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RANGE INCLUSION OF TOEPLITZ AND HANKEL OPERATORS

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For any bounded measurable function φ defined on the unit circle, let M_φ denote the operator "multiplication by φ" defined on L^2 of the unit circle. Let P denote the orthogonal projection from L^2 onto the Hardy space H^2.

If φ and ψ belong to L^∞, the ranges of the Toeplitz operator $T_\varphi = PM_\varphi$ defined on H^2 and the Hankel operator $H_\psi = PM_\psi$ defined on $(H^2)^\perp$ are compared and the following results obtained:

THEOREM A. *Suppose that φ and ψ belong to L^∞. Then* range $T_\varphi \subset$ range H_ψ *if and only if $\varphi = 0$.*

THEOREM B. *The following statements are equivalent for a nonzero function φ belonging to L^∞:*

(i) range $H_\varphi \subset$ range T_φ.

(ii) $\bar{\varphi} = ug$, *where g is an outer function in H^∞ and u is a unimodular function such that* $\operatorname{dist}(u, H^\infty) < 1$.

Theorem B answers in the affirmative a question posed by Joe Ball and Bill Helton at the 1988 AMS Summer Research Institute on Operator Theory/Operator Algebras and Applications as to whether a nonzero φ for which range $H_\varphi \subset$ range T_φ must be log-integrable.

1. INTRODUCTION

This paper discusses when range inclusion occurs between two related operators defined on classical function spaces. We will be working in the Lebesgue space L^2 of the unit circle, and all unsubscripted norms and inner products will refer to this space. Also, all integrals that appear will be taken over the unit circle with respect to normalized Lebesgue measure. We write the Fourier series of a function

f belonging to L^2 as

$$f = \sum_{k=-\infty}^{\infty} \hat{f}(k)\zeta^k.$$

Inside L^2 lies the Hardy space H^2 consisting of those f all of whose negatively indexed Fourier coefficients vanish. The orthogonal complement of H^2 in L^2 will be denoted by H^2_-. Let P denote the orthogonal projection of L^2 onto H^2; P is given by the formula

$$P\left(\sum_{k=-\infty}^{\infty} \hat{f}(k)\zeta^k\right) = \sum_{k=0}^{\infty} \hat{f}(k)\zeta^k.$$

It follows from this formula that $\lim_{n\to\infty} \|P(\varphi\zeta^n)\| = \|\varphi\|$ for any φ belonging to L^2.

Suppose that φ belongs to L^∞. We define three operators associated with φ. Let M_φ denote the operator on L^2 defined by $(M_\varphi f)(\zeta) = \varphi(\zeta)f(\zeta)$ for $f \in L^2$; in other words, M_φ is the operator of pointwise multiplication by φ on the unit circle. Obviously, M_φ is a bounded operator and its norm is no greater than $\|\varphi\|_\infty$. The other two operators are defined in terms of M_φ: the Toeplitz operator with symbol φ, denoted by T_φ, acts on H^2 and is defined to be the operator $PM_\varphi|H^2$, and the Hankel operator with symbol φ, denoted by H_φ, maps H^2_- to H^2 and is defined to be the operator $PM_\varphi|H^2_-$.

We should note here that this definition of the Hankel operator is different from but equivalent to the usual definition (we have essentially taken adjoints). Ordinarily, the Hankel operator is defined with domain H^2 and range H^2_-; here, the Hankel operator maps H^2_- to H^2. The change is made in order to give meaning to questions about range inclusion.

Given φ and ψ belonging to L^∞, when is the range of T_φ contained in the range of H_ψ? When is the range of H_ψ contained in the range of T_φ? We answer the first question completely and the second in the special case when $\varphi = \psi$.

2. A REVIEW OF BASICS

Suppose that φ belongs to L^∞. The following formulas involving T_φ and H_φ are well known [2]:

$$T_\varphi^* = T_{\bar{\varphi}};$$

$$H_\varphi^* = (1 - P)M_{\bar{\varphi}}|H^2;$$

$$H_\varphi H_\varphi^* = T_{|\varphi|^2} - T_\varphi T_{\bar{\varphi}}.$$

We are now ready to tackle the task of range inclusion.

3. RANGE $T_\varphi \subset$ RANGE H_ψ

The statement of Theorem A that appears here is stronger than our original result. The improvement was suggested by Carl Cowen.

