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THE F. AND M. RIESZ THEOREM FOR C^*-ALGEBRAS

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INTRODUCTION

The F. and M. Riesz theorem [10] states that every analytic measure on the circle group is absolutely continuous with respect to the Haar measure. This result is among the deepest facts of classical harmonic analysis and is the basis for much of the work that has been done in this field since its publication in 1916.

A rich environment where the ideas of classical harmonic analysis have been applied is the theory of operator algebras and, in special, the theory of non-selfadjoint algebras. A paper by Arveson [1] for example contains a generalization of the inner-outer factorization theorem for analytic functions as well as Jensen's inequality to the context of certain operator algebras called sub-diagonal algebras.

In the present paper we take another step in this direction proving an extension of the F. and M. Riesz theorem to C^*-algebras containing a special kind of non-selfadjoint subalgebras which we call analytic subalgebras. Analytic subalgebras are the C^*-counterparts of Arveson's sub-diagonal algebras.

Generalizations of the F. and M. Riesz theorem appear quite often in the literature. The reader will find related results in [6] and [7]. We believe nevertheless that ours is the first such result in a non-commutative context.

Our desire to search for a generalized F. and M. Riesz theorem grew out of our interest in the theory of C^*-algebras of right ordered groups which, as much as Arveson's theory of sub-diagonal algebras, presents a wide area where ideas of Classical Harmonic Analysis can be searched for.

The precise problem that motivated the present work came up in connection with our previous work on Hankel matrices over right ordered goups [4] and is described as follows.

Let G be a discrete right ordered group and denote by $C^*_r(G)$ its reduced C^*-algebra. Let $CH_0^\infty(G)$ (resp. $H_0^\infty(G)$) be the norm closed (resp. ultra-weakly closed) algebra of operators on $\ell_2(G)$ generated by $\{\lambda(g): g > e\}$ where λ is the left regular representation of G. Note of course that $CH_0^\infty(G)$ and $H_0^\infty(G)$ are non-selfadjoint algebras.

Is it true that

$$\operatorname{dist}(a, H_0^\infty(G)) = \operatorname{dist}(a, CH_0^\infty(G))$$

for all a in $C_r^*(G)$?

Although perhaps somewhat technical this turns out to be a deep question. When G is the group of integers the answer is yes and it says that if f is a continuous complex valued function on the unit disk then its distance to the classical Hardy space H^∞ equals the distance from f to the disk algebra $A(D)$. This fact is at the same time the key ingredient of the proof of Sarason's theorem on the closedness of $H^\infty + C$ [11].

For the case of amenable groups the answer is also yes ([4], Theorem 14) and one can use it to generalize Sarason's theorem: $C_r^*(G) + H_0^\infty(G)$ is closed.

We do not know whether the above distance formula holds for an arbitrary right ordered group. Nevertheless the methods introduced in the present work apply to give quite a clear picture of the general situation. If one denotes by $D(z)$ (resp. $d(z)$) the distance of $a + z$ to $H_0^\infty(G)$ (resp. to $CH_0^\infty(G)$) where z is a complex number, we shall prove that $d = D$ except possibly on a convex open subset of the complex plane where d is constant and attains its minimum.

This paper is organized as follows. In the first part we develop the necessary generalizations of absolute continuity and singularity for states (and, more generally linear functionals) on C^*-algebras and their relationship to representation theory After this is accomplished we present a non-commutative version of the Lebesgue decomposition theorem for measures. The results in this first part are not new and were first obtained by Henle [8]. In the second part we introduce the notion of analytic subalgebras and prove our main result, the F. and M. Riesz theorem for C^*-algebras. In the third and final part we present the application discussed above.

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1. LEBESGUE DECOMPOSITION

In this first part we shall be concerned with non-commutative generalizations of some aspects of measure theory which will lead us to a theorem on decomposition of linear functionals on C^*-algebras resembling the Lebesgue decomposition theorem for measures. See also [8]. For this purpose we must first define absolute continuity and singularity for a pair of linear functionals on a given C^*-algebra. We shall do so based on the following observation: given two finite measures μ and ν on a space X, a necessary and sufficient condition for ν to be absolutely continuous with respect to μ is that integration with respect to ν gives a normal linear functional on $L^\infty(X, \mu)$.

Let A be a unital C^*-algebra. It is well known that the enveloping von Neumann algebra A'' of A is naturally isomorphic to the second dual of A and moreover that every continuous linear functional φ on A has a unique normal extension φ'' to A'' with the same norm ([9], 3.7.8).

Whenever φ is a continuous linear functional on A we let $|\varphi|$ be its absolute value ([2], 12.2.8) and $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS representation of A constructed from $|\varphi|$. Given such a φ we denote by A''_φ the weak closure of the range of π_φ. By the universal property of A'' ([9], 3.7.7) there exists exactly one normal epimorphism π''_φ from A'' to A''_φ extending π_φ.

Since π''_φ is normal, its kernel is a weakly closed ideal of A'' so there exists a unique central idempotent e_φ in A'' such that

$$\operatorname{Ker}(\pi''_\varphi) = (1 - e_\varphi)A''.$$

One may easily veiify that $\varphi''(x) = \varphi''(xe_\varphi)$ for all x in A''.

The restriction of π''_φ to $e_\varphi A''$ is then an isomorphism onto A''_φ whose inverse we denote by ρ_φ. It follows that $\pi''_\varphi\rho_\varphi$ is the identity on A''_φ while

$$\rho_\varphi\pi''_\varphi(x) = e_\varphi x$$

for all x in A''.

1. Proposition. *If φ and ψ are continuous linear functionals on A then the following are equivalent.*

i) $e_\varphi \leqslant e_\psi$;

ii) $\varphi'' = 0$ *on* $(1 - e_\psi)A''$;

iii) *There exists a normal linear functional* $\bar{\varphi}$ *on* A''_ψ *such that* $\bar{\varphi}\pi_\psi = \varphi$;

iv) π_φ *is equivalent to a subrepresentation of the direct sum of infinitely many copies of* π_ψ.

