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STABILITY OF INVOLUTORY $*$-ANTIAUTOMORPHISMS IN UHF ALGEBRAS

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1. INTRODUCTION

In [4] Connes introduced the concept of stability for a periodic automorphism α of a C^*-algebra A: an automorphism α of order n is said to be *stable* if any unitary u satisfying $u\alpha(u)\ldots\alpha^{n-1}(u)=1$ is of the form $v^*\alpha(v)$ with v unitary. For a periodic antiautomorphism Φ the analogous concept is obtained by requiring the automorphism $\Phi\circ *$ to be stable; in particular an involutory $*$-antiautomorphism Φ is said to be stable if each unitary u satisfying $\Phi(u)=u$ is of the form $v\Phi(v)$ for some unitary v. Stability of involutory $*$-antiautomorphisms is closely related to questions of conjugacy; perhaps the simplest such connection (given in Proposition 1.2 of [9]) is that, for a stable involutory $*$-antiautomorphism Φ, the antiautomorphisms Φ and $(\mathrm{Ad}\,u)\circ\Phi$ are conjugate for each unitary u satisfying $\Phi(u)=u$. Other, more complicated, conjugacy theorems from [9] and [20] rely implicitly on this notion of stability, but need not do so explicitly since, as remarked by Giordano in Lemma 1.1 of [9], every involutory $*$-antiautomorphism Φ on a von Neumann algebra is stable. This is true because whenever u is a unitary with $\Phi(u)=u$ we can find a square root v of u in the von Neumann algebra generated by u and then $u=v^2=v\Phi(v)$. This argument cannot be applied in other C^*-algebras (because some unitaries with spectrum S^1 may not have square roots in the C^*-algebra they generate) and the purpose of the present paper is to provide an alternative approach to stability, valid in a class of C^*-algebras which includes all UHF algebras with infinitely many 2×2 matrix factors.

The approach to stability taken here is based on Connes' original argument in Corollary 2.6 of [4]. The appropriate analogy for involutory antiautomorphisms is to show that the two projections e_{11} and e_{22} (where these denote the obvious matrix units) are equivalent in the real algebra in $M_2(A)$ determined by the involutory $*$-antiautomorphism

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}\begin{pmatrix}\Phi(a) & \Phi(c)\\ \Phi(b) & \Phi(d)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & u^*\end{pmatrix}.$$

We will demonstrate this result using methods from K-theory. Using the fact (Proposition 2.2) that $M_2(\{a \in A : \Phi(a) = a^*\})$ is isomorphic to a crossed product $A \times_\alpha \mathbf{Z}_2$ for $\alpha = \Phi \circ *$ and results about the K-theory of crossed products, such as a real version of the Pimsner-Voiculescu exact sequence, we will show (in Theorem 4.2) that the K-theory complexification map is injective for a certain class of C^*-algebras. Then using an appropriate cancellation theorem (Theorem 5.2) we will deduce (in Theorem 6.4) the equivalence of e_{11} and e_{22} from the equality of the real K_0 classes $[e_{11}]$ and $[e_{22}]$. Finally, we will conclude the paper with an application to the existence of Φ-invariant matrix subalgebras in a UHF algebra with infinitely many 2×2 matrix factors (Theorem 7.4).

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2. ELEMENTARY RESULTS ON REAL CROSSED PRODUCTS

If α is a real-linear action of a locally compact group G on a real (or complex) C^*-algebra A, then the real crossed product $A \times_\alpha G$ is defined exactly as for the complex case (described in 7.6 of [15]). The action α extends to a complex linear action (still denoted α) of G on the complexification $A^{\mathbf{C}}$ of A and thus gives rise to a complex crossed product $A^{\mathbf{C}} \times_\alpha G$. The following result, which is closely related to the von Neumann algebra results given in Proposition 1.13 of [10] and Proposition 1.1 of [19], describes the relationship between $A \times_\alpha G$ and $A^{\mathbf{C}} \times_\alpha G$.

PROPOSITION 2.1. *Let A be a real C^*-algebra with complexification $A^{\mathbf{C}}$, let Ψ be the associated involutory $*$-antiautomorphism of $A^{\mathbf{C}}$ (defined by $\Psi(a + ib) = a^* + ib^*$ for $a, b \in A$) and let α denote both the action of a locally compact group G on A and the complexified action on $A^{\mathbf{C}}$. Then Ψ extends to an involutory $*$-antiautomorphism $\tilde{\Psi}$ of $A^{\mathbf{C}} \times_\alpha G$, for which the associated real algebra $\{x \in A^{\mathbf{C}} \times_\alpha G : \tilde{\Psi}(x) = x^*\}$ is $*$-isomorphic to the real crossed product $A \times_\alpha G$.*

Proof. Define $\tilde{\Psi}$ on the linear space $K(G, A^{\mathbf{C}})$ of continuous functions from G to $A^{\mathbf{C}}$ with compact supports by

$$(\tilde{\Psi} f) = \Delta(g)^{-1}\alpha_g[\Psi(f(g^{-1}))],$$

where Δ is the modular function on G. Routine computations show that $\tilde{\Psi}$ is an involutory $*$-antiautomorphism of $K(G, A^{\mathbf{C}})$, that $\tilde{\Psi}$ is isometric in the $L_1(G, A^{\mathbf{C}})$ norm and hence extends to $L_1(G, A^{\mathbf{C}})$ and finally that $\tilde{\Psi}$ extends to an involutory $*$-antiautomorphism of $A^{\mathbf{C}} \times_\alpha G$ with the required properties.

The next proposition, which shows how the 2×2 matrix algebra over a real C^*-algebra R can be viewed as a crossed product of a $\mathbf{Z}_2$ action on its complexification A, is the key observation underlying Theorem 4.2 (which gives conditions for the complexification map $K_0(R) \to K_0(A)$ to be injective).

PROPOSITION 2.2. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A, let $\alpha = \Phi \circ *$ and let $R = \{a \in A: \Phi(a) = a^*\}$.*

(i) *There is a real-linear $*$-isomorphism from the complexification $A^{\mathbf{C}}$ $(=A\otimes_{\mathbf{R}}\mathbf{C})$ of A onto $A \oplus A$ under which $\alpha(a, b) = (b, a)$ and $\Psi(a, b) = (\Phi(b), \Phi(a))$, where Ψ is the antiautomorphism of $A^{\mathbf{C}}$ associated with the real algebra A.*

(ii) *There is a real-linear $*$-isomorphism from $A^{\mathbf{C}} \times_\alpha \mathbf{Z}_2$ onto $M_2(A)$ under which the dual action $\hat{\alpha}$ is given by $\hat{\alpha} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ and the antiautomorphism $\tilde{\Psi}$ of Proposition 2.1 is given by $\tilde{\Psi}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} \Phi(d) & \Phi(b) \\ \Phi(c) & \Phi(a)\end{pmatrix}$.*

(iii) *There is a real-linear $*$-isomorphism from $A \times_\alpha \mathbf{Z}_2$ onto $M_2(R)$ under which*

$$\alpha = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

Proof. (i) The mapping taking $a + ib$ to $(a, \alpha(a)) + i(b, \alpha(b))$ for each a, $b \in A$ is an appropriate isomorphism.

