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ANALYTICITY OF THE SEMISPECTRAL MEASURES ON THE BITORUS AND THE SZEGÖ OPERATOR

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1. INTRODUCTION

The one variable Szegö-Kolmogorov-Krein theorem in its scalar form (see [11], [15]) as well as in the operatorial form (see [12], [13]) is an important tool in the study of a single non-normal operator (see [16]). This is why an analogous result in several complex variables would be useful in the study of commuting systems of non-normal operators.

The first attempt in this direction for two variables was made by Helson and Lowdenslager in [6]. This result holds for all positive measures on the bitorus $\mathbf{T}^2$, but it has the disadvantage that an analyticity with respect to the so called augmented halfplanes is involved. The natural concept of analyticity (i.e. with respect to the first quarter) appears in the recent extension given by A. G. Miamee [8], but with the disadvantage that it holds only for a restricted class of positive measures on $\mathbf{T}^2$.

It is our aim to obtain a Szegö-Kolmogorov-Krein type theorem in two variables even in an operatorial form, in which we try to remove the two mentioned disadvantages.

Namely such a result involving a larger class of measures on $\mathbf{T}^2$ in the context of the natural concept of analyticity, which improves the quoted result from [8] will be obtained in Section 5 (Theorem 3), as the scalar case of an operatorial form of such a theorem in two variables (Theorem 1).

We shall work with semispectral measures (as in [14]) on $\mathbf{T}^2$, but the L^2-bounded analytic functions will be replaced with the Wold-type analytic functions on $\mathbf{D}^2$, which were introduced in [5]. We also use the terminology and notation of [5].

2. PRELIMINARIES

Let $\mathscr{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on the separable complex Hilbert space $\mathscr{H}$, F a $\mathscr{B}(\mathscr{H})$-valued semispectral measure on the bitorus

$\mathbf{T}^2$ and $[\mathscr{K}, V, E]$ a minimal spectral dilation of F. Denote by U_1, U_2 the unitary operators on $\mathscr{K}$ which correspond to the spectral measure E and by V_1, V_2 their restrictions to the subspace

$$\mathscr{K}^+ := \bigvee_{m,n\geq 0} U_1^m U_2^n V\mathscr{H}.$$

Then V_1 and V_2 are commutative isometries on $\mathscr{K}^+$ and determine a Wold decomposition of $\mathscr{K}^+$ under the form

$$\mathscr{K}^+ = \mathscr{K}_{\mathrm{u}} \oplus \mathscr{K}_{\mathrm{s}} \oplus \mathscr{K}_{\mathrm{m}} \oplus \mathscr{K}_{\mathrm{e}}, \tag{1}$$

such that the four summands reduce V_1 and V_2, and the pair (V_1, V_2) is unitary on $\mathscr{K}_{\mathrm{u}}$, a (unilateral) shift on $\mathscr{K}_{\mathrm{s}}$, a modified (unilateral) shift on $\mathscr{K}_{\mathrm{m}}$, i.e. we respectively have

$$\mathscr{K}_{\mathrm{s}} = \bigoplus_{m,n\geq 0} V_1^m V_2^n \mathscr{R}, \quad \mathscr{K}_{\mathrm{m}} = \bigoplus_{\substack{m>0\\ \text{or}\\ n>0}} U_1^m U_2^n \mathscr{L}, \tag{2}$$

where $\mathscr{R} \subset \mathscr{K}^+$ and $\mathscr{L} \subset \mathscr{K} \ominus \mathscr{K}^+$ are wandering subspaces for (U_1, U_2) (see [2], [3], [4]). $\mathscr{K}_{\mathrm{e}}$ has the property that it does not contain any nontrivial subspace reducing (V_1, V_2) to a pair of one of the first three just mentioned kinds. The restriction of (V_1, V_2) on $\mathscr{K}_{\mathrm{e}}$ is called *the ultraevanescent part* of the given isometric pair. We also consider the orthogonal decomposition of $\mathscr{K}_{\mathrm{e}}$

$$\mathscr{K}_{\mathrm{e}} = \mathscr{K}_{\mathrm{e1}} \oplus \mathscr{K}_{\mathrm{e0}}, \tag{3}$$

where $\mathscr{K}_{\mathrm{e1}}$ is generated by the subspaces of $\mathscr{K}_{\mathrm{e}}$ reducing V_1, respectively V_2 to unitary operators, while $\mathscr{K}_{\mathrm{e0}}$ is the intersection of the shift subspaces of V_1 and V_2 in $\mathscr{K}_{\mathrm{e}}$.

From (2) and by Proposition 1 and 2 in [2] it follows that $\mathscr{K}$ contains the following reducing subspace for (U_1, U_2)

$$\mathscr{K}_0 := \bigoplus_{m,n\in\mathbf{Z}} U_1^m U_2^n \mathscr{R} \oplus \bigoplus_{m,n\in\mathbf{Z}} U_1^m U_2^n \mathscr{L} = \bigoplus_{m,n\in\mathbf{Z}} U_1^m U_2^n (\mathscr{R} \oplus \mathscr{L}).$$