If φ is positive then (i) *through* (iv) *are equivalent to*

v) $\varphi''(1 - e_\psi) = 0$.

Proof. To prove that (i) implies (ii) let x be in A''. Then

$$\varphi''((1 - e_\psi)x) = \varphi''((1 - e_\psi)xe_\varphi) = \varphi''(0) = 0.$$

Next assume (ii) and define $\bar{\varphi}$ on A''_ψ by $\bar{\varphi} = \varphi''\rho_\psi$. It is clear that $\bar{\varphi}$ is normal. We have for all a in A

$$\bar{\varphi}\pi_\psi(a) = \varphi''\rho_\psi\pi_\psi(a) = \varphi''\rho_\psi\pi''_\psi(a) = \varphi''(ae_\psi) = \varphi''(a) = \varphi(a).$$

So (iii) follows.

If φ is as in (iii) then there exists a square summable sequence (ξ_n) of elements in H_ψ such that

$$|\varphi|(x) = \sum_{n=1}^{\infty} \langle x\xi_n, \xi_n\rangle \quad \forall x \in A''_\psi .$$

The vector $\xi = (\xi_n)$ is then in the space H_ψ^∞, the direct sum of infinitely many copies of H_ψ, and we have for all a in A

$$|\varphi|(a) = |\varphi|(\pi_\psi(a)) = \sum_{n=1}^{\infty} \langle \pi_\psi(a)\xi_n, \xi_n\rangle = \langle \pi_\psi^\infty(a)\xi, \xi\rangle$$

where π_ψ^∞ is the direct sum of infinitely many copies of π_ψ. Therefore π_φ is equivalent to a subrepresentation of π_ψ^∞.

To prove that (iv) implies (i) let V be the subspace of H_ψ^∞ corresponding to the subrepresentation π_φ. Then

$$\pi_\varphi''(1 - e_\psi) = \pi_\psi^{\infty\,\prime\prime}(1 - e_\psi)|V = 0;$$

therefore $1 - e_\psi$ is in $\mathrm{Ker}(\pi_\varphi'')$ so $1 - e_\psi \leqslant 1 - e_\varphi$ proving $e_\varphi \leqslant e_\psi$.

Now assume that φ is positive. Given that $\varphi''(1 - e_\psi) = 0$ we have for all x in A''

$$|\varphi''((1 - e_\psi)x)| \leqslant \varphi''(1 - e_\psi)^{1/2}\varphi''(x^*x)^{1/2} = 0$$

proving that (v) implies (ii). The converse is clear. ∎

2. Definition. If the equivalent conditions of Proposition 1 are satisfied we say that φ is *absolutely continuous with respect to* ψ and write $\varphi \ll \psi$.

Note that by (iii) the set of all φ's which are absolutely continuous with respect to ψ is in correspondence with the predual of A''_ψ, a fact that is somewhat related to the Radon-Nikodym theorem. A deeper relation will be provided by Theorem 4 below. Before that we need the following result (compare [5], p. 219).

3. Lemma. *Let W be a von Neumann algebra of operators on a Hilbert space H. Suppose there is a vector ξ in H whose associated vector state is a faithful trace on W. Then*

a) *every normal state on W is a vector state,*

b) *every normal linear functional on W is of the form $\varphi(x) = \langle x(\zeta), \eta\rangle$ where ζ and η are vectors in H and*

c) *the weak and σ-weak topologies coincide on W.*

Proof. It is clear that (a) implies (b) (by polar decomposition [2], 12.2.4) and (c) so it is enough to prove (a).

Let τ be defined for all a in W by $\tau(a) = \langle a\xi, \xi\rangle$.

Given a normal state φ on W we have that for all $x \geqslant 0$ in W

$$\tau(x) = 0 \Rightarrow \varphi(x) = 0$$

since τ is faithful by hypothesis. Therefore by the (non-commutative) Radon-Nikodym theorem (see [9], 5.3.11 and 5.3.12, [3] and [12]) there exists a (possibly unbounded) positive operator h affiliated with W such that

$$\varphi(x) = \tau(hx)$$

for all $x \geqslant 0$ in W. The meaning of $\tau(hx)$ above should perhaps be better explained. Let p_n be the spectral projection of h corresponding to the interval $]n-1, n]$. Then $K_r = h^{1/2}p_n$ is in W_+. For all x in W_+ we define

$$\tau(hx) = \sum_{n=1}^{\infty} \tau(k_n x k_n).$$

Observe that

$$\varphi(1) = \tau(h1) = \sum_{n=1}^{\infty} \tau(k_n^2) = \sum_{n=1}^{\infty} \|k_n(\xi)\|^2 ;$$

therefore, since the vectors $k_n(\xi)$ are mutually orthogonal, the series $\sum_{n=1}^{\infty} k_n(\xi)$ is summable. Let η be its sum. We then have for all $x \geqslant 0$ in W

$$\langle x(\eta), \eta\rangle = \sum_n \sum_m \langle xk_n(\xi), k_m(\xi)\rangle = \sum_n \sum_m \tau(k_m x k_n) =$$

$$= \sum_n \tau(k_n x k_n) = \tau(hx) = \varphi(x).$$

▨

Our next result describes the linear functionals which are absolutely continuous with respect to a trace.

4. THEOREM. *Let τ be a positive trace on A. Then for every continuous linear functional φ on A the following are equivalent:*

i) *φ is absolutely continuous with respect to τ;*

ii) *π_φ is equivalent to a subrepresentation on π_τ;*

iii) *there is a vector ζ in H_τ such that for all a in A*

$$|\varphi|(a) = \langle \pi_\tau(a)\zeta, \zeta\rangle.$$

In this case there exists another vector η in H_τ such that

$$\varphi(a) = \langle \pi_\tau(a)\zeta, \eta \rangle$$

for all a in A.