(ii) The mapping taking $(a, b)\delta_0 + (c, d)\delta_1$ to $ae_{11} + ce_{12} + de_{21} + be_{22}$, where δ_0, δ_1 denote the appropriate Kronecker delta functions in $L_1(\mathbf{Z}_2, A^{\mathbf{C}})$, is easily checked to be a $*$-isomorphism from $A^{\mathbf{C}} \times_\alpha \mathbf{Z}_2$ onto $M_2(A)$. The formulae for $\hat{\alpha}$ and $\tilde{\Psi}$ follow from their effects on the generators of $A^{\mathbf{C}} \times_\alpha \mathbf{Z}_2$, which are given by $\hat{\alpha}[(a, b)\delta_0] = (a, b)\delta_0$, $\hat{\alpha}(\delta_1) = -\delta_1$, $\tilde{\Psi}[(a, b)\delta_0] = (\Phi(b), \Phi(a))\delta_0$ and $\tilde{\Psi}(\delta_1) = \delta_1^* = \delta_1$.

(iii) From Proposition 2.1 and parts (i) and (ii) above, $A \times_\alpha \mathbf{Z}_2$ is $*$-isomorphic to the real algebra in $M_2(A)$ associated with the involutory $*$-antiautomorphism defined by

$$\tilde{\Psi}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} \Phi(d) & \Phi(b) \\ \varphi(c) & \Phi(a)\end{pmatrix}.$$

Then

$$\beta\tilde{\Psi}\beta^{-1}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} \Phi(a) & \Phi(c) \\ \Phi(b) & \Phi(d)\end{pmatrix}$$

where β is the $*$-isomorphism of $M_2(A)$ defined by

$$\beta = \mathrm{Ad}\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i\end{pmatrix}.$$

It then follows immediately that β is a $*$-isomorphism from the real algebra associated with $\tilde{\Psi}$ onto $M_2(R)$. From (ii) the dual action is therefore given by

$$\operatorname{Ad}\beta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \operatorname{Ad}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} = \operatorname{Ad}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

as required.

Although Proposition 2.2 tells us that the K-theory of R is the same as that of $A\times_\alpha \mathbf{Z}_2$, the K-theory of A is more closely linked to that of $A\times_\alpha \mathbf{Z}$ than $A\times_\alpha \mathbf{Z}_2$. The next two propositions provide a link between $A\times_\alpha \mathbf{Z}$ and $A\times_\alpha \mathbf{Z}_2$; this link is not as close as in the complex case where, as described in 10.3.3 of [1], $A\times_\alpha \mathbf{Z}$ is isomorphic to the mapping torus of $\hat{\alpha}$ on $A\times_\alpha \mathbf{Z}_2$, but it will nevertheless be sufficient for our purposes.

PROPOSITION 2.3. *Let A be a unital real C^*-algebra with complexification $A^{\mathbf{C}}$, let Ψ be the associated involutory $*$-antiautomorphism of $A^{\mathbf{C}}$ and let α be a $\mathbf{Z}_2$ action on both A and $A^{\mathbf{C}}$. Then $A\times_\alpha \mathbf{Z}$ is $*$-isomorphic to*

$$M_\Psi = \{f\in C([0,1],\ A^{\mathbf{C}}\times_\alpha \mathbf{Z}_2) : f(1) = \hat{\alpha}(f(0)),\ f(t) = (\tilde{\Psi}\hat{\alpha})(f(1-t)^*)$$
$$\text{for each } 0 \leqslant t \leqslant 1\},$$

where $\hat{\alpha}$ denotes the $\mathbf{Z}_2$ action on $A^{\mathbf{C}}\times_\alpha \mathbf{Z}_2$ which is dual to α.

Proof. By 10.3.3 of [1], $A^{\mathbf{C}}\times_\alpha \mathbf{Z}$ is $*$-isomorphic to the mapping torus

$$M = \{f\in C([0,1],\ A^{\mathbf{C}}\times_\alpha \mathbf{Z}_2) : f(1) = \hat{\alpha}(f(0))\}.$$

Then, by Proposition 2.1, $A\times_\alpha \mathbf{Z}$ is $*$-isomorphic to

$$M_\Psi = \{f\in M : \tilde{\Psi}(f) = f^*\},$$

where $\tilde{\Psi}$ is used to denote both the antiautomorphism of $A\times_\alpha \mathbf{Z}$ defined in Proposition 2.1 and the corresponding antiautomorphism of M. The map $\tilde{\Psi}$ is determined as an antiautomorphism by the equations

$$\tilde{\Psi}(x\delta_0) = \Psi(x)\delta_0$$
$$\tilde{\Psi}(\delta_1) = \delta_1^*,$$

which describe its effect on the generators of $A^{\mathbf{C}}\times_\alpha \mathbf{Z}$. Since the map Γ defined by

$$(\Gamma f)(t) = (\tilde{\Psi}\hat{\alpha})(f(1-t))$$

is clearly a $*$-antiautomorphism of M, it suffices to show that, under the isomorphism of 10.3.3 of [1], it has the same effect as $\tilde{\Psi}$ on the generators of $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$. Now the isomorphism from $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$ onto M takes $x\delta_0 \in L_1(\mathbf{Z}, A^{\mathbf{C}})$ to the constant function with value $x\delta_0 \in L_1(\mathbf{Z}_2, A^{\mathbf{C}})$ and takes $\delta_1 \in L_1(\mathbf{Z}, A^{\mathbf{C}})$ to the function $f: t \mapsto e^{i\pi t}\delta_1$. However

$$(\Gamma x\delta_0)(t) = (\tilde{\Psi}\hat{\alpha})(x\delta_0) = \tilde{\Psi}(x\delta_0) = \Psi(x)\delta_0$$

and

$$(\Gamma f)(t) = (\tilde{\Psi}\hat{\alpha})(e^{i\pi(1-t)}\delta_1) = -\tilde{\Psi}(e^{i\pi(1-t)}\delta_1) = -e^{i\pi(1-t)}\delta_1^* = e^{-i\pi t}\delta_1^* = f^*(t)$$

so that Γ and $\tilde{\Psi}$ do indeed have the same effect on the generators of $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$, as required.