The functional analogous of this space will play a central role in what follows. It is the range of the Fourier representation Φ on $\mathscr{K}_0$ (i.e. it is isomorphic as Hilbert space with $\mathscr{K}_0$) and consists of all $\mathscr{R} \oplus \mathscr{L}$-valued (strongly) measurable functions on $\mathbf{T}^2$, whose $\mathscr{R} \oplus \mathscr{L}$-norm is square integrable on $\mathbf{T}^2$ with respect to the normalized Lebesgue measure m_2. This space will be denoted by $L^2(\mathbf{T}^2; \mathscr{R} \oplus \mathscr{L})$ or briefly by $L^2(\mathscr{R} \oplus \mathscr{L})$. In general, if, for $\alpha \in \{+, -\}$, $\mathscr{H}_\alpha$ is a separable complex Hilbert space, we shall denote by $L^2_\alpha(\mathscr{H}_\alpha)$ the subspace of all functions $f \in L^2(\mathscr{H}_\alpha) \subset$ $\subset L^2(\mathscr{H}_+ \oplus \mathscr{H}_-)$, whose Fourier coefficients satisfy $\hat{f}(m, n) = 0$ for $(m, n) \notin \mathbf{Z}^2_+$, if α means $+$, respectively $\hat{f}(m, n) = 0$ for $(m, n) \in -\mathbf{Z}_+^2$, if α means $-$. By the above mentioned Fourier representation, the spaces $L_+^2(\mathscr{R})$ and $L_-^2(\mathscr{L})$ correspond to the subspaces $\mathscr{K}_s$ and $\mathscr{K}_m$, respectively. We also denote by $H^2(\mathscr{H}_+)$ and $M^2(\mathscr{H}_-)$ the range of the Poisson representation in $L^2(\mathscr{H}_+ \oplus \mathscr{H}_-)$ (see [9], [5]) of $L_+^2(\mathscr{H}_+)$ and respectively $L_-^2(\mathscr{H}_-)$, regarded as Hilbert spaces of $\mathscr{H}_+ \oplus \mathscr{H}_-$-valued functions on $\mathbf{D}^2$ with the norm

$$\|g\|^2 := \sup_{0 \leqslant \rho < 1} \int_{\mathbf{T}^2} \|g(\rho w)\|^2 \, \mathrm{d}m_2(w).$$

As in the scalar case ([9], [10]), for every $g \in H^2(\mathscr{H}_+)$, respectively $g \in M^2(\mathscr{H}_-)$ and for m_2-almost all $w \in \mathbf{T}^2$, there exists the radial limit $g_\alpha(w) := \lim\limits_{\rho \to 1} g(\rho w)$ which belongs to $L_\alpha^2(\mathscr{H}_\alpha)$ (for $\alpha = +$, respectively $-$).

Let us recall that a *Wold-type function* on $\mathbf{D}^2$ is a couple $\Theta = [(\mathscr{H}, \mathscr{H}_+; \Theta_+), (\mathscr{H}, \mathscr{H}_-; \Theta_-)]$, where $\Theta_\alpha : \mathbf{D}^2 \to \mathscr{B}(\mathscr{H}, \mathscr{H}_\alpha)$, $\alpha \in \{+, -\}$ are functions of the form

$$\Theta_+(z) = \sum_{m,n \geqslant 0} z_1^m z_2^n \Theta_{mn}^+ ,$$

$$\Theta_-(z) = \sum_{m>0 \vee n>0} z_1^{(m)} z_2^{(n)} \Theta_{mn}^- , \quad (z = (z_1, z_2) \in \mathbf{D}^2),$$

where $z_j^{(m)} = |z_j|^{|m|-m} z_j^m$ $(z_j \neq 0)$, the series being norm-convergent, such that for every $h \in \mathscr{H}$, we have $\Theta_+(\cdot)h \in H^2(\mathscr{H}_+)$, $\Theta_-(\cdot)h \in M^2(\mathscr{H}_-)$ and $\sup\{\|\Theta_\alpha(\cdot)h\|, \|h\| \leqslant 1\} < \infty$, $\alpha \in \{+, -\}$. The function $(\mathscr{H}, \mathscr{H}_+; \Theta_+)$ will be called the *analytic part* of Θ. In the terminology of [14] this is an L^2-bounded analytic function (on $\mathbf{D}^2$).

We associate now to the Wold-type function Θ the (bounded linear) operator $V_\Theta : \mathscr{H} \to L_+^2(\mathscr{H}_+) \oplus L_-^2(\mathscr{H}_-)$ defined by

$$V_\Theta h := (\Theta_+(\cdot)h)_+ \oplus (\Theta_-(\cdot)h)_- \quad (h \in \mathscr{H}).$$

We also associate with Θ, the $\mathscr{B}(\mathscr{H})$-valued semispectral measure F_Θ on $\mathbf{T}^2$ defined by

$$F_\Theta(\sigma) := V_\Theta^* E^\times(\sigma) V_\Theta \quad (\sigma \text{ -- a Borel set in } \mathbf{T}^2),$$

where $E^\times$ is the spectral measure corresponding to the multiplication with coordinate functions in $L^2(\mathscr{H}_+ \oplus \mathscr{H}_-)$.

Returning to the given $\mathscr{B}(\mathscr{H})$-valued semispectral measure F on $\mathbf{T}^2$, we also work with the subspaces $\mathscr{K}_0$, $\mathscr{K}^+$, $\mathscr{R}$, $\mathscr{L}$ of the dilation space $\mathscr{K}$. We denote by P_0 the orthogonal projection of $\mathscr{K}$ onto $\mathscr{K}_0$, and $V_0 = \Phi P_0 V$. Then $V_0\mathscr{H} \subset L_+^2(\mathscr{R}) \oplus$ $\oplus L_-^2(\mathscr{L})$ and, as was shown in [5], V_0 induces, via the Poisson representation, a Wold-type function $\Theta = [(\mathscr{H}, \mathscr{R}; \Theta_+), (\mathscr{H}, \mathscr{L}; \Theta_-)]$ on $\mathbf{D}^2$ with $V_\Theta = V_0$, which verifies

(a) $\bigvee_{m,n\geq 0} w_1^m w_2^n V_0 \mathcal{H} = L^2_+(\mathcal{R}) \oplus L^2_-(\mathcal{L})$;

(b) $F_\Theta \leqslant F$ (the equality being true iff $\mathcal{H}_u = \mathcal{H}_e = \{0\}$).