Proof of Theorem A. If $\varphi = 0$, then $T_\varphi = 0$, so trivially $\operatorname{range} T_\varphi = \{0\} \subset \subset \operatorname{range} H_\psi$.

Suppose now that $\operatorname{range} T_\varphi \subset \operatorname{range} H_\psi$. By a theorem of R. G. Douglas [1], there is a constant $c > 0$ such that

$$T_\varphi T_\varphi^* \leqslant c H_\psi H_\psi^*$$

which means that

$$T_\varphi T_\varphi^* \leqslant c(T_{|\varphi|^2} - T_\psi T_\psi^*)$$

or that

$$T_\psi T_\psi^* + c^{-1} T_\varphi T_\varphi^* \leqslant T_{|\psi|^2}.$$

Fix f in H^2. It follows then that

$$\langle T_\psi T_\psi^* f, f\rangle + c^{-1}\langle T_\varphi T_\varphi^* f, f\rangle \leqslant \langle T_{|\psi|^2} f, f\rangle. \tag{1}$$

Now

$$\langle T_\psi T_\psi^* f, f\rangle = \|T_\psi^* f\|^2 = \|P(\bar\psi f)\|^2 \tag{2}$$

and similarly for φ. Also

$$\langle T_{|\psi|^2} f, f\rangle = \langle P(|\psi| f^2), f\rangle = \langle |\psi|^2 f, f\rangle = \int |\psi f|^2 = \int |\bar\psi f|^2 = \|\bar\psi f\|^2. \tag{3}$$

Substituting these into (1), we obtain

$$\|P(\bar\psi f)\|^2 + c^{-1}\|P(\bar\varphi f)\|^2 \leqslant \|\bar\psi f\|^2. \tag{4}$$

If we substitute $f = \zeta^n$ in (4) and let n tend to infinity, we obtain

$$\|\bar\psi\|^2 + c^{-1}\|\bar\varphi\|^2 \leqslant \|\bar\psi\|^2. \tag{5}$$

But $c^{-1} > 0$, so (5) is possible only if $\|\varphi\| = 0$, that is, if $\varphi = 0$. ▨

4. RANGE $H_\varphi \subset$ RANGE T_φ

We now turn to the possibility that $\operatorname{range} H_\varphi \subset \operatorname{range} T_\varphi$. If $\varphi = 0$, then trivially $\operatorname{range} H_\varphi \subset \operatorname{range} T_\varphi$, so we assume for the rest of the section that $\varphi \neq 0$.

If A is a subset of L^2, denote the closure of A in L^2 by $\mathrm{cl}(A)$.

THEOREM 1. *Suppose that φ belongs to L^∞ and is not identically zero. Then the following are equivalent:*

(i) range $H_\varphi \subset$ range T_φ.

(ii) *P is bounded below on* $\mathrm{cl}(\bar{\varphi}H^2)$.

Proof. Suppose that range $H_\varphi \subset$ range T_φ. Once again, by Douglas's Theorem [1] there is a constant $c \geqslant 0$ such that

$$H_\varphi H_\varphi^* \leqslant cT_\varphi T_\varphi^*$$

which, after substituting for $H_\varphi H_\varphi^*$, yields

$$T_{|\varphi|^2} - T_\varphi T_\varphi^* \leqslant cT_\varphi T_\varphi^*$$

or

$$T_{|\varphi|^2} \leqslant (1 + c)T_\varphi T_\varphi^* .$$

In other words, for f in H^2

$$\langle T_{|\varphi|^2}f, f\rangle \leqslant (1 + c)\langle T_\varphi T_\varphi^* f, f\rangle$$

which, using (2) and (3), means that

$$\|\bar{\varphi}f\|^2 \leqslant (1 + c)\|P(\bar{\varphi}f)\|^2. \tag{6}$$

So P is bounded below on the linear manifold $\bar{\varphi}H^2$, and hence on its closure $\mathrm{cl}(\bar{\varphi}H^2)$. This proves that (i) implies (ii).