Proof. Clearly (iii) implies (ii) and (ii) implies (i) so it is enough to prove that (i) implies (iii). Define $\bar{\tau}$ on A_τ'' by

$$\bar{\tau}(x) = \langle x(\xi_\tau), \xi_\tau \rangle.$$

Then it is clear that $\bar{\tau}$ is a normal trace on A_τ''. Observe that $\bar{\tau}$ is faithful because if x is in A_τ'' and $\bar{\tau}(x^*x) = 0$ then for all a and b in A

$$|\langle x\pi_\tau(a)\xi_\tau, \pi_\tau(b)\xi_\tau\rangle| = |\bar{\tau}(\pi_\tau(b)^* x\pi_\tau(a))| = |\bar{\tau}(\pi_\tau(a)\pi_\tau(b)^* x)| \leqslant$$

$$\leqslant \bar{\tau}(x^* x)^{1/2}\tau(\pi_\tau(a)\pi_\tau(b)^*\pi_\tau(b)\pi_\tau(a)^*)^{1/2} = 0$$

whence $x = 0$. Let φ be as in (1.iii). If we now use Lemma 3 we are able to find ζ in H_τ such that

$$|\varphi|(a) = |\varphi|(\pi_\tau(a)) = \langle \pi_\tau(a)\zeta, \zeta \rangle.$$

The last part follows from the polar decomposition applied to φ''. ∎

We now study the notion of mutual singularity of linear functionals.

5. Proposition. *Let φ and ψ be continuous linear functionals on A. Then the following are equivalent:*

i) $e_\varphi e_\psi = 0$;

ii) $\varphi'' = 0$ *on* $e_\psi A''$;

iii) $\psi'' = 0$ *on* $e_\varphi A''$;

iv) π_φ *and* π_ψ *are disjoint representations of* A *(cf.* [2], 5.2.2.*).*

If φ (resp. ψ) is positive then the conditions above are again equivalent to

v) $\varphi''(e_\psi) = 0$ *(resp.* $\psi''(e_\varphi) = 0$*).*

Proof. For all x in A''

$$\varphi''(e_\psi x) = \varphi''(e_\psi x e_\varphi)$$

so (i) implies (ii). To prove the converse of this implication let $\varphi'' = u|\varphi''|$ be the polar decomposition of φ''. Then for all x and y in A''

$$\langle \pi_\varphi''(e_\psi)\pi_\varphi''(x)\xi_\varphi, \pi_\varphi''(y)\xi_\varphi\rangle = |\varphi''|(y^* e_\psi x) = \varphi''(u^* y^* e_\psi x) = \varphi''(e_\psi u^* y^* x) = 0.$$

So $\pi_\varphi''(e_\psi) = 0$ hence $e_\psi \leqslant 1 - e_\varphi$ proving (i).

We next prove that (i) implies (iv). For this consider the representation $\pi = \pi_\varphi \oplus \pi_\psi$ of A. Let f_φ and f_ψ be the projections in the commutator $\pi(A)'$ of $\pi(A)$ corresponding to π_φ and π_ψ respectively.

According to ([2], 5.2.1) we must prove that the central supports of f_φ and f_ψ in $\pi(A)'$ are mutually orthogonal.

Observe that

$$\pi''(e_\varphi) = \pi''_\varphi(e_\varphi) \oplus \pi''_\psi(e_\varphi) = 1 \oplus 0 = f_\varphi$$

and similarly that $\pi''(e_\psi) = f_\psi$. Therefore both f_φ and f_ψ belong to the center of $\pi(A)'$. This says that f_φ and f_ψ are respectively identical to their central supports which are then mutually orthogonal.

The proof that (iv) implies (i) goes as follows. Given that π_φ and π_ψ are disjoint, if we let $\pi = \pi_\varphi \oplus \pi_\psi$ with corresponding projections f_φ and f_ψ, we have that the central supports of f_φ and f_ψ are orthogonal. But since $f_\varphi + f_\psi = 1$ it follows that f_φ and f_ψ coincide, respectively, with their central supports hence f_φ and f_ψ belong to $\pi(A)'' = \pi''(A'')$. We may therefore find an orthogonal pair of projections E_φ and E_ψ in A'' which are mapped by π'' to f_φ and f_ψ respectively (note that lifting of mutually orthogonal projections through an epimorphism of von Neumann algebras is always possible). In other words we have

$$\pi''_\varphi(E_\varphi) = 1, \quad \pi''_\varphi(E_\psi) = 0,$$

$$\pi''_\psi(E_\varphi) = 0, \quad \pi''_\psi(E_\psi) = 1.$$

Therefore $\pi''_\varphi(1 - E_\varphi) = 0$ so $1 - E_\varphi \in (1 - e_\varphi)A''$ hence $e_\varphi \leqslant E_\varphi$.

Sjmilarly $e_\psi \leqslant E_\psi$. Therefore e_φ and e_ψ are mutually orthogonal.

The remaining implications are of easy verification and are left to the reader. ▨

6. Definition. If the conditions above are satisfied we say that φ and ψ are *mutually singular* and write $\varphi \perp \psi$.

Our next result is a non-commutative analogue of the Lebesgue decomposition theorem for measures. It was first obtained by Henle in [8].

7. Proposition. *Let φ and ψ be continuous linear functionals on A. Then φ can be uniquely decomposed as a sum*

$$\varphi = \varphi_\alpha + \varphi_\sigma$$

where $\varphi_\alpha \ll \psi$ and $\varphi_\sigma \perp \psi$.