If in addition $\mathcal{H}_{e0} = \{0\}$, then Θ also satisfies the following maximality property

(c) For each Wold-type function $\Theta' = [(\mathcal{H}, \mathcal{R}'; \Theta'_+), (\mathcal{H}, \mathcal{L}'; \Theta'_-)]$ on $\mathbf{D}^2$, $F_{\Theta'} \leqslant F$ implies $F_{\Theta'} \leqslant F_\Theta$.

The properties (a), (b), (c) determine each of the functions Θ_+ and Θ_- uniquely up to a left side unitary constant factor. The condition (a) says that the function Θ is outer. Clearly this is equivalent with the conditions

$$\bigvee_{m,n\geq 0} w_1^m w_2^n P_+ V_0 \mathcal{H} = L^2_+(\mathcal{R}), \qquad \bigvee_{\substack{m>0 \\ \text{or} \\ n>0}} w_1^m w_2^n P_- V_0 \mathcal{H} = L^2_-(\mathcal{L}),$$

P_+ and P_- being the projections from $L^2_+(\mathcal{R}) \oplus L^2_-(\mathcal{L})$ onto $L^2_+(\mathcal{R})$ and $L^2_-(\mathcal{L})$ respectively.

We need also the following preliminary result.

PROPOSITION 1. *For each outer L^2-bounded analytic function $(\mathcal{H}, \mathcal{H}_+; \Theta_+)$ on $\mathbf{D}^2$, $\Theta_+(z)\mathcal{H}$ is dense in $\mathcal{H}_+$ for every $z \in \mathbf{D}^2$.*

Proof. Let $V_+ : \mathcal{H} \to H^2(\mathcal{H}_+)$ defined by

$$(V_+h)(z) := \Theta_+(z)h \quad (h \in \mathcal{H};\ z \in \mathbf{D}^2).$$

Then for each $h \in \mathcal{H}$ and $h_1 \in \mathcal{H}_+$ the function $z \mapsto ((V_+h)(z), h_1)$ belongs to the Hardy class $H^2(\mathbf{D}^2)$ and consequently it has radial limit ([10]) m_2-a.e. on $\mathbf{T}^2$. By applying the Cauchy formula ([9], 3.4.3 and 3.4.4) for every $z \in \mathbf{D}^2$ we obtain

$$(\Theta_+(z)h, h_1) = \int_{\mathbf{T}^2} \lim_{\rho\to 1}((V_+h)(\rho w_1, \rho w_2), h_1)(1 - z_1\bar{w}_1)^{-1}(1 - z_2\bar{w}_2)^{-1}\,\mathrm{d}m_2(w) =$$

$$= \lim_{\rho\to 1}\int_{\mathbf{T}^2}((V_+h)(\rho w_1, \rho w_2), (1 - \bar{z}_1 w_1)^{-1}(1 - \bar{z}_2 w_2)^{-1}h_1)\,\mathrm{d}m_2(w) =$$

$$= ((V_+h)_+, \overline{C(z, \cdot)}h_1)_{L^2_+(\mathcal{H}_+)},$$

where $C(z, w)$ $(z \in \mathbf{D}^2,\ w \in \mathbf{T}^2)$ is the bidimensional Cauchy kernel. Thus if $h_1 \in \mathcal{H}_+$ is orthogonal to $\Theta_+(z)\mathcal{H}$, for an $z \in \mathbf{D}^2$, then $\overline{C(z, \cdot)}h_1$ is orthogonal to $V_{\Theta_+}\mathcal{H}$ in $L^2_+(\mathcal{H}_+)$, which, since Θ_+ is outer, leads to $h_1 = 0$.

3. THE SZEGÖ OPERATOR

Following the idea of [12], we define the Szegö operator $\Delta[F]:\mathscr{H}\to\mathscr{H}$ attached to the semispectral measure F by

$$(4)\qquad (\Delta[F]h,h):=\inf\sum_{\substack{0\leqslant m,j\leqslant p\\ 0\leqslant n,k\leqslant q}}\int_{\mathbf{T}^2} w_1^{m-j}w_2^{n-k}\,\mathrm{d}(F(w)h_{mn},h_{jk})$$

$h\in\mathscr{H}$, the infimum being taken over all finite systems $\{h_{mn}\}\subset\mathscr{H}$ with $h_{00}=h$. If U_1, U_2 are the unitary operators associated with F as in the previous section, we put

$$\mathscr{K}_0^+:=\bigvee_{\substack{m,n\geqslant 0\\ (m,n)\neq(0,0)}} U_1^mU_2^nV\mathscr{H}$$

and denote by P_0^+ the projection of $\mathscr{K}^+$ onto $\mathscr{K}_0^+$. Now, for each $h\in\mathscr{H}$, we have

$$(\Delta[F]h,h)=\inf\sum_{m,n,j,k}\int w_1^{m-j}w_2^{n-k}\mathrm{d}(E(w)Vh_{mn},Vh_{jk})=$$

$$=\inf\sum_{m,n,j,k}(U_1^{m-j}U_2^{n-k}Vh_{mn},Vh_{jk})=\inf\|\sum_{m,n}U_1^mU_2^nVh_{mn}\|^2=$$

$$=\inf\|Vh-\sum_{\substack{m,n\\ (m,n)\neq(0,0)}}U_1^mU_2^nVh_{mn}\|^2=\|(I_{\mathscr{K}^+}-P_0^+)Vh\|^2=$$

$$=(V^*(I_{\mathscr{K}^+}-P_0^+)Vh,h).$$

Hence, $\Delta[F]=V^*(I_{\mathscr{K}^+}-P_0^+)V$ and thus $\Delta[F]$ is a positive operator on $\mathscr{H}$.