PROPOSITION 2.4. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A, let $\alpha = \Phi \circ *$ and let $R = \{a \in A : \Phi(a) = a^*\}$. Then there exists a ideal $\mathcal{J}$ in $A \times_\alpha \mathbf{Z}$ such that $\mathcal{J}$ is $*$-isomorphic to $C_0((0, 1), M_2(A))$ and $(A \times_\alpha \mathbf{Z})/\mathcal{J}$ is $*$-isomorphic to $M_2(R) \oplus (R \otimes \mathcal{H})$, where $\mathcal{H}$ denotes the algebra of quaternions.*

Proof. By Propositions 2.3 and 2.2 (ii), $A \times_\alpha \mathbf{Z}$ is $*$-isomorphic to

$$M_\Psi = \{f \in C([0, 1], M_2(A)) : f(1) = \hat{\alpha}(f(0)),\ f(t) = (\tilde{\Psi}\hat{\alpha})(f(1-t)^*) \quad \text{for each } 0 \leqslant t \leqslant 1\},$$

where

$$\hat{\alpha}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

and

$$\tilde{\Psi}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Phi(d) & \Phi(b) \\ \Phi(c) & \Phi(a) \end{pmatrix}$$

for each a, b, c, d in A. Let $\mathcal{J}$ be the ideal in $A \times_\alpha \mathbf{Z}$ corresponding to the kernel of the $*$-homomorphism $f \mapsto (f(0), f(1/2))$ on M_Ψ. Since each element of M_Ψ is determined by its values on $[0, 1/2]$, it is easily seen that $\mathcal{J}$ is $*$-isomorphic to $C_0((0, 1), M_2(A))$. It is also clear that $(A \times_\alpha \mathbf{Z})/\mathcal{J}$ is $*$-isomorphic to

$$\{x \in M_2(A) : x = \tilde{\Psi}(x^*)\} \oplus \{x \in M_2(A) : x = (\tilde{\Psi}\hat{\alpha})(x^*)\}.$$

By Proposition 2.2 (iii), the first summand is $*$-isomorphic to $M_2(R)$. The second summand is the real algebra associated with the involutory $*$-antiautomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \Phi(d) & -\Phi(b) \\ -\Phi(c) & \Phi(a) \end{pmatrix}$$

of $M_2(A)$. This antiautomorphism can be identified with $\Phi \otimes \Phi_{\mathscr{H}}$ on $A \otimes M_2(\mathbf{C})$, where

$$\Phi_{\mathscr{H}}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the real algebra associated with $\Phi_{\mathscr{H}}$ is $*$-isomorphic to the algebra $\mathscr{H}$ of quaternions, the second summand is $*$-isomorphic to $R \otimes \mathscr{H}$, as required.

3. A PIMSNER-VOICULESCU EXACT SEQUENCE FOR REAL C^*-ALGEBRAS

In this section we will show that Cuntz's refinement [7] of the proof given by Pimsner and Voiculescu in [16] carries over to produce an exact sequence for crossed products of real C^*-algebras. The result is considerably weaker than the complex one (since real Bott periodicity is of order 8 rather than 2) but it is sufficient to obtain useful information about $K_0(A \times_\alpha \mathbf{Z})$, where α is the action discussed in Proposition 2.4.

Throughout this section O_2 will denote the C^*-algebra generated by two isometries S_1, S_2 (on an infinite dimensional Hilbert space) satisfying $S_1^* S_1 = S_2^* S_2 = 1$ and $S_1 S_1^* + S_2 S_2^* = 1$. It was shown in [6] that O_2 is an infinite simple C^*-algebra which does not depend on the choice of the isometries S_1 and S_2 satisfying the given relations. Γ will denote the involutory $*$-antiautomorphism of O_2 specified on the generators S_1, S_2 by $\Gamma(S_i) = S_i^*$; the associated real algebra is the real C^*-algebra generated by S_1 and S_2.

Let A be a real C^*-algebra with complexification $A^{\mathbf{C}}$ and let α be a $\mathbf{Z}$-action on both A and $A^{\mathbf{C}}$. Following [7], $\mathscr{E}$ will denote the C^*-subalgebra of $(A^{\mathbf{C}} \times_\alpha \mathbf{Z}) \otimes O_2$ generated by $\{x\delta_0 \otimes 1, \delta_1 \otimes S_1 : x \in A^{\mathbf{C}}\}$ and $\hat{\mathscr{E}}$ will denote the C^*-subalgebra of $(A^{\mathbf{C}} \times_\alpha \mathbf{Z}) \otimes O_2$ generated by $\mathscr{E}$ and $\varphi(\mathscr{E})$, where φ is the $*$-homomorphism from $\mathscr{E}$ into $(A^{\mathbf{C}} \times_\alpha \mathbf{Z}) \otimes O_2$ specified by

$$\varphi(x\delta_0 \otimes 1) = x\delta_0 \otimes 1$$

and

$$\varphi(\delta_1 \otimes S_1) = \delta_1 \otimes (S_1^2 S_1^* + S_2 S_1 S_2^*)$$

for each $x \in A^{\mathbf{C}}$. Furthermore, also following [7], $\beta: \mathscr{E} \to \hat{\mathscr{E}}$ will be the $*$-homomorphism defined for each $x \in \mathscr{E}$ by

$$\beta(x) = (\delta_0 \otimes S_2)x(\delta_0 \otimes S_2)^*,$$

$\mathscr{I}$ will be the closed ideal in $\mathscr{E}$ generated by $\{\beta(x\delta_0 \otimes 1): x \in A^{\mathbf{C}}\}$ and $\hat{\mathscr{I}}$ will be the closed ideal in $\hat{\mathscr{E}}$ generated by $\{\beta(y) : y \in \mathscr{E}\}$. Note that, as remarked in [7], $\hat{\mathscr{E}}$ is the C^*-subalgebra of $(A^{\mathbf{C}} \times_\alpha \mathbf{Z}) \otimes O_2$ generated by $\mathscr{E}$ and $\beta(\mathscr{E})$.