Moreover if $\varphi'' = u|\varphi''|$, $\varphi''_\alpha = u_\alpha|\varphi''_\alpha|$ *and* $\varphi''_\sigma = u_\sigma|\varphi''_\sigma|$ *are the corresponding polar decompositions we have*

i) $|\varphi| = |\varphi_\alpha| + |\varphi_\sigma|$, $u_\alpha = ue_\psi$, $u_\sigma = u(1 - e_\psi)$,

ii) $\|\varphi\| = \|\varphi_\alpha\| + \|\varphi_\sigma\|$,

iii) $|\varphi_\alpha| = |\varphi|_\alpha$,

iv) $|\varphi_\sigma| = |\varphi|_\sigma$.

Proof. For all a in A define $\varphi_\alpha(a) = \varphi''(ae_\psi)$ and $\varphi_\sigma(a) = \varphi''(a(1 - e_\psi))$.

Clearly $\varphi = \varphi_\alpha + \varphi_\sigma$. It is also clear that φ''_α vanishes on $(1 - e_\psi)A''$ and that φ''_σ vanishes on $e_\psi A''$ so $\varphi_\alpha \ll \psi$ and $\varphi_\sigma \perp \psi$. If $\varphi = \varphi_1 + \varphi_2$ is another such decomposition then for all a in A

$$\varphi_1(a) = \varphi''_1(ae_\psi) = \varphi''_1(ae_\psi) + \varphi''_2(ae_\psi) = \varphi''(ae_\psi) = \varphi_\alpha(a).$$

Thus $\varphi_1 = \varphi_\alpha$ and consequently $\varphi_2 = \varphi_\sigma$ proving the uniqueness of the decomposition.

Fact (i) follows from ([2], 12.2.4) and clearly (ii) follows from (i). Finally (iii) and (iv) are consequences of the uniqueness of the decomposition together with (i). ▨

8. Proposition. *If* φ, φ_1, φ_2, ψ *and* χ *denote continuous linear functionals on* A *we have:*

a) *If* φ_1 *and* φ_2 *are absolutely continuous (resp. singular) with respect to* ψ *then so is any linear combination of* φ_1 *and* φ_2.

b) *If* $\varphi \ll \psi$ *and* $\psi \perp \chi$ *then* $\varphi \perp \chi$.

c) *If* $0 \leq \varphi \leq \psi$ *then* $\varphi \ll \psi$.

Proof. Follows from Propositions 1 and 5. ▨

Now consider a pair of mutually singular states φ_1 and φ_2 and put $\psi = \varphi_1 + \varphi_2$. Let (π, H, ξ) be the GNS representation of A associated to ψ. Because each $\varphi_i \leq \psi$ there are positive operators P_i in $\pi(A)'$ such that

$$\varphi_i(a) = \langle P_i\pi(a)\xi, \xi\rangle$$

for all a in A.

9. Lemma. $P_i = \pi''(e_{\varphi_i})$ *for* $i = 1, 2$. *Hence* P_i *is a central projection in* $\pi(A)'' = A''_\psi$. *If* $j = 3 - i$ *and* $\bar\varphi_j$ *is given by* (1.iii) *then* $\bar\varphi_j(P_i) = 0$.

Proof. Let $Q_i = \pi''(e_{\varphi_i})$. Then Q_i is a central projection in $\pi(A'')$ and for all a in A

$$\langle Q_i\pi(a)\xi, \xi\rangle = \langle\pi''(e_{\varphi_i}a)\xi, \xi\rangle = \varphi''(e_{\varphi_i}a) = \varphi''_i(a) = \varphi_i(a).$$

From the uniqueness of P_i as defined above it follows that $Q_i = P_i$. Note that $\overline{\varphi}_j$ is given by

$$\overline{\varphi}_j(x) = \langle P_j x(\xi), \xi \rangle$$

for all x in A''_ψ hence

$$\overline{\varphi}_j(P_i) = \langle \pi''(e_{\varphi_j})\pi''(e_{\varphi_i})\xi, \xi \rangle = 0$$

because $e_{\varphi_i} e_{\varphi_j} = 0$. ▨

10. Lemma. *Let τ be a positive trace on A. Suppose φ is a state on A which is absolutely continuous with respect to τ. If there exists a vector ζ in H_φ such that*

$$\langle \pi_\varphi(a)\zeta, \zeta \rangle = \tau(a)$$

for all a in A then ζ is a cyclic vector for π_φ.

Proof. Let V_1 be the cyclic subspace of H_φ generated by ζ. It is then clear that $\pi_\varphi | V_1$ is equivalent to π_τ.

Since φ is absolutely continuous with respect to τ it follows by Theorem 4 that π_φ is equivalent to a subrepresentation of π_τ. So H_τ contains a subspace V_2 which is covariantly isomorphic to H_φ. If we identify H_φ and V_2 we may write $\pi_\tau | V_1 \approx \pi_\tau$. So there exists an isometry u from H_τ to V_1 lying in the commutator of π_τ. But this commutator is anti-isomorphic to A''_τ by (a very special case of) the Tomita-Takesaki theory hence it is a finite von Neumann algebra. Therefore u must be a unitary operator. This shows that V_1 is equal to H_τ. In particular we have $V_1 = H_\varphi$ which is what we wanted to prove. ▨

2. THE F. and M. RIESZ THEOREM

Let A be a unital C^*-algebra equipped with a positive normalized trace τ. A subalgebra B of A is called *analytic* if B contains the unit of A, $B + B^*$ is dense in A and the restriction of τ to B is multiplicative.