We shall now write $\Delta[F]$ as a sum of two positive operators. For, let us combine the decompositions (1) and (3) under the form

$$\mathscr{K}^+=\bigoplus_{\alpha\in\mathscr{A}}\mathscr{K}_\alpha;\quad \mathscr{A}=\{\mathrm{u},\mathrm{m},\mathrm{s},\mathrm{e1},\mathrm{e0}\},$$

which means that the corresponding projections P_α are orthogonal to each other and $I_{\mathscr{K}^+}=\sum_{\alpha\in\mathscr{A}}P_\alpha$. From here, since for $\alpha\in\{\mathrm{u},\mathrm{m},\mathrm{e1}\}$ holds $\mathscr{K}_\alpha\subset\mathscr{K}_0^+$ and consequently $P_\alpha P_0^+=P_\alpha$, we obtain $I_{\mathscr{K}^+}-P_0^+=(I_{\mathscr{K}^+}-P_0^+)(P_\mathrm{s}+P_{\mathrm{e0}})$ and finally:

$$(5)\qquad \Delta[F]=V^*(I_{\mathscr{K}^+}-P_0^+)P_\mathrm{s}V+V^*(I_{\mathscr{K}^+}-P_0^+)P_{\mathrm{e0}}V.$$

It remains to prove that P_s and P_{e0} commutes with P_0^+. For, since from the above relations, P_α commutes with P_0^+ for each $\alpha\in\{\mathrm{u},\mathrm{m},\mathrm{e1}\}$, it suffices to see that P_s

commutes with P_0^+. With that end in view let $\mathscr{K}_s = \mathscr{R} \oplus \mathscr{K}_s^0$ and observe that $\mathscr{K}_s^0 \subset \mathscr{K}_0^+$, while $\mathscr{R} \subset \mathscr{K}^+ \ominus \mathscr{K}_0^+$. This leads to the commutativity of P_s^0 to P_0^+ and of $P_{\mathscr{R}}$ to $I_{\mathscr{K}^+} - P_0^+$. Thus $P_s = P_{\mathscr{R}} + P_s^0$ commutes with P_0^+.

From (5) we now deduce that $\Delta[F] = 0$ iff $P_s V\mathscr{H} \subset \mathscr{K}_0^+$ and $P_{e0} V\mathscr{H} \subset \mathscr{K}_0^+$, or, because $\mathscr{K}^+ = V\mathscr{H} \vee \mathscr{K}_0^+$, equivalently $\mathscr{K}_s \oplus \mathscr{K}_e \subset \mathscr{K}_0^+$. It is obvious that the condition $\mathscr{K}_s \subset \mathscr{K}_0^+$ is equivalent to $\mathscr{K}_s = \mathscr{R} = \{0\}$, which makes the semispectral measure F without any analytical character.

Now let $\Theta = [(\mathscr{H}, \mathscr{R}; \Theta_+), (\mathscr{H}, \mathscr{L}; \Theta_-)]$ be the Wold-type function associated with F, and F_Θ the semispectral measure associated to Θ as in Section 2. By using condition (a) and defining the projection P_Θ^+ in $L_+^2(\mathscr{R}) \oplus L_-^2(\mathscr{L})$ similarly with P_0^+ in $\mathscr{K}^+$, for each $h \in \mathscr{H}$, we have

$$(V^*(I_{\mathscr{K}^+} - P_0^+)P_s Vh, h) = \|(I_{\mathscr{K}^+} - P_0^+)P_s Vh\|^2 =$$

$$= \|\Phi(P_s - P_0^+ P_s)P_0 Vh\|^2 = \|(P_+ - P_\Theta^+ P_+)\Phi P_0 Vh\|^2 =$$

$$= (V_0^*(I_{L_+^2 \oplus L_-^2} - P_\Theta^+)V_0 h, h) =$$

$$= (\Delta[F_\Theta]h, h) = \inf_{h_{mn}} \|V_0 h - \sum_{\substack{m,n \\ (m,n)\neq(0,0)}} w_1^m w_2^n V_0 h_{mn}\|^2_{L_+^2(\mathscr{R})\oplus L_-^2(\mathscr{L})} =$$

$$= \inf_{\substack{u\in H^2(\mathscr{R}), u(0)=0 \\ v\in M^2(\mathscr{L}), v(0)=0}} \|\Theta_+(\cdot)h \oplus \Theta_-(\cdot)h - u \oplus v\|^2_{H^2(\mathscr{R})\oplus M^2(\mathscr{L})} =$$

$$= \inf_{u\in H^2(\mathscr{R}), u(0)=0} \|\Theta_+(\cdot)h - u\|^2_{H^2(\mathscr{R})} = \|\Theta_+(0)h\|^2 =$$

$$= (\Theta_+(0)^*\Theta_+(0)h, h).$$

Thus, the first operator in the decomposition (5) of $\Delta[F]$ gives him an analytical character. The second operator in (5) is a perturbation term, which is absent in the case of semispectral measures on $\mathbf{T}$ ([13], [12]). We also note that if Θ' is another Wold-type function satisfying $F_{\Theta'} \leqslant F$ and if Θ'' is the Wold-type outer function associated to $F_{\Theta'}$, then by (b) we have $F_{\Theta''} \leqslant F_{\Theta'}$, and by a calculation as above we deduce

$$\Theta_+''(0)^*\Theta_+''(0) = \Delta[F_{\Theta''}] \leqslant \Delta[F_{\Theta'}] \leqslant \Delta[F].$$

Now, by using Proposition 1, we can unify all facts from above about the operator $\Delta[F]$ in the following Szegö-Kolmogorov-Krein result.

