extension of the considerations in Section 6 of [11], it is clear that if $X: \mathscr{H} \to \mathscr{H}'$ is an invertible operator such that $XS = S'X$, then Index S = Index S'. The last assertion of Proposition 12 is now obvious. ∎

REMARK 4. The Szegö tuple S is a well-known example of an essentially normal spherical isometry with Index $S = -1$ [10]. Actually, by modifying the proofs in [4], it can be shown that the formula Index $S = (-1)^{m+1}$Index S' is valid for any Fredholm subnormal tuple whose dual S' is also Fredholm [12]. The Cauchy tuple S is a well-known example of a Fredholm toral isometry with Index $S = -1$ [10]. For $m = 2$, the Fredholmness of S' was verified by explicit computations in [4]; the considerations there generalize to arbitrary dimensions. In particular, for m even, the Cauchy tuple cannot be similar to its dual either. Note that for $m = 1$, both the Cauchy tuple and the Szegö tuple degenerate into the unilateral shift, which is known to be unitarily equivalent to its dual [9]. The author is admittedly not clear-headed regarding the intertwining of toral or spherical isometries with their duals, even in the special cases of the Cauchy tuple and the Szegö tuple. Indeed, two lines of thought suggest themselves:

(1) Investigate conditions under which a toral or spherical isometry is quasi-similar (similar, unitarily equivalent) to its dual.

(2) Investigate conditions under which the intertwining of two toral (spherical) isometries S and T through a non-zero map X guarantees the intertwining of S' and T' through a non-zero map X' and correlate properties of X and X'.

These considerations can be generalized to arbitrary subnormal tuples, but in view of Proposition 6 and the remarks preceding Proposition 12, they seem to be particularly relevant for toral or spherical isometries.

Finally we comment that the key to the entire analysis above was to require the joint spectrum $\sigma(M)$ of the minimal normal extension M of a subnormal tuple S to lie in a lower-dimensional surface in $\mathbf{C}^m$. One may, for example, require $\sigma(M)$ to lie in the topological boundary $\partial\mathbf{D}^m$ of the unit disk $\mathbf{D}^m$. It is left to the reader to verify using Proposition 7 in [5] that these $\partial\mathbf{D}^m$-isometries are characterized by the conditions

$$\left(\prod_{i=1}^{m}(1 - z_i w_i)^{n_i}\right)(S, S^*) \geqslant 0 \quad (0 \leqslant n_i)$$

and

$$((1 - z_1 w_1)(1 - z_2 w_2) \ldots (1 - z_m w_m))(S, S^*) = 0.$$

However, in the absence of any adequate function theory in several variables, even the study of such interesting and elementary objects as $\partial \mathbf{D}^m$-isometries is bound to prove rather frustrating.

QUESTION 2. Does every operator X in the commutant of a $\partial \mathbf{D}^m$-isometry S have a norm-preserving lift to an operator $\tilde{X}$ in the commutant of the minimal normal extenson M of S?

REMARK 5. It has been pointed out to the author by the referee that it is implicit in the work of S. W. Drury [14] that any commuting operator tuple $S = (S_1, \ldots, S_m)$ satisfying the condition $I - S_1^* S_1 - \ldots - S_m^* S_m \geqslant 0$ has a normal dilation M with $\sigma(M)$ contained in $\mathbf{S}^{2m-1}$. Thus the conclusion of Proposition 11 holds with the less restrictive assumption of $I - S_1^* S_1 - \ldots - S_m^* S_m \geqslant 0$.

Acknowledgments. The author is thankful to John B. Conway and Bernard Morrel for some useful comments.

REFERENCES

1. ATHAVALE, A., Holomorphic kernels and commuting operators, *Trans. Amer. Math. Soc.* 304(1987), 101–110.
2. ATHAVALE, A., Subnormal tuples quasisimilar to the Szegö tuple, *Michigan Math. J.*, to appear.
3. ATHAVALE, A., Model theory on the unit ball in $\mathbf{C}^m$, preprint.
4. ATHAVALE, A., On the duals of subnormal tuples, *Integral Equations Operator Theory*, to appear.
5. ATHAVALE, A.; PEDERSEN, S., Moment problems and subnormality, *J. Math. Anal. Appl.*, to appear.
6. BREHMER, S., Über vertauschbare kontraktionen des Hilbertschen Raumes, *Acta Sci. Math. (Szeged)*, 22(1961), 106–111.
7. CLARY, W. S., *Quasi-similarity and subnormal operators*, Ph. D. Thesis, University of Michigan, 1973.
8. CONWAY, J. B., On quasisimilarity for subnormal operators, *Illinois J. Math.*, 24(1980), 689–702.
9. CONWAY, J. B., The dual of a subnormal operator, *J. Operator Theory*, 5(1981), 195–211.
10. CURTO, R. E., Fredholm and invertible n-tuples of operators. The deformation problem, *Trans. Amer. Math. Soc.*, 266(1981), 129–159.

11. CURTO, R. E., Applications of several complex variables to multiparameter spectral theory, in *Surveys of some recent results in operator theory*, Vol. II., eds. J. B. Conway and B. B. Morrel, Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, UK, 1988, pp. 25–90.
12. CURTO, R. E., Private communication.
13. DOUGLAS, R. G., On the operator equation $S^*XT = X$ and related topics, *Acta Sci. Math. (Szeged)*, **30**(1969), 19–32.
14. DRURY, S. W., A generalization of von Neumann's inequality to the complex ball, *Proc. Amer. Math. Soc.*, **68**(1978), 300–304.
15. ERNST, J., *Charting the operator terrain*, Mem. Amer. Math. Soc., **171**(1976).
16. HALPERIN, I., Sz.-Nagy–Brehmer dilations, *Acta Sci. Math. (Szeged)*, **23**(1962), 279–289.
17. HASTINGS, W. W., Commuting subnormal operators simultaneously quasisimilar to unilateral shifts, *Illinois J. Math.*, **22** (1978), 506–519.
18. HOOVER, T. B., Quasi-similarity of operators, *Illinois J. Math.*, **16**(1972), 678–686.
19. ITO, T., On the commutative family of subnormal operators, *J. Fac. Sci. Hokkaido Univ.*, **14**(1958), 1–15.
20. MLAK, W., Intertwining operators, *Studia Math.*, **43**(1972), 219–233.
21. RUDIN, W., *New constructions of functions holomorphic in the unit ball of* $\mathbf{C}^n$, CBMS Regional Conference Series in Mathematics, AMS, Providence, Rhode Island, 1986.
22. RUDIN, W., *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
23. SŁOCINSKI, W., On the Wold-type decomposition of a pair of commuting isometries, *Ann. Polon. Math.*, **37**(1980), 255–262.
24. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North-Holland, Amsterdam, 1970.

AMEER ATHAVALE
Department of Mathematics,
University of Poona,
Ganeshkhind, Pune-411007,
India.

Received January 24, 1989 ; revised July 7, 1989.