Although in some pathological examples B may be a selfadjoint subalgebra of A we are mostly interested in the case where B is not. The basic example of this situation (which the reader should keep in the back of his mind) is the following: A is the algebra of continuous functions on the unit circle, τ is the trace corresponding to the Haar measure on the circle and B is the disc algebra, that is, the subalgebra of A consisting of all functions that admit an analytic extension to the unit disc. Since, in this case, $B + B^*$ contains the trigonometric polynomials it is clear that $B + B^*$ is dense in A. Using the Cauchy integral formula one sees that the trace of an element of B equals the value of its analytic extension at the origin from which one can easily verify that τ is multiplicative on B. It is through this example that our main results relate to the classical F. and M. Riesz theorem

From now on we fix a unital C^*-algebra A where a positive normalized trace τ is defined and let B be a fixed analytic subalgebra of A.

Since $\tau|B$ is multipicative, its kernel, which we denote by B_0, is an ideal of B hence a subalgebra of A. Clearly $B = B_0 + C1$.

Let (π, H, ξ) denote the GNS representation of A associated with τ and let

$$H^+ = \overline{\pi(B)\xi}$$

$$H^- = \overline{\pi(B_0^*)\xi}.$$

Observe that if $x \in B$ and $y \in B_0$ then

$$\langle \pi(x)\xi, \pi(y^*)\xi \rangle = \tau(yx) = \tau(y)\tau(x) = 0.$$

So H^+ and H^- are orthogonal subspaces of H. Since $B + B_0^* = B + B^*$ is dense in A it follows that $H = H^+ \oplus H^-$.

11. DEFINITION. A continuous linear functional φ on A is called *analytic* if φ vanishes on B_0.

The following is the main result of this paper. Compare [6] as well as [7 for similar results on (commutative) function algebras.

12. THEOREM. *Let φ be a continuous linear functional on A and let $\varphi = \varphi_\alpha + \varphi_\sigma$ be the Lebesgue decomposition of φ with respect to τ. If φ is analytic then so are φ_α and φ_σ. Moreover $\varphi_\sigma(1) = 0$.*

Proof. Let $\psi = \tau + |\varphi_\alpha| + |\varphi_\sigma| = \tau + |\varphi|$ and form the GNS representation (π, H, ξ) associated with ψ. Let V be the linear subspace of H given by

$$V = \overline{\pi(B_0)\xi}$$

It is clear that V is invariant under B_0. By elementary Hilbert space techniques here exists a unique vector η in V which is closest to ξ. The vector $\xi - \eta$ is then orthogonal to V.

Claim 1. $\|\xi - \eta\| \geqslant 1$.

Since $\tau \leqslant \psi$ there exists a unique operator P in the commutator of π such that $0 \leqslant P \leqslant 1$ and

$$\tau(a) = \langle P\pi(a)\xi, \xi \rangle$$

for all a in A.

For every vector ζ of the form $\zeta = \pi(b)\xi$ where b is in B_0 we have

$$\|\xi - \zeta\|^2 \geqslant \|P^{1/2}(\xi - \zeta)\|^2 = \langle P(\xi - \zeta), \xi - \zeta\rangle = \langle P\pi(1 - b)\xi, \pi(1 - b)\xi\rangle =$$

$$= \tau((1 - b)^*(1 - b)) = \tau(1 - b - b^* + b^*b) = 1 + \tau(b^*b) \geqslant 1$$

since $\tau(b) = \tau(b^*) = 0$.

The set of all such ζ's is dense in V so $\|\xi - \zeta\| \geqslant 1$ for all ζ in V. This proves our claim.

Let $c = \|\xi - \eta\|$.

Claim 2. $\langle\pi(a)(\xi - \eta), \xi - \eta\rangle = c^2\tau(a) \quad \forall a \in A$.

Since $\xi - \eta$ is orthogonal to V and since for all b in B_0 we have that $\pi(b)(\xi - \eta)$ is in V it follows that

$$\langle\pi(b)(\xi - \eta), \xi - \eta\rangle = 0 = c^2\tau(b)$$

for all $b \in B_0$. By taking adjoints we conclude that the above equality holds also for all $b \in B_0^*$. On the other hand

$$\langle\pi(1)(\xi - \eta), \xi - \eta\rangle = c^2 = c^2\tau(1).$$

This proves claim (2) since A is as the closure of $B_0 + B_0^* + C1$.

Since $|\varphi_\sigma| \leqslant \psi$ there exists a positive operator S commuting with the range of π such that

$$|\varphi_\sigma|(a) = \langle S\pi(a)\xi, \xi\rangle$$

for all a in A.

Claim 3. $S(\xi - \eta) = 0$.

By (8.a & b) $\tau + |\varphi_\alpha|$ is singular with respect to $|\varphi_\sigma|$. Since ψ is the sum of $\tau + |\varphi_\alpha|$ and $|\varphi_\sigma|$ we may use Lemma 9 to conclude that S is a central projection in $\pi(A)''$ and that $\bar{\tau}(S) = 0$ (see (1.iii) for a definition of $\bar{\tau}$).

It is clear that $\bar{\tau}$ is given for every x in $\pi(A)''$ by

$$\bar{\tau}(x) = c^{-2}\langle x(\xi - \eta), \xi - \eta\rangle.$$

So

$$0 = \bar{\tau}(S) = c^{-2}\langle S(\xi - \eta), \xi - \eta\rangle = c^{-2}\|S(\xi - \eta)\|^2$$

proving claim (3).

Let Q be the unique positive operator in the commutator of π such that

$$|\varphi_\alpha|(a) = \langle Q\pi(a)\xi, \xi\rangle$$

for all a in A. Also let $\varphi'' = u|\varphi''|$ be the polar decomposition of φ''.

Recall that π'' denotes the unique normal extension of π to A''.

Claim 4. $\langle Q\pi(b)(\xi - \eta), \pi''(u^*)\xi\rangle = 0$ for all b in B_0.