THEOREM 1. *Let F be a $\mathscr{B}(\mathscr{H})$-valued semispectral measure on $\mathbf{T}^2$ and $\Delta[F]$ the corresponding Szegö operator. With the above notation, the following statements* *hold:*

(i) $\Delta[F] = 0$ *iff* $\mathcal{K}_{e0} \subset \mathcal{K}_0^+$ *and there exists no Wold-type function* Θ *on* $\mathbf{D}^2$ *having nontrivial analytic part and satisfying* $F_\Theta \leqslant F$.

(ii) *If* $\Delta[F] \neq 0$ *and* $\mathcal{K}_{e0} \subset \mathcal{K}_0^+$, *there is a Wold-type outer function* $\Theta = [(\mathcal{H}, \mathcal{R}; \Theta_+), (\mathcal{H}, \mathcal{L}; \Theta_-)]$ *on* $\mathbf{D}^2$, *such that* $F_\Theta \leqslant F$, $\dim \mathcal{R} = \dim(\Delta[F]\mathcal{H})$ *and*

$$\Delta[F] = \Delta[F_\Theta] = \Theta_+(0)^*\Theta_+(0). \tag{6}$$

Moreover, if $\mathcal{K}_{e0} = \{0\}$, *then each of the functions* Θ_+ *and* Θ_- *are uniquely determined up to a left side constant unitary factor.*

REMARK. 1. Both situations $\Delta[F] = 0$ and $\Delta[F] \neq 0$ from (i) and (ii) may effectively appear. Indeed:

a) In the Example from [5] we have that the isometries V_1 and V_2 associated to the semispectral measure do not doubly commute on $\mathcal{K}^+$, but they doubly commute on $\mathcal{K}_e$. This implies by Theorem 8 form [4] that $\mathcal{K}_{e0} = \{0\}$. Moreover, the analytic part of the associated Wold-type outer function is trivial. It follows then by (i) that $\Delta[F] = 0$.

b) The case $\Delta[F] \neq 0$ appears, for example, if V_1, V_2 doubly commute on $\mathcal{K}^+$ (which implies $\mathcal{K}_{e0} = \{0\}$). Moreover, by Theorem 6 from [4] and by Theorem 2 from [5] the corresponding Wald-type outer function reduces to its analytic part.

REMARK 2. The restriction $\mathcal{K}_{e0} \subset \mathcal{K}_0^+$ in (ii) seems to depend on our method of proof. It remains unknown what happens if this condition is not satisfied. However, when a weaker condition is not satisfied the conclusion in (ii) fails (see Example after Theorem 3).

4. SEMISPECTRAL MEASURES AND SCALAR MULTIPLES OF WOLD TYPE ASSOCIATED FUNCTION

We are now interested in finding necessary and/or sufficient conditions in order that the analytic part of the associated Wold-type outer function would have a scalar multiple, in terms of the given semispectral measure F. As we shall see, this problem is connected to subordination problem with functions from $H^2(\mathbf{D}^2)$ of the m_2-absolutely continuous part of F. Let us mention that, by Theorem 6 from [3], the measure F_Θ from (6) is even absolutely continuous with respect to m_2. Before we state the principal result of this section, let us recall that an L^2-bounded analytic function $(\mathcal{H}, \mathcal{H}_1; \Theta)$ on $\mathbf{D}^2$ has a *scalar multiple*, if there is a scalar non-null function $g \in H^2(\mathbf{D}^2)$ and a contractive analytic function $(\mathcal{H}_1, \mathcal{H}; \Theta_1)$ on $\mathbf{D}^2$, such that

$$\Theta_1(z)\Theta(z) = g(z)I_{\mathcal{H}}, \quad \Theta(z)\Theta_1(z) = g(z)I_{\mathcal{H}_1} \quad (z \in \mathbf{D}^2).$$

THEOREM 2. *Let F be a $\mathscr{B}(\mathscr{H})$-valued semispectral measure on $\mathbf{T}^2$ and, for each $h \in \mathscr{H}$, denote $f_h := \frac{\mathrm{d}(F(\cdot)h, h)}{\mathrm{d}m_2}$. Assume that $\Delta[F] \neq 0$ and $\mathscr{H}_{e0} = \{0\}$. Then the following are equivalent:*

(i) *There is a Wold-type function Θ on $\mathbf{D}^2$ having non-trivial analytic part, with scalar multiple, and satisfying $F_\Theta \lesssim F$.*

(ii) *The analytic part of the Wold-type outer function Θ on $\mathbf{D}^2$ corresponding to F by Theorem 1 (ii) has a scalar multiple.*