LEMMA 3.1. *Let A be a real C^*-algebra with complexification $A^{\mathbf{C}}$, let Ψ be the associated involutory $*$-antiautomorphism of $A^{\mathbf{C}}$, let α be a $\mathbf{Z}$ action on both A and $A^{\mathbf{C}}$ and let $\tilde{\Psi}$ be the involutory $*$-antiautomorphism of $A^{\mathbf{C}}\times_\alpha \mathbf{Z}$ defined in Proposition 2.1. Then $\tilde{\Psi} \otimes \Gamma$ on $(A^{\mathbf{C}}\times_\alpha \mathbf{Z})\otimes O_2$ restricts to involutory $*$-antiautomorphisms of the subalgebras $\mathscr{E}$, $\hat{\mathscr{E}}$, $\mathscr{I}$ and $\hat{\mathscr{I}}$.*

Proof. The effect of $\tilde{\Psi} \otimes \Gamma$ on the appropriate generators is given by

$$(\tilde{\Psi} \otimes \Gamma)(x\delta_0 \otimes 1) = \Psi(x)\delta_0 \otimes 1$$

$$(\tilde{\Psi} \otimes \Gamma)(\delta_1 \otimes S_1) = (\delta_1 \otimes S_1)^*$$

$$(\tilde{\Psi} \otimes \Gamma)\varphi(\delta_1 \otimes S_1) = \varphi(\delta_1 \otimes S_1)^*$$

$$(\tilde{\Psi} \otimes \Gamma)\beta(x\delta_0 \otimes 1) = \beta(\Psi(x)\delta_0 \otimes 1)$$

$$(\tilde{\Psi} \otimes \Gamma)\beta(y) = \beta(\tilde{\Psi} \otimes \Gamma)(y)$$

for each $x \in A^{\mathbf{C}}$ and $y \in \mathscr{E}$.

PROPOSITION 3.2. *Let A, α, β, $\tilde{\Psi}$, Γ, $\mathscr{E}$ and $\hat{\mathscr{E}}$ be as in Lemma 3.1 and let the real algebras associated with the restrictions of $\tilde{\Psi} \otimes \Gamma$ to $\mathscr{E}$ and $\hat{\mathscr{E}}$ be $\mathscr{E}_{\mathbf{R}}$ and $\hat{\mathscr{E}}_{\mathbf{R}}$. Then there exists a diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K}_{\mathbf{R}} \otimes_{\mathbf{R}} A & \longrightarrow & \mathscr{E}_{\mathbf{R}} & \longrightarrow & A \times_\alpha \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathscr{K}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathscr{E}_{\mathbf{R}} & \longrightarrow & \hat{\mathscr{E}}_{\mathbf{R}} & \longrightarrow & A \times_\alpha \mathbf{Z} & \longrightarrow & 0 \end{array}$$

which is commutative with exact rows, where $\mathscr{K}_{\mathbf{R}}$ denotes the algebra of compact operators on a separable real Hilbert space, j denotes the inclusion from $\mathscr{E}_{\mathbf{R}}$ into $\hat{\mathscr{E}}_{\mathbf{R}}$ and k denotes the mapping $x \mapsto x\delta_0 \otimes 1$ from A into $\mathscr{E}_{\mathbf{R}}$.

Furthermore, the K-theoretic map from $\mathrm{K}_*(\mathscr{E}_{\mathbf{R}})$ into $\mathrm{K}_*(\hat{\mathscr{E}}_{\mathbf{R}})$ arising from the second row can be identified with β_*.

Proof. This is a simple adaptation of the proof of Proposition 1.1 of [7]. The major points are as follows. From Proposition 1.1 of [7] there exists a $*$-isomorphism from $\mathscr{I}$ onto $\mathscr{K} \otimes A^{\mathbf{C}}$ defined by

$$(\delta_1 \otimes S_1)^i\beta(x\delta_0 \otimes 1)(\delta_1 \otimes S_1)^{*j} \mapsto e_{ij} \otimes x$$

for each $x \in A^{\mathbf{C}}$ and a $*$-isomorphism from $\hat{\mathscr{I}}$ onto $\mathscr{K} \otimes \mathscr{E}$ given by

$$(\delta_1 \otimes S_1)^i \beta(y)(\delta_1 \otimes S_1)^{*j} \mapsto e_{ij} \otimes y$$

for each $y \in \mathscr{E}$, where $(e_{ij})_{i,j\geq 0}$ is a set of matrix units in $\mathscr{K}$. In the first case $\tilde{\Psi} \otimes \Gamma$ corresponds to $\mathrm{Tr} \otimes \Psi$ and in the second case to $\mathrm{Tr} \otimes (\tilde{\Psi} \otimes \Gamma)$, where Tr is the involutory $*$-antiautomorphism of $\mathscr{K}$ defined by $\mathrm{Tr}(e_{ij}) = e_{ji}$; hence the associated real algebras are $*$-isomorphic to $\mathscr{K}_{\mathbf{R}} \otimes_{\mathbf{R}} A$ and $\mathscr{K}_{\mathbf{R}} \otimes_{\mathbf{R}} \mathscr{E}_{\mathbf{R}}$.

The $*$-isomorphism from $\mathscr{E}/\mathscr{I}$ onto $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$ defined in Proposition 1.1 of [7] takes $(x\delta_0 \otimes 1) + \mathscr{I}$ to $x\delta_0$ and $(\delta_1 \otimes S_1) + \mathscr{I}$ to δ_1 for each $x \in A^{\mathbf{C}}$, from which it follows that $\tilde{\Psi} \otimes \Gamma$ on $\mathscr{E}$ gives rise to $\tilde{\Psi}$ on $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$. Similarly $\tilde{\Psi} \otimes \Gamma$ on $\hat{\mathscr{E}}$ also gives rise to $\tilde{\Psi}$ on $A^{\mathbf{C}} \times_\alpha \mathbf{Z}$ and so, by Proposition 2.1, the associated real algebra is $*$-isomorphic to $A \times_\alpha \mathbf{Z}$.