Write $\eta = \lim \pi(b_n)\xi$ where (b_n) is a sequence in B_0. Then for all b in B_0 we have

$$0 = \lim \varphi(b(1 - b_n)) = \lim |\varphi''|(ub(1 - b_n)) = \lim(|\varphi''_a| + |\varphi''_\sigma|)(ub(1 - b_n)) =$$

$$= \lim\langle Q\pi''(ub(1 - b_n))\xi, \xi\rangle + \lim\langle S\pi''(ub(1 - b_n))\xi, \xi\rangle =$$

$$= \langle Q\pi(b)(\xi - \eta), \pi''(u^*)\xi\rangle + \langle S\pi(b)(\xi - \eta), \pi''(u^*)\xi\rangle =$$

$$= \langle Q\pi(b)(\xi - \eta), \pi''(u^*)\xi\rangle.$$

This completes the proof of claim (4).

It is clear that $P + Q + S = 1$ and we know by Lemma 9 that both S and $P + Q$ are central projections in $\pi(A)''$. Also from Lemma 9 it follows that $P + Q = \pi''(e_\tau)$. This gives a decomposition of π in the direct sum of two cyclic subrepresentations π_1 and π_2 corresponding, respectively, to $P + Q$ and S. Let $H_1 = (P + Q)H$ so H_1 is the space of π_1.

A cyclic vector for π_1 is clearly $\xi_1 = (P + Q)\xi$. Observe that $S(\xi - \eta) = 0$ implies that $\xi - \eta$ is in H_1.

Claim 5. There exists a unitary operator U on H_1, commuting with π_1, such that

$$UP^{1/2}\xi_1 = c^{-1}(\xi - \eta).$$

The vector state associated to ξ_1 is clearly $\tau + |\varphi_a|$ which is absolutely continuous with respect to τ.

Since $\langle \pi_1(a)(\xi - \eta), \xi - \eta\rangle = c^2\tau(a)$ for all a in A we may apply Lemma 10 to conclude that $c^{-1}(\xi - \eta)$ is cyclic for π_1. This implies in particular that π_1 is equivalent to π_τ.

Another application of Lemma 10 proves that $P^{1/2}\xi_1$ is also a cyclic vector for π_1 since

$$\langle \pi_1(a)P^{1/2}\xi_1, P^{1/2}\xi_1\rangle = \tau(a)$$

for all a in A.

Put together, these last two facts imply that the cyclic representations $(\pi_1, c^{-1}(\xi - \eta))$ and $(\pi_1, P^{1/2}\xi_1)$ are equivalent in the sense that the equivalence preserves the prescribed cyclic vectors. This proves claim (5).

Claim 6. ξ_1 is in the closure of $UP^{1/2}(V + \mathbf{C}\xi)$.

The key fact to prove this last claim is that $(\pi_1, P^{1/2}\xi_1)$ is equivalent to (π_τ, ξ_τ) as cyclic representations. This of course follows from claim (5).

Let H_1^+ and H_1^- be given by

$$H_1^+ = \overline{\pi_1(B)P^{1/2}\xi_1} = \overline{\pi_1(B_0 + C1)P^{1/2}\xi_1}$$

$$H_1^- = \overline{\pi_1(B_0^*)P^{1/2}\xi_1}.$$

From the observation above we may conclude that H_1 is the orthogonal direct sum of H_1^+ and H_1^-.

For all b in B_0 we have

$$\langle U^*\xi_1, \pi_1(b^*)P^{1/2}\xi_1\rangle = \langle \pi_1(b)\xi_1, UP^{1/2}\xi_1\rangle =$$

$$= \langle \pi(b)\xi, UP^{1/2}\xi_1\rangle = c^{-1}\langle \pi(b)\xi, \xi - \eta\rangle = 0$$

since $\pi(b)\xi$ is in V and $\xi - \eta$ is orthogonal to V. We then conclude that $U^*\xi_1$ is orthogonal to H_1^- so $U^*\xi_1$ is in H_1^+. We may then write

$$U^*\xi_1 = \lim \pi_1(\lambda_n + b_n)P^{1/2}\xi_1$$

where the λ_n's are complex numbers and the b_n's are in B_0. So

$$\xi_1 = \lim UP^{1/2}\pi_1(\lambda_n + b_n)\xi_1 = \lim UP^{1/2}\pi(\lambda_n + b_n))\xi$$

proving claim (6).

Claim 7. φ_α and φ_σ are analytic.

Let $\varphi_\alpha'' = u_\alpha|\varphi_\alpha''|$ be the polar decomposition of φ_α''. We know from Proposition 7 that $u_\alpha = ue_\tau$. Write

$$\xi_1 = \lim UP^{1/2}\pi(\lambda_n + b_n)\xi$$

as above. For all b in B_0 we have

$$\varphi_\alpha(b) = \varphi_\alpha''(b) = |\varphi_\alpha''|(u_\alpha b) = \langle Q\pi''(u_\alpha)\pi(b)\xi, \xi\rangle = \langle Q\pi(b)\xi_1, \pi''(u_\alpha^*)\xi\rangle =$$

$$= \lim\langle Q\pi(b)UP^{1/2}\pi(\lambda_n + b_n)\xi, \pi''(u_\alpha^*)\xi\rangle =$$

$$= c^{-1}\lim\langle Q\pi(b(\lambda_n + b_n))(\xi - \eta), \pi''(u_\alpha^*)\xi\rangle =$$

$$= c^{-1}\lim\langle Q\pi(b(\lambda_n + b_n))(\xi - \eta), \pi''(e_\tau u^*)\xi\rangle =$$

$$= c^{-1}\lim\langle Q\pi''(e_\tau)\pi(b(\lambda_n + b_n))(\xi - \eta), \pi''(u^*)\xi\rangle.$$

Recall that $\pi''(e_\tau) = P + Q$ so $Q\pi''(e_\tau) = Q$. The above then equals

$$c^{-1}\lim\langle Q\pi(b(\lambda_n + b_n))(\xi - \eta),\ \pi''(u^*)\xi\rangle$$

which is zero by claim (4).