Assume now that P is bounded below on the subspace $\mathrm{cl}(\bar{\varphi}H^2)$ of L^2. Then there is some constant $c' \geqslant 0$ such that

$$\|\bar{\varphi}f\|^2 \leqslant c'\|P(\bar{\varphi}f)\|^2 \tag{7}$$

for every f belonging to H^2. We claim that $c' \geqslant 1$; the proof is similar to the argument at the end of the proof of Theorem A. Let n be any nonnegative integer. Setting $f = \zeta^n$ in (7) and letting n tend to infinity as before, we obtain

$$\|\bar{\varphi}\|^2 \leqslant c'\|\bar{\varphi}\|^2.$$

Since φ is assumed to be nonzero, it follows that $c' \geqslant 1$. Hence (6) holds for all f in H^2 with $c = c' - 1$. Now the argument that (i) implies (ii) up to that point can be reversed, and hence it follows that (ii) implies (i). ▨

We can go further and characterize those φ for which P is bounded below on $\mathrm{cl}(\bar{\varphi}H^2)$, but first we need to know more about the space $\mathrm{cl}(\bar{\varphi}H^2)$. For a subset E of the unit circle, let χ_E denote the characteristic function of E, taking the value 1 on E and the value 0 off E. The following characterization of $\mathrm{cl}(\psi H^2)$ for ψ in L^∞ can be found in [3, Chapter 3].

THEOREM 2. *Let* $\psi \in L^\infty$.

(i) *If* $\int \log|\psi| = -\infty$, *then* $\mathrm{cl}(\psi H^2) = \chi_E L^2$, *where* $E = \{\psi = 0\}$.

(ii) *If* $\int \log|\psi| > -\infty$, *then* $\mathrm{cl}(\psi H^2) = uH^2$, *where, letting g denote the outer function in H^∞ such that $|\psi| = |g|$, $u = \psi/g$ is unimodular.*

LEMMA 3. *If* $\int \log|\varphi| = -\infty$, *then P is not bounded below on* $\mathrm{cl}(\bar{\varphi}H^2)$.

Proof. By the above theorem, $\mathrm{cl}(\bar{\varphi}H^2) = \chi_E L^2$, where $E = \{\varphi = 0\}$. Clearly, since $\varphi \neq 0$, the space $\mathrm{cl}(\bar{\varphi}H^2)$ is nontrivial. Take any nonzero $\psi \in \mathrm{cl}(\bar{\varphi}H^2)$. It follows from the description of $\mathrm{cl}(\bar{\varphi}H^2)$ that $\bar{\zeta}^n\psi$ is in $\mathrm{cl}(\bar{\varphi}H^2)$ as well. Now $\|\bar{\zeta}^n\psi\| = \|\psi\|$ is a positive constant, and

$$\|P(\bar{\zeta}^n\psi)\|^2 = \sum_{k=n}^{\infty} |\hat{\psi}(k)|^2,$$

which tends to zero as n tends to infinity. It follows directly that P is not bounded below on $\mathrm{cl}(\bar{\varphi}H^2)$. ▨

LEMMA 4. *Suppose that $\varphi \in L^\infty$ and that $\int \log|\varphi| > -\infty$. Let g be the outer function in H^∞ such that $|\varphi| = |g|$ and let $u = \bar{\varphi}/g$ be the unimodular quotient. Then P is bounded from below on $\mathrm{cl}(\bar{\varphi}H^2)$ if and only if T_u is left invertible.*

Proof. We have the representation $\mathrm{cl}(\bar{\varphi}H^2) = uH^2$. Now P is bounded from below on $\mathrm{cl}(\bar{\varphi}H^2)$ if and only if there is a constant $c \geqslant 0$ such that

$$\|uf\| \leqslant c\|P(uf)\| \quad \text{for } f \in H^2. \tag{8}$$

Suppose that f belongs to H^2. Since u is unimodular, we have $\|uf\| = \|f\|$. Also, $P(uf) = T_u f$ by definition. Hence (8) holds for some $c \geqslant 0$ if and only if

$$\|f\| \leqslant c\|T_u f\| \quad \text{for } f \in H^2$$

for the same c, that is, exactly when T_u is bounded below on H^2. Since an operator is bounded below if and only if it is left-invertible, the lemma follows. ▨

Putting Theorem 1, Lemma 3, and Lemma 4 together with the fact that a Toeplitz operator with unimodular symbol u is left invertible if and only if $\mathrm{dist}(u, H^\infty) < 1$ [2, page 187] gives Theorem B of this paper.

REFERENCES

1. DOUGLAS, R. G., On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, 17(1966), 413–415.
2. DOUGLAS, R. G., *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.
3. HELSON, H., *Harmonic analysis*, Addison-Wesley, Reading, Massachusetts, 1983.

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