PROPOSITION 3.3. *Let $k: A \to \mathscr{E}_{\mathbf{R}}$ be defined by $k(x) = x\delta_0 \otimes 1$. Then the homomorphism $k_*: \mathrm{K}_*(a) \to \mathrm{K}_*(\mathscr{E}_{\mathbf{R}})$ is injective.*

Proof. The argument that $k_*: \mathrm{K}_1(A) \to \mathrm{K}_1(\mathscr{E}_{\mathbf{R}})$ is injective is identical to that in Proposition 1.2 of [7]. To obtain the result for other K_n let $C(\mathrm{S}^1)_* = \{f \in C(\mathrm{S}^1) : f^*(t) = f(\bar{t})$ for each $t \in \mathrm{S}^1\}$ and, following the notation of §4 of [5], let $C_0^{\mathbf{R}}(\mathrm{i}\mathbf{R})$ denote the kernel of the evaluation map at 1 on $C(\mathrm{S}^1)_*$. Then, for each real C^*-algebra B, $\mathrm{K}_n(C(\mathrm{S}^1)_* \otimes_{\mathbf{R}} B)$ is isomorphic to $\mathrm{K}_n(\mathbf{B}) \oplus \mathrm{K}_n(C_0^{\mathbf{R}}(\mathrm{i}\mathbf{R}) \otimes_{\mathbf{R}} B)$ and, by the remark at the end of §4 of [5], $\mathrm{K}_n(C_0^{\mathbf{R}}(\mathrm{i}\mathbf{R}) \otimes_{\mathbf{R}} B)$ is isomorphic to $\mathrm{K}_{n-1}(B)$. The map $\mathrm{id} \otimes k_*$ from $\mathrm{K}_n(C(\mathrm{S}^1)_* \otimes_{\mathbf{R}} A)$ to $\mathrm{K}_n(C(\mathrm{S}^1)_* \otimes_{\mathbf{R}} \mathscr{E}_{\mathbf{R}})$ corresponds to the map $k_* \oplus k_*$ from $\mathrm{K}_n(A) \oplus \mathrm{K}_{n-1}(A)$ into $\mathrm{K}_n(\mathscr{E}_{\mathbf{R}}) \oplus \mathrm{K}_{n-1}(\mathscr{E}_{\mathbf{R}})$ and it therefore follows that k_* is injective on $\mathrm{K}_r(A)$ for each $r \leq 1$. The result for $r \geq 1$ can be obtained by periodicity or from a consideration of $\mathrm{K}_n(C(\mathrm{S}^1, \mathbf{R}) \otimes_{\mathbf{R}} B)$.

PROPOSITION 3.4. *The homomorphism $\varphi \oplus j$ from $\mathscr{E}_{\mathbf{R}}$ to $M_2(\hat{\mathscr{E}}_{\mathbf{R}})$ defined by*

$$(\varphi \oplus j)(x) = \begin{pmatrix} \varphi(x) & 0 \\ 0 & j(x) \end{pmatrix}$$

is homotopic to $j \oplus j$ in the topology of pointwise norm convergence in $M_2(\hat{\mathscr{E}}_{\mathbf{R}})$.

Proof. As in Proposition 1.3 of [7] let $W = \sum S_i S_j S_i^* S_j^* \in O_2$. As noted in 1.4 of [6] the summands of W form part of a self-adjoint system of 4×4 matrix units in O_2; for definiteness we will take the set defined by

$$e_{11} = S_1^2 S_1^{*2},\ e_{12} = S_1^2 S_2^* S_1^*,\ e_{13} = S_1^2 S_1^* S_2^*,\ e_{14} = S_1^2 S_2^{*2},$$

$$e_{21} = S_1 S_2 S_1^{*2},\ e_{22} = S_1 S_2 S_2^* S_1^*,\ e_{23} = S_1 S_2 S_1^* S_2^*,\ e_{24} = S_1 S_2 S_2^{*2},$$

$$e_{31} = S_2 S_1 S_1^{*2},\ e_{32} = S_2 S_1 S_2^* S_1^*,\ e_{33} = S_2 S_1 S_1^* S_2^*,\ e_{23} = S_2 S_1 S_2^{*2},$$

$$e_{41} = S_2^2 S_1^{*2},\ e_{42} = S_2^2 S_2^* S_1^*,\ e_{43} = S_2^2 S_1^* S_2^*,\ e_{44} = S_2^2 S_2^{*2}.$$

Then $W = e_{11} + e_{23} + e_{32} + e_{44}$; let $X = e_{11} + e_{22} - e_{33} + e_{44}$ and, for $0 \leqslant t \leqslant \pi/2$, let u_t be the element of $M_2(O_2)$ defined by

$$u_t^{11} = e_{11} + e_{22}\sin t + e_{23}\cos t + e_{32}\cos 2t \cos t - e_{33}\cos 2t \sin t + e_{44}$$

$$u_t^{12} = - e_{33}\sin 2t$$

$$u_t^{21} = -e_{32}\sin 2t \cos t + e_{33}\sin 2t \sin t$$

$$u_t^{22} = e_{11} + e_{22} - e_{33}\cos 2t + e_{44}.$$

The elements u_t $(0 \leqslant t \leqslant \pi/2)$ form a continuous path of unitaries in $M_2(O_2)$ with

$$u_0 = \begin{pmatrix} W & 0 \\ 0 & X \end{pmatrix} \quad \text{and} \quad u_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It was shown in Proposition 1.3 of [7] that each of $\delta_0 \otimes e_{11}$, $\delta_0 \otimes e_{23}$, $\delta_0 \otimes e_{32}$ and $\delta_0 \otimes e_{44}$ belong to $\hat{\mathscr{E}}$ (and in fact to $\hat{\mathscr{E}}_{\mathbf{R}}$). Similarly,

$$\delta_0 \otimes e_{22} = \delta_0 \otimes S_1(1 - S_1S_1^*)S_1^* \in \hat{\mathscr{E}}_{\mathbf{R}}$$

and

$$\delta_0 \otimes e_{33} = \beta(\delta_0 \otimes S_1S_1^*) \in \hat{\mathscr{E}}_{\mathbf{R}}.$$

Hence $\delta_0 \otimes u_t \in M_2(\hat{\mathscr{E}}_{\mathbf{R}})$ for each t and so $\delta_0 \otimes u_0$ is homotopic to the identity in $M_2(\hat{\mathscr{E}}_{\mathbf{R}})$.

As in Proposition 1.3 of [7] we define homomorphisms $\varphi_t \colon \mathscr{E}_{\mathbf{R}} \to M_2(\hat{\mathscr{E}}_{\mathbf{R}})$ by

$$\varphi_t(x\delta_0 \otimes 1) = \operatorname{diag}(x\delta_0 \otimes 1,\ x\delta_0 \otimes 1)$$

and

$$\varphi_t(\delta_1 \otimes S_1) = (\delta_0 \otimes u_t)\operatorname{diag}(\delta_1 \otimes S_1,\ \delta_1 \otimes S_1)$$

for $x \in A$. Then $\{\varphi_t : 0 \leqslant t \leqslant \pi/2\}$ is a continuous path of homomorphisms from $\mathscr{E}_{\mathbf{R}}$ to $M_2(\hat{\mathscr{E}}_{\mathbf{R}})$ with

$$\varphi_0(x\delta_0 \otimes 1) = (\varphi \oplus j)(x\delta_0 \otimes 1),$$

$$\varphi_0(\delta_1 \otimes S_1) = \operatorname{diag}(\delta_1 \otimes WS_1,\ \delta_1 \otimes XS_1) = (\varphi \oplus j)(\delta_1 \otimes S_1),$$

$$\varphi_1(x\delta_0 \otimes 1) = (j \oplus j)(x\delta_0 \oplus 1),$$

$$\varphi_1(\delta_1 \otimes S_1) = (j \oplus j)(\delta_1 \otimes S_1)$$

for each $x \in A$. Thus $\varphi_0 = \varphi \oplus j$ and $\varphi_1 = j \oplus j$, as required.