(iii) *There is a function $g \in H^2(\mathbf{D}^2)$ such that, for each $h \in \mathscr{H}$, $\|h\| = 1$, the following inequality holds*

$$|g_+|^2 \leqslant f_h \quad m_2\text{-a.e. } on \ \mathbf{T}^2.$$

Proof. (i) $\Rightarrow$ (iii). Let $\Theta = [(\mathscr{H}, \mathscr{H}_+ ; \Theta_+), (\mathscr{H}, \mathscr{H}_- ; \Theta_-)]$ be a Wold-type function with $\mathscr{H}_+ \neq \{0\}$, such that $(\mathscr{H}, \mathscr{H}_+ ; \Theta_+)$ has a scalar multiple. Let $g \in H^2(\mathbf{D}^2)$ be the scalar multiple and $(\mathscr{H}_+, \mathscr{H}; \Theta'_+)$ be the corresponding contractive factor. Then, for each polynomial p on $\mathbf{T}^2$ and for each $h \in \mathscr{H}$, we have

$$\int |p(w)|^2 \mathrm{d}(F(w)h, h) \geqslant \int |p(w)|^2 \mathrm{d}(F_\Theta(w)h, h) =$$

$$= \int |p(w)|^2 \mathrm{d}(E^\Theta(w)V_0 h, V_0 h) = \|pV_0 h\|^2_{L^2(\mathscr{H}_+ \oplus \mathscr{H}_-)} =$$

$$= \|p\Theta_+(\cdot)h\|^2_{H^2(\mathscr{H}_+)} + \|p\Theta_-(\cdot)h\|^2_{M^2(\mathscr{H}_-)} \geqslant \|p\Theta_+(\cdot)h\|^2_{H^2(\mathscr{H}_+)} =$$

$$= \sup_{0 \leqslant \rho < 1} \int |p(\rho w)|^2 \|\Theta_+(\rho w)h\|^2 \mathrm{d}m_2 \geqslant \sup_{0 \leqslant \rho < 1} \int |p(\rho w)|^2 \|\Theta'_+(\rho w)\Theta_+(\rho w)h\|^2 \mathrm{d}m_2 =$$

$$= \sup_{0 \leqslant \rho < 1} \int |p(\rho w)|^2 |g(\rho w)|^2 \|h\|^2 \,\mathrm{d}m_2 = \int |p(w)|^2 |g_+(w)|^2 \|h\|^2 \,\mathrm{d}m_2 .$$

Since the non negative continuous functions on $\mathbf{T}^2$ can be uniformly approximated by moduli of analytic polynomials (see [14]), (iii) follows from the previous inequality.

(iii) $\Rightarrow$ (ii). Let g be as in (iii). We shall prove that g is just the scalar multiple required in (ii). To this end we first construct a contraction operator $T: \mathscr{H}^+ \to L^2_+(\mathscr{H})$, satisfying $T = TP_0|\mathscr{H}^+$. For, let us observe that (iii) implies that there is a Baire null-set $\sigma \subset \mathbf{T}^2$ such that

$$|g_+(w)|^2 \|h\|^2 \leqslant f_h(w) \quad (h \in \mathscr{H}, \ w \in \mathbf{T}^2 \setminus \sigma).$$

If in this inequality instead of h we write $p(w) = \sum_{m,n} w_1^m w_2^n h_{mn}$, where $w = (w_1, w_2) \in \mathbf{T}^2 \setminus \sigma$ and $\{h_{mn}; 0 \leqslant m \leqslant p, 0 \leqslant n \leqslant q\}$ is an arbitrary finite system in $\mathscr{H}$, and then we integrate with respect to m_2, we obtain

$$\|g_+ p\|^2_{L^2(\mathscr{H})} \leqslant \sum_{\substack{0 \leqslant m,j \leqslant p \\ 0 \leqslant n,k \leqslant q}} \int_{\mathbf{T}^2} w_1^{m-j} w_2^{n-k} \mathrm{d}(F(w)h_{mn}, h_{jk}) =$$

$$= \sum_{m,n,j,k} (U_1^{m-j} U_2^{n-k} V h_{mn}, V h_{jk}) = \| \sum_{m,n} U_1^m U_2^n V h_{mn} \|^2.$$

It follows that we can define a contraction $T: \mathscr{K}^+ \to L^2_+(\mathscr{H})$ by

$$T(\sum_{m,n} U_1^m U_2^n V h_{mn}) = g_+ p.$$

From the obvious equalities $TU_j = w_j T$ $(j = 1, 2)$ we have

$$T\mathscr{K}_{\mathrm{u}} \subset T(\bigcap_{m \geqslant 0} U_1^m \mathscr{K}^+ \vee \bigcap_{n \geqslant 0} U_2^n \mathscr{K}^+) \subset \bigcap_{m \geqslant 0} TU_1^m \mathscr{K}^+ \vee \bigcap_{n \geqslant 0} TU_2^n \mathscr{K}^+ =$$

$$= \bigcap_{m \geqslant 0} w_1^m L^2_+(\mathscr{H}) \vee \bigcap_{n \geqslant 0} w_2^n L^2_+(\mathscr{H}) = \{0\},$$

the last equality being motivated by the complete nonunitarity of operators of multiplication by coordinate functions (see Proposition 3 from [14]). Therefore $T\mathscr{K}_{\mathrm{u}} = \{0\}$. Similarly $T\mathscr{K}_{\mathrm{e}1} = \{0\}$, while $T\mathscr{K}_{\mathrm{e}0} = \{0\}$ from the hypothesis $\mathscr{K}_{\mathrm{e}0} = \{0\}$. Thus, having in mind the obvious inclusion $\mathscr{K}_{\mathrm{s}} \oplus \mathscr{K}_{\mathrm{m}} \subset \mathscr{K}^+ \cap \mathscr{K}_0$, we obtain $T = TP_0 | \mathscr{K}^+$. Now let $(\mathscr{K}, \mathscr{R}; \Theta_+)$ be the analytic part of the Wold-type outer function Θ corresponding to F by Theorem 1 (ii). The contraction operator $S: L^2_+(\mathscr{R}) \to L^2_+(\mathscr{H})$ defined by $Sf = T\Phi^{-1}f$ induces, via the Poisson representation $P[\cdot]$, a contraction from $H^2(\mathscr{R})$ into $H^2(\mathscr{H})$. Then we can define an analytic contractive function $(\mathscr{R}, \mathscr{H}; \Theta'_+)$ on $\mathbf{D}^2$ by $\Theta'_+(z)r = P[Sr](z)$ for $z \in \mathbf{D}^2$ and $r \in \mathscr{R}$ (see [5]). So, by using the above structure of T, for each $h \in \mathscr{H}$ and $z \in \mathbf{D}^2$, we obtain