This proves that φ_α is analytic and therefore that φ_σ is analytic too.

Claim 8. $\varphi_\sigma(1) = 0$.

Let $\varphi''_\sigma = u_\sigma|\varphi''_\sigma|$ be the polar decomposition of φ''_σ. Write $\eta = \lim \pi(b_n)\xi$ with b_n in B_0 and note that by claim (7)

$$0 = \lim \varphi_\sigma(b_n) = \lim \varphi''_\sigma(b_n) = \lim |\varphi''_\sigma|(u_\sigma b_n) = \lim\langle S\pi''(u_\sigma b_n)\xi,\ \xi\rangle =$$

$$= \lim\langle S\pi''(u_\sigma)\pi(b_n)\xi,\ \xi\rangle = \langle S\pi''(u_\sigma)\eta,\ \xi\rangle.$$

By claim (3) we have $S(\eta) = S(\xi)$ so

$$0 = \langle S\pi''(u_\sigma)\xi,\ \xi\rangle = |\varphi''_\sigma|(u_\sigma) = \varphi''_\sigma(1) = \varphi_\sigma(1).$$

∎

A group G is said to be right ordered if G is equipped with a linear order which is invariant under multiplication on the right by elements of G.

Given a discrete group G we let $C^*_{(r)}(G)$ be either the reduced or the full C^*-algebra of G. The canonical trace on $C^*_{(r)}(G)$ will be denoted by τ.

To each group element g there corresponds, in a canonical way, a unitary element U_g in $C^*_{(r)}(G)$ in such a way that the map $g \in G \mapsto U_g \in C^*_{(r)}(G)$ is a representation of G. For further references to the theory of group C^*-algebras the reader should consult [9].

13. Corollary. *Let G be a discrete right ordered group and let φ be a continuous linear functional on $C^*_{(r)}(G)$. Write the Lebesgue decomposition of φ with respect to τ as $\varphi = \varphi_\alpha + \varphi_\sigma$. If $\varphi(U_g) = 0$ for all $g > e$ (e denoting the identity element of G) then*

i) $\varphi_\alpha(U_g) = \varphi_\sigma(U_g) = 0$ for all $g > e$ and

ii) $\varphi_\sigma(1) = 0$.

Proof. Let B be the (non-selfadjoint) subalgebra of $C^*_{(r)}(G)$ generated by $\{U_g : g \geqslant e\}$ and apply Theorem 12. ∎

Concluding this section we should mention an example to show that the classical version of the F. and M. Riesz theorem does not apply in full generality (that is, one cannot conclude that φ is absolutely continuous in the theorem above). Consider the direct sum of two copies of the group of all integers with lexicographic order. The C^*-algebra of this group is the algebra of continuous functions on the 2-torus. On this algebra consider the linear functional φ given by

$$\varphi(f) = \int f(z, 1)z\, dz$$

where z denotes the first torus variable and dz is the Haar measure on the circle.

The reader may easily verify that φ is analytic and singular thus contradicting what should be expected from the classical F. and M. Riesz theorem applied to this case.

3. A DISTANCE FORMULA

In this final section we shall present an application of Theorem 12 above. Fix throughout a C^*-algebra A, positive normalized trace τ on A and an analytic subalgebra B of A. Denote by B_0 the kernel of $\tau|B$ and by π the GNS representation of A associated with τ.

Let $\mathscr{A}$ be the von Neumann algebra generated by the range of π. Also let $\mathscr{B}_0$ be the ultra-weak closure of $\pi(B_0)$. For a fixed a in A let for every complex number z

$$d(z) = \text{dist}(a + z, B_0)$$

and

$$D(z) = \text{dist}(\pi(a) + z, \mathscr{B}_0).$$

We propose to prove the following

14. THEOREM. *There exist a (possibly empty) convex open subset Ω of the complex plane such that:*

i) $\Omega \subset \{z \in \mathbf{C} : |z| \leqslant 2\|a\|\}$,

ii) *for* $z \in \Omega$, $D(z) < d(z) = \inf\{d(w) : w \in \mathbf{C}\}$,

iii) *for* $z \notin \Omega$, $D(z) = d(z)$.

The proof will be presented in a number of steps. Initially observe that for b in B_0 and a in A one has

$$\text{dist}(\pi(a), \mathscr{B}_0) \leqslant \text{dist}(\pi(a), \pi(B_0)) \leqslant \|\pi(a) - \pi(b)\| \leqslant \|a - b\|$$

so it is clear that $\text{dist}(\pi(a), \mathscr{B}_0) \leqslant \text{dist}(a, B_0)$.

A standard use of the Hahn-Banach extension theorem provides a continuous inear functional φ on A of norm one which vanishes on B_0 and such that $\varphi(a) = \text{dist}(a, B_0)$. Let $\varphi = \varphi_\alpha + \varphi_\sigma$ be its Lebesgue decomposition with respect to τ as in Proposition 7. Applying Theorem 12 we conclude that φ_α and φ_σ both vanish on B_0 and moreover that $\varphi_\sigma(1) = 0$.