COROLLARY. *Let* $j\colon \mathscr{E}_{\mathbf{R}} \to \hat{\mathscr{E}}_{\mathbf{R}}$ *and* $\varphi\colon \mathscr{E}_{\mathbf{R}} \to \hat{\mathscr{E}}_{\mathbf{R}}$ *be as defined above. Then* $j_* = \varphi_*$ *on* $\mathrm{K}_*(\mathscr{E}_{\mathbf{R}})$.

THEOREM 3.5. *Let* α *be an automorphism of a unital real* C^**-algebra* A. *Then there exists an exact sequence*

$$\begin{array}{ccc} \mathrm{K}_0(A) \xrightarrow{\theta} \mathrm{K}_0(A) \to \mathrm{K}_0(A \times_\alpha \mathbf{Z}) \to \mathrm{K}_7(A) \xrightarrow{\theta} \mathrm{K}_7(A) \to \mathrm{K}_7(A \times_\alpha \mathbf{Z}) \to \ldots \to & & \mathrm{K}_5(A \times_\alpha \mathbf{Z}) \\ \uparrow & & \downarrow \\ \mathrm{K}_1(A \times_\alpha \mathbf{Z}) \leftarrow \mathrm{K}_1(A) \xleftarrow{\theta} \mathrm{K}_1(A) \leftarrow \mathrm{K}_1(A \times_\alpha \mathbf{Z}) \leftarrow \mathrm{K}_2(A) \xleftarrow{\theta} \mathrm{K}_2(A) \leftarrow \ldots \leftarrow & & \mathrm{K}_4(A) \end{array}$$

where each map θ *is given by* $\mathrm{id} - \alpha_*^{-1}$.

Proof. As in the proof of Theorem 1.5 of [7] the exact sequences of Proposition 3.2 give rise to the following long exact sequences.

$$\begin{array}{ccccccccccccc} \ldots \to & \mathrm{K}_1(A) & \to & \mathrm{K}_1(\mathscr{E}_{\mathbf{R}}) & \to & \mathrm{K}_1(A \times_\alpha \mathbf{Z}) & \to & \mathrm{K}_0(A) & \to & \mathrm{K}_0(\mathscr{E}_{\mathbf{R}}) & \to & \mathrm{K}_0(A \times_\alpha \mathbf{Z}) & \to \ldots \\ & \downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & & \downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & \\ \ldots \to & \mathrm{K}_1(\mathscr{E}_{\mathbf{R}}) & \xrightarrow{\beta_*} & \mathrm{K}_1(\hat{\mathscr{E}}_{\mathbf{R}}) & \to & \mathrm{K}_1(A \times_\alpha \mathbf{Z}) & \to & \mathrm{K}_0(\mathscr{E}_{\mathbf{R}}) & \xrightarrow{\beta_*} & \mathrm{K}_0(\hat{\mathscr{E}}_{\mathbf{R}}) & \to & \mathrm{K}_0(A \times_\alpha \mathbf{Z}) & \to \ldots \end{array}$$

Furthermore, defining $\hat{\alpha} = \mathrm{Ad}(\delta_1 \otimes 1)$ on $\hat{\mathscr{E}}_{\mathbf{R}}$, the proof of Proposition 1.4 of [7] applies without change to yield $\beta_* = \varphi_* - \hat{\alpha}_*^{-1} j_*$ on $\mathrm{K}_0(\mathscr{E}_{\mathbf{R}})$, from which it follows by the corollary above that $\beta_* = j_* - \hat{\alpha}_*^{-1} j_*$. The argument in Proposition 1.4 of [7] also yields this formula on $\mathrm{K}_1(\mathscr{E}_{\mathbf{R}})$, but we will instead use an argument similar to that used in Proposition 3.3 to obtain the formula on $\mathrm{K}_n(\mathscr{E}_{\mathbf{R}})$ for all n.

If we replace A by $A \otimes_{\mathbf{R}} C(\mathrm{S}^1)_*$ where $C(\mathrm{S}^1)_* = \{f \in C(\mathrm{S}^1) : f^* = f \circ \mathrm{id}^*\}$ and α by $\alpha \otimes \mathrm{id}$ then $\mathscr{E}_{\mathbf{R}}$ is replaced by $\mathscr{E}_{\mathbf{R}} \otimes_{\mathbf{R}} C(\mathrm{S}^1)_*$, $\hat{\mathscr{E}}_{\mathbf{R}}$ by $\hat{\mathscr{E}}_{\mathbf{R}} \otimes_{\mathbf{R}} C(\mathrm{S}^1)_*$, φ by $\varphi \otimes \mathrm{id}$, β by $\beta \otimes \mathrm{id}$, j by $j \otimes \mathrm{id}$ and $\hat{\alpha}$ by $\hat{\alpha} \otimes \mathrm{id}$. Hence $(\beta \otimes \mathrm{id})_* = (j \otimes \mathrm{id})_* - (\hat{\alpha} \otimes \mathrm{id})_*^{-1}(j \otimes \mathrm{id})_*$ on $\mathrm{K}_0(\mathscr{E}_{\mathbf{R}})$. However, as in the proof of Proposition 3.3, for any homomorphism $\gamma\colon \mathscr{E}_{\mathbf{R}} \to \hat{\mathscr{E}}_{\mathbf{R}}$, $(\gamma \otimes \mathrm{id})_*$ on $\mathrm{K}_n(\mathscr{E}_{\mathbf{R}} \otimes_{\mathbf{R}} C(\mathrm{S}^1)_*)$ corresponds to $\gamma_* \oplus \gamma_*$ on $\mathrm{K}_n(\mathscr{E}_{\mathbf{R}}) \oplus \mathrm{K}_{n-1}(\mathscr{E}_{\mathbf{R}})$. Therefore, by successive applications of this technique and periodicity, $\beta_* = j_* - \hat{\alpha}_*^{-1} j_*$ on $\mathrm{K}_n(\mathscr{E}_{\mathbf{R}})$ for all n.