$$\Theta'_+(z)\Theta_+(z)h = P[S(\Theta_+(\cdot)h)^+](z) = P[T\Phi^{-1}P_+ V_0 h](z) =$$

$$= P[TP_0 \Phi^{-1} V_0 h](z) = P[TP_0 V h](z) = P[TVh](z) = g(z)h,$$

i.e. $\Theta'_+(z)\Theta_+(z) = g(z)I_{\mathscr{H}}$, $z \in \mathbf{D}^2$. From here, it follows immediately

$$[g(z)I_{\mathscr{R}} - \Theta_+(z)\Theta'_+(z)]\Theta_+(z) = 0 \quad (z \in \mathbf{D}^2).$$

Since, by Proposition 1, Θ is outer, the previous equality implies $\Theta_+(z)\Theta'_+(z) = g(z)I_{\mathscr{R}}$, $z \in \mathbf{D}^2$. Consequently g is a scalar multiple for Θ_+.

(ii) $\Rightarrow$ (i) being trivial, the proof is finished.

Let us mention that the hypothesis $\mathscr{K}_{e0} = \{0\}$ was needed only for the implication (iii) $\Rightarrow$ (ii). We can also characterize the situation $f_h \geqslant |f_+|^2 + |g_-|^2$ with $f \in H^2$ and $g \in M^2$ by using the scalar multiple for Θ_-.

5. THE SCALAR CASE

If $F = \mu$ ($\mathscr{K} = \mathbf{C}$) is a Baire finite positive measure on $\mathbf{T}^2$, then the corresponding unitary operators U_1 and U_2 are given by the multiplication with coordinate functions in $L^2(\mu)$, and $\mathscr{K}^+$ is the $L^2(\mu)$-closure of the analytic polynomials which, as usually, will be denoted by $H^2(\mu)$. In this case $\Delta[\mu] = \inf\int|1 - p|^2 \mathrm{d}\mu$, the infimum being taken over all analytic polynomials vanishing at the origin. Let us remark that the $H^2(\mu)$-closure of such polynomials plays the role of the previous space $\mathscr{K}_0^+$, which we shall denote by $H_0^2(\mu)$.

We shall also denote by $H_\alpha^2(\mu)$ the coresponding subspaces from the decompositions (1) and (3). In this case a function $g \in H^2(\mathbf{D}^2)$ is outer (in the sense of our definition from Section 2, which differs from that one given in [9] or [10]), if the smallest invariant subspace containing g_+ is $H^2(m_2)$. We say that, if this is the case, g_+ is outer too.

Now the scalar version of Theorem 1 can be stated as follows:

THEOREM 3. *Let μ be a finite positive Baire measure on $\mathbf{T}^2$ and $f = \mathrm{d}\mu/\mathrm{d}m_2$ its Radon-Nikodym derivative. Suppose that $H_{e0}^2(\mu) \subset H_0^2(\mu)$. Then*

$$\Delta[\mu] = \Delta[f\mathrm{d}m_2] = \exp\int \log f\,\mathrm{d}m_2. \tag{7}$$

Moreover, $\Delta[\mu] > 0$ iff $\log f \in L^1(\mathbf{T}^2)$ or equivalently, iff there is an outer function $g \in H^2(\mathbf{D}^2)$, such that $|g_+|^2 \leqslant f$ m_2-a.e. and $\Delta[\mu] = |g(0)|^2$.

Proof. By Theorem 1 from [6] we have

$$\exp\int \log f\,\mathrm{d}m_2 \leqslant \Delta[f\mathrm{d}m_2] \leqslant \Delta[\mu].$$

For the reverse inequality obviously it remains the case $\Delta[\mu] > 0$. By Theorem 1 (ii) and Theorem 2 ((i) $\Rightarrow$ (iii)) it follows that there is an outer function $g \in H^2(\mathbf{D}^2)$ such that $|g_+|^2 \leqslant f$ m_2-a.e. and $\Delta[\mu] = |g(0)|^2$. Since g is outer, by Theorem 4.4.6 from

[9], we obtain

$$\Delta[\mu] = |g(0)|^2 = \exp\left(2 \log \left|\int g_+ \, dm_2\right|\right) = \exp \int \log|g_+|^2 dm_2,$$

from which $\Delta[\mu] \leqslant \exp \int \log f \, dm_2$, as claimed.

We now mention an example in which a stronger condition as $H^2_{e0}(\mu) \not\subset \not\subset H^2_0(\mu)$ is satisfied and the equality (7) fails (see Remark 2 Section 3).

EXAMPLE. Let λ be the measure on $\mathbf{T}^2$ concentrated on its diagonal, defined by Lebesgue one-dimensional measure m_1, i.e.

$$\int_{\mathbf{T}^2} f \, d\lambda = \int_{\mathbf{T}} f(w, w) dm_1(w) \quad (f \in C(\mathbf{T}^2)).$$

Since λ is m_2-singular, by Proposition 3.1 from [1], it follows that the multiplication operators with coordinate functions S_1 and S_2 on $H^2(\lambda)$ are commuting unilateral shifts. Also, by Corollary 7 from [3], the pair (S_1, S_2) on $H^2(\lambda)$ is ultraevanescent. From the definition of λ, it is easy to see that $H^2(\lambda) = \mathbf{C} \oplus H^2_0(\lambda)$. Hence $H^2_{e0}(\lambda) = H^2(\lambda) \supsetneqq H^2_0(\lambda)$. On the other hand $\Delta[\mu] > 0$, while $\exp\left(\int \log (d\lambda/dm_2) dm_2\right) = 0$.