15. LEMMA. $|\varphi_\alpha(a)| = \|\varphi_\sigma\|\mathrm{dist}(a, B_0)$.

Proof. Let ε be positive and choose b in B_0 such that $\|a - b\| \leqslant \mathrm{dist}(a, B_0) + \varepsilon$. We have

$$\mathrm{dist}(a, B_0) = |\varphi(a)| \leqslant |\varphi_\alpha(a)| + |\varphi_\sigma(a)| = |\varphi_\alpha(a - b)| + |\varphi_\sigma(a - b)| \leqslant$$

$$\leqslant \|\varphi_\alpha\| \|a - b\| + \|\varphi_\sigma\| \|a - b\| \leqslant$$

$$\leqslant \|\varphi_\alpha\| (\mathrm{dist}(a, B_0) + \varepsilon) + \|\varphi_\sigma\| (\mathrm{dist}(a, B_0) + \varepsilon) = \mathrm{dist}(a, B_0) + \varepsilon.$$

It follows that $\|\varphi_\alpha\| \mathrm{dist}(a, B_0) \leqslant |\varphi_\alpha(a)|$. Since the converse inequality is obviously true, the lemma is proved. ∎

16. LEMMA. *If* $\varphi_\alpha \neq 0$ *then* $\mathrm{dist}(a, B_0) = \mathrm{dist}(\pi(a), \mathscr{B}_0)$.

Proof. Given that $\varphi_\alpha \neq 0$ let $\psi = \|\varphi_\alpha\|^{-1}\varphi_\alpha$ so that ψ is absolutely continuous with respect to τ, has norm one, vanishes on B_0 and satisfies $|\psi(a)| = \mathrm{dist}(a, B_0)$. In other words, we may have taken φ to be absolutely continuous at the start. Therefore by Proposition 1 there exists a normal linear functional $\bar{\psi}$ on $\mathscr{A}$ with norm one such that $\bar{\psi}\pi = \psi$. It follows that $\bar{\psi}(\mathscr{B}_0) = 0$ so for all x in $\mathscr{B}_0$ we have

$$\|\pi(a) - x\| \geqslant |\bar{\psi}(\pi(a) - x)| = |\bar{\psi}(\pi(a))| = |\psi(a)| = \mathrm{dist}(a, B_0)$$

whence $\mathrm{dist}(\pi(a), \mathscr{B}_0) \geqslant \mathrm{dist}(a, B_0)$ concluding the proof. ∎

Denote for every complex number z, $a_z = a + z$.

17. LEMMA. *If* $|z| > 2\|a\|$ *then*

$$\mathrm{dist}(a_z, B_0) = \mathrm{dist}(\pi(a_z), \mathscr{B}_0).$$

Proof. Let φ be a norm one continuous linear functional on A, vanishing on B_0, such that $|\varphi(a_z)| = \mathrm{dist}(a_z, B_0)$. In view of the last lemma it clearly suffices to show that $\varphi_\alpha \neq 0$. Suppose this is not the case so that $\|\varphi_\sigma\| = 1$ and $|\varphi_\sigma(a_z)| = \mathrm{dist}(a_z, B_0)$. Recall that by Theorem 12 we have $\varphi_\sigma(1) = 0$ so

$$\mathrm{dist}(a_z, B_0) = |\varphi_\sigma(a_z)| = |\varphi_\sigma(a)| \leqslant \|a\|.$$

On the other hand

$$\mathrm{dist}(a_z, B_0) = \mathrm{dist}(a + z, B_0) \geqslant \mathrm{dist}(z, B_0) - \mathrm{dist}(a, B_0) \geqslant |z| - \|a\|.$$

Comparing the last two inequalities we conclude that $|z| \leqslant 2\|a\|$ contradicting the hypothesis. ∎

Observe that we have just proven that for a sufficiently large z one has $d(z) = D(z)$. Otherwise we have the following

18. LEMMA. *Suppose $d(z) \neq D(z)$. Then d attains its minimum at z.*

Proof. Choose φ as in the proof above. By Lemma 16 we must have $\varphi_\alpha = 0$. Observe that for every complex number μ and for every b in B_0

$$d(z) = \operatorname{dist}(a + z, B_0) = |\varphi_\sigma(a + z)| = |\varphi_\sigma(a + \mu - b)| \leqslant \|a + \mu - b\|.$$

Taking the infimum for b in B_0 we get $d(z) \leqslant \operatorname{dist}(a + \mu, B_0) = d(\mu)$. ▨

Collecting our previous results we may now prove Theorem 14.

Proof of Theorem 14. Let $k = \inf\{d(w) : w \in \mathbf{C}\}$ and $\Omega = \{w \in \mathbf{C} : D(w) < k\}$. It is clear that Ω is open and convex (even if it is empty). For z in Ω we have $D(z) < k \leqslant d(z)$ so Lemma 18 gives $d(z) = k$ and Lemma 17 gives $|z| \leqslant 2\|a\|$.

For z not in Ω we must have $d(z) = D(z)$ since otherwise Lemma 18 would imply that $D(z) < d(z) = k$ which would say that $z \in \Omega$. ▨

Unfortunately we have not been able to find an example to show that Ω may be non-empty. One could therefore conjecture that Ω is always empty, a fact which can be rephased as $\operatorname{dist}(a, B_0) = \operatorname{dist}(\pi(a), \mathscr{B}_0)$ for every a in A. This would certainly be a much nicer result since it would imply Sarason's theorem as mentioned in the ntroduction.

Concluding let us study the case of right ordered groups. If G is such a group et λ be its left regular representation on $\ell_2(G)$. Denote by $CH_0^\infty(G)$ (resp. $H_0^\infty(G)$) the norm closed (resp. ultra-weakly closed) algebra of operators on $\ell_2(G)$ generated by $\{\lambda(g) : g > e\}$. As an immediate consequence of Theorem 14 we have the following:

19. COROLLARY. *For every a in the reduced group C^*-algebra of G we have*

$$\operatorname{dist}(a + z,\ CH_0^\infty(G)) = \operatorname{dist}(a + z,\ H_0^\infty(G))$$

except possibly for z in a convex open subset $\Omega \subset \{w \in \mathbf{C} : |w| \leqslant 2\|a\|\}$ where $\operatorname{dist}(a + z,\ CH_0^\infty(G))$ attains its minimum.

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