Finally the homological algebra used in the proof of Theorem 1.5 of [7] shows that the vertical maps k_* and j_* in the diagram above are both isomorphisms. Identifying $\mathrm{K}_*(A)$ with $\mathrm{K}_*(\mathscr{E}_{\mathbf{R}})$ via k_* and defining $\theta = k_*^{-1} j_*^{-1} \beta_* k_*$, the top line of the diagram combined with real Bott periodicity gives the required result.

REMARK. An alternative, somewhat shorter, proof of Theorem 3.5 can be obtained by adapting Propositions 5.5 and 5.6 of [5] to the real context, bearing in mind the comment at the end of § 4 of [5] about the real versions of the results in that section. However such a proof relies on the heavy machinery of Kasparov's (real) KK-theory [14] and we have preferred a simpler if somewhat longer approach.

4. THE COMPLEXIFICATION AND REALIFICATION MAPS IN K-THEORY

Let Φ be an involutory $*$-antiautomorphism of a C^*-algebra A and let $R = \{a \in A\colon \Phi(a) = a^*\}$. The embedding of R in A and the map $r + is \mapsto \begin{pmatrix} r & s \\ -s & r \end{pmatrix}$ from A into $M_2(R)$ give rise to the K-theoretic complexification map $c\colon \mathrm{K}_0(R) \to \mathrm{K}_0(A)$ and realification map $r\colon \mathrm{K}_0(A) \to \mathrm{K}_0(R)$. As described in Theorem III.2.7 of [13] (which covers the commutative case but carries over to the current situation), the composite maps are given by $r \circ c = 2\mathrm{id}$ and $c \circ r = \mathrm{id} + \alpha_*$, where $\alpha = \Phi \circ *$. If $\mathrm{K}_0(R)$ contains no non-zero 2-torsion elements then the equation $r \circ c = 2\mathrm{id}$ implies that c is injective; if $\alpha_* = \mathrm{id}$ and $\mathrm{K}_0(A)$ contains no non-zero 2-torsion elements then, similarly, the equation $c \circ r = \mathrm{id} + \alpha_*$ implies that r is injective. If $\alpha_* \neq \mathrm{id}$ then, as illustrated by the involutory $*$-antiautomorphism $(a, b) \mapsto (b, a)$ on $A = \mathbf{C}^2$ (for which $r\colon \mathbf{Z}^2 \to \mathbf{Z}$ is given by $r(a, b) = a + b$), the realification map need not be injective even if $\mathrm{K}_0(A)$ is torsion free. It may also be the case, as illustrated by $\Phi = \mathrm{id}$ on $A = C_0(\mathbf{R}^2, \mathbf{C})$ (for which, by Theorem III.5.19 of [13], $c\colon \mathbf{Z}_2 \to \mathbf{Z}$ and for which $\alpha_* = -\mathrm{id}$) that $\mathrm{K}_0(A)$ is torsion free but c is not injective. Nevertheless, we will prove, using Proposition 2.4 and Theorem 3.5, that if $\mathrm{K}_0(A)$ contains no non-zero 2-torsion elements, $\mathrm{K}_1(A) = 0$ and $\alpha_* = \mathrm{id}$ then both r and c are injective. The first step is to note a K-theoretical consequence of Proposition 2.4.

PROPOSITION 4.1. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A, let $\alpha = \Phi \circ *$ and let $R = \{a \in A\colon \Phi(a) = a^*\}$. Then there exists an exact sequence*

$$\mathrm{K}_1(A) \to \mathrm{K}_0(A \times_\alpha \mathbf{Z}) \to \mathrm{K}_0(R) \oplus \mathrm{K}_0(R \otimes \mathscr{H}) \to \mathrm{K}_0(A).$$

Proof. By Proposition 2.4 there exists an exact sequence

$$0 \to C_0((0, 1), M_2(A)) \to A \times_\alpha \mathbf{Z} \to M_2(R) \oplus (R \otimes \mathscr{H}) \to 0.$$

The long exact sequence of K-theory obtained from this sequence includes the terms

$$\mathrm{K}_0(C_0(0, 1),\ M_2(A)) \to \mathrm{K}_0(A \times_\alpha \mathbf{Z}) \to \mathrm{K}_0(M_2(R)) \oplus \mathrm{K}_0(R \otimes \mathscr{H}) \to$$
$$\to \mathrm{K}_{-1}(C_0(0, 1),\ M_2(A)).$$

From Theorem 8.2.2 of [1], $\mathrm{K}_0(C_0(0, 1), M_2(A))$ is isomorphic to $\mathrm{K}_1(M_2(A))$ and K_{-1} is defined using Bott periodicity so that $\mathrm{K}_{-1}(C(0, 1), M_2(A))$ is isomorphic to $\mathrm{K}_0(M_2(A))$. The result then follows from the stability of K_0 and K_1.

THEOREM 4.2. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A such that $\mathrm{K}_1(A) = 0$, let $\alpha = \Phi \circ *$, let $R = \{a \in A\colon \Phi(a) = a^*\}$ and let $\alpha_* = \mathrm{id}$ on $\mathrm{K}_0(A)$.*

(i) *There exists an exact sequence*

$$0 \to \mathrm{K}_0(A) \xrightarrow{\theta} \mathrm{K}_0(R) \oplus \mathrm{K}_0(R \otimes \mathscr{H}) \xrightarrow{\pi} \mathrm{K}_0(A).$$

(ii) *If in addition* $\mathrm{K}_0(A)$ *has no non-zero 2-torsion elements then the complexification map* $c: \mathrm{K}_0(R) \to \mathrm{K}_0(A)$ *and the realification map* $r: \mathrm{K}_0(A) \to \mathrm{K}_0(R)$ *are both injective.*

Proof. (i) This is an immediate consequence of Theorem 3.5 and Proposition 4.1.

(ii) If $\mathrm{K}_0(A)$ has no 2-torsion elements and $\alpha_* = \mathrm{id}$ then it follows from the equation $c \circ r = \mathrm{id} + \alpha_*$ that r is injective. From the exact sequence in (i), any 2--torsion element in $\mathrm{K}_0(R) \oplus \mathrm{K}_0(R \otimes \mathscr{H})$ must lie in the kernel of π and hence in the image of θ. However, since θ is an isomorphism, there are no non-zero 2-torsion elements in the image of θ. Hence $\mathrm{K}_0(R)$ has no 2-torsion elements and then, from the equation $r \circ c = 2\mathrm{id}$, it follows that the complexification map c is injective.