Let us observe that Theorem 3 can be applied in the particular case in which the isometries $V_1 = U_1|H^2(\mu)$ and $V_2 = U_2|H^2(\mu)$ are doubly commuting, since in this case $H^2_{e0}(\mu) = \{0\}$ (see [4]). So, our Theorem 3 improves Theorem 4.2 from [8], since the commuta ivity property which is required for μ, means that V_1 and V_2 double commute. We also mention the following:

COROLLARY. 1. *Let $f \in L^1(m_2)$. Then $f \in L \log L(m_2)$ iff there is $g \in H^2(m_2)$ so that $|g|^2 \leqslant |f|$* m_2-a.e.

Note that in [8] (Theorems 3.1 and 3.5) conditions under which a function $f \in L^2(m_2)$ verifies $|f| = |g|^2$ m_2-a.e. with $g \in H^2(m_2)$ were given. The following result is connected to this problem and improves. Theorems 4.1 of [7], 2.4 of [8] and 9.24 (iii) of [11]. Let us firstly recall that a finite positive Baire measure μ on $\mathbf{T}^2$ is said to be *Szegö-total* (see [11]), if μ is not a point mass and every function from $H^2(\mu)$ vanishing on a set of positive measure μ, vanishes μ-a.e. on $\mathbf{T}^2$. Now we can state:

THEOREM 4. *Let μ be a finite positive Baire measure on $\mathbf{T}^2$ and V_1, V_2 be the multiplication operators with the coordinate functions on $H^2(\mu)$. Then the following* *are equivalent:*

(i) μ *is a Szegö-total measure and the unilateral shift part of the pair* (V_1, V_2) *is non-trivial.*

(ii) *The pair* (V_1, V_2) *generates a unilateral shift in* $H^2(\mu)$.

(iii) *There is a uniquely determined outer function* f *in* $H^2(m_2)$ *such that* $d\mu = |f|^2 dm_2$.

(iv) μ *has the commutativity property (*i.e. V_1, V_2 *doubly commute) and there is a function* $f \in L^2(m_2)$ *satisfying* $\hat{f}(m, n) = 0$ *for* $(m, n) \notin \mathbf{Z}_+^2$, *such that* $d\mu = |f| dm_2$.

(v) V_1 *and* V_2 *are doubly commuting and* $d\mu = |f|^2 dm_2$ *with* $f \in H^2(m_2)$.

Proof. (i) $\Rightarrow$ (ii) results from Theorem 9.24 (iii) in [11].

(ii) $\Rightarrow$ (iii). Under hypothesis (ii) we have $H_s^2(\mu) = H^2(\mu)$ in decomposition (1). Hence the Wold-type outer function on $\mathbf{D}^2$ associated with μ (by conditions (a)–(b)) reduces to the analytic part $g \in H^2(\mathbf{D}^2)$ and we have $d\mu = |g_+|^2 dm_2$ too.

(iii) $\Rightarrow$ (iv) results from Theorem 4.4.6 of [9], Lemma 1.3 and Theorem 2.18 of [10] and Theorem 2.4 of [8].

(iv) $\Rightarrow$ (v) uses Theorems 2.4 and 3.5 from [8].

For (v) $\Rightarrow$ (i) observe that the double commutativity of V_1, V_2 implies $H_m^2(\mu) = H_{e0}^2(\mu) = \{0\}$.

Since $\log |f| \in L^1(m_2)$, from Theorems 3 and 1 it follows that the outer function $g \in H^2(\mathbf{D}^2)$ corresponding to μ by (a), (b), verifies $|g_+| = |f|$ m_2-a.e. But, the existence of g is connected with the nontriviality of the right defect space $\mathscr{R}$ in $H_s^2(\mu)$ (see (2)). Now, let $h \in H^2(\mu)$ and $\sigma \subset \mathbf{T}^2$ with $\mu(\sigma) = 0$ such that $h = 0$ on σ. Then $\chi_\sigma h = 0$ (χ_σ means the characteristic function of σ), therefore $\chi_{\sigma'} h = h$, where $\sigma' = \mathbf{T}^2 \setminus \sigma$. If $h \neq 0$ in $H^2(\mu)$, since g is outer, we have $0 \neq hg_+ \in H^2(m_2)$ and also $hg_+ = \chi_{\sigma'} hg_+ \in \chi_{\sigma'} L^2(m_2)$. So $(\chi_{\sigma'} L^2(m_2)) \cap H^2(m_2) \neq \{0\}$ and, since m_2 is Szegö-total, it follows that $m_2(\sigma') = 1$. Also $m_2(\sigma) = 0$, and then $\mu(\sigma) = 0$, which is a contradiction. It results $h = 0$ μ-a.e. and so μ is Szegö-total, which proves the last implication.

From this theorem and by Corollary 10 of [5] we infer

COROLLARY 2. *A function* $f \in H^2(m_2)$ *has a factorization of the form* $f = \varphi g$, *wehere* $\varphi, g \in H^2(m_2)$, $|\varphi| = 1$ m_2-a.e. *and* g *is outer, iff the measure* $|f|^2 dm_2$ *has the commutativity property.*

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