5. A CANCELLATION THEOREM FOR REAL K-THEORY

If e, f are projections in a real or complex unital C^*-algebra A for which $[e] = [f]$ in $\mathrm{K}_0(A)$ then there exists a projection x in some matrix algebra over A such that the matrices

$$\begin{pmatrix} e & 0 \\ 0 & x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f & 0 \\ 0 & x \end{pmatrix}$$

are equivalent. In order to deduce that e is equivalent to f a cancellation theorem is required. The foundations for such theorems in C^*-algebras were laid in [17], where the topological stable range $\mathrm{tsr}(A)$ of A was defined. The simplest situation occurs when $\mathrm{tsr}(A) = 1$ (i.e. when the invertible elements in A are dense). However the property of having topological stable rank one does not transfer to real structures in A, as is illustrated by the real algebra $C([0, 1], \mathbf{R})$ in $C([0, 1], \mathbf{C})$. Since $[0, 1]$ is one dimensional it follows from Proposition 1.7 of [17] that $\mathrm{tsr}(C([0, 1], \mathbf{C})) = 1$; however the function $t \mapsto 2t - 1$ cannot be approximated by invertible elements in $C([0, 1], \mathbf{R})$ and so $\mathrm{tsr}(C([0, 1], \mathbf{R})) \neq 1$. Nevertheless, there is a connection between the topological stable rank of a real algebra and its complexification, given by the following proposition.

PROPOSITION 5.1. *Let* Φ *be an involutory* $*$*-antiautomorphism in a unital* C^**-algebra* A, *let* $\alpha = \Phi \circ *$ *and let* $R = \{a \in A : \Phi(a) = a^*\}$. *If* $\mathrm{tsr}(A) \leqslant k$ *then* $\mathrm{tsr}(R) \leqslant 2k + 1$.

Proof. The proof uses a number of results from [17] which, although proved for complex algebras, remain true, with the same proofs, for real C^*-algebras. We will refer to such results as the 'real versions' of the corresponding theorems from [17].

By the real version of Theorem 7.1 of [17], $\text{tsr}(A \times_\alpha \mathbf{Z}) \leqslant k + 1$. However, by Proposition 2.4, there is an ideal I in $A \times_\alpha \mathbf{Z}$ with quotient $M_2(R)$ and hence, by the real version of Theorem 4.3 of [17], $\text{tsr}(M_2(R)) \leqslant k + 1$. The real version of Theorem 6.1 of [17] then yields $\text{tsr}(R) \leqslant 2k + 1$, as required.

Proposition 5.1 enables the following version of the cancellation theorem of [2] to be proved.

THEOREM 5.2. *Let A be a simple unital stably finite C^*-algebra for which there is a constant $k < \infty$ such that* $\text{tsr}(pM_n(A)p) \leqslant k$ *for all n and all projections p in $M_n(A)$. Let Φ be an involutory $*$-antiautomorphism of A and let $R = \{a \in A : \Phi(a) = a^*\}$. If* $\mathrm{K}_0(R)$ *contains arbitrarily small positive elements, then R has cancellation.*

Proof. Let n be a natural number and let p be a projection in $M_n(R)$. Then, applying Proposition 5.1 to the real algebra $pM_n(R)p$ in $pM_n(A)p$, $\text{tsr}(pM_n(R)p) \leqslant \leqslant 2k + 1$ and hence, by the real version of Theorem 2.3 of [17], the Bass stable rank $\text{Bsr}(pM_n(R)p)$ of $pM_n(R)p$ satisfies $\text{Bsr}(pM_n(R)p) \leqslant 2k + 1$.

It is clear that R is a simple unital stably finite real C^*-algebra and hence it obeys the conditions of Theorem A.1 in [2]. However the proof of that theorem (which is just the proof of Theorem 2.2 of [18]) applies to real as well as complex algebras, giving the required result.

6. STABILITY OF INVOLUTORY $*$-ANTIAUTOMORPHISMS

We are now in a position to establish the stability of each involutory $*$-antiautomorphism in certain C^*-algebras. As outlined in the introduction, the method is to adapt to antiautomorphisms the technique used by Connes in the proof of Corollary 2.6 of [4].

LEMMA 6.1. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A and let u be a unitary in A satisfying $\Phi(u) = u$. Then the map Ψ defined by*

$$\Psi = \left[\mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\right] \circ (\Phi \otimes \mathrm{Tr}),$$

where Tr *is the transpose map on $M_2(\mathbf{C})$, is an involutory $*$-antiautomorphism of $M_2(A)$.*

Proof. Clearly Ψ is a $*$-antiautomorphism. It is involutory since, for each $a, b, c, d \in A$,

$$\Psi^2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Psi\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} \Phi(a) & \Phi(c) \\ \Phi(b) & \Phi(d) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} =$$

$$= \Psi\begin{pmatrix} \Phi(a) & \Phi(c)u^* \\ u\Phi(b) & u\Phi(d)u^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} a & b\Phi(u) \\ \Phi(u^*)c & \Phi(u^*)d\Phi(u) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} =$$

$$= \begin{pmatrix} a & b\Phi(u)u^* \\ u\Phi(u^*)c & u\Phi(u^*)d\Phi(u)u^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

LEMMA 6.2. *Let Φ be an involutory $*$-antiautomorphism of a unital C^*-algebra A, let u be a unitary in A satisfying $\Phi(u) = u$ and let Ψ be the involutory $*$-antiautomorphism of $M_2(A)$ defined in Lemma 6.1. If the projections e_{11} and e_{22} are equivalent in the real algebra $\{x \in M_2(A)\colon \Psi(x) = x^*\}$ then there exists a unitary v in A such that $u = v\Phi(v)$.*

Proof. A simple calculation shows that if $yy^* = e_{11}$ and $y^*y = e_{22}$ (where e_{11} and e_{22} denote the obvious matrix units in $M_2(A)$), then y must be of the form

$$y = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

where x is a unitary in A. The condition $\Psi(y) = y^*$ corresponds to the equation $u\Phi(x) = x^*$ so, setting $v = x^*$, $u = v\Phi(v)$, as required.

LEMMA 6.3. *Let Φ be an involutory $*$-antiautomorphism of a* UHF *algebra A with infinitely many 2×2 matrix factors and let $R = \{a \in A : \Phi(a) = a^*\}$. Then $K_0(R)$ contains arbitrarily small positive elements* i.e. *for each $u > 0$ in $K_0(R)$ there exists $v > 0$ in $K_0(R)$ with $2v \leqslant u$.*

Proof. It is shown at the end of Chapter 6 of [8] that $K_0(A)$ is isomorphic to the subgroup $G(n) = \{a/b\colon a \in \mathbf{Z},\ b \in \mathbf{N},\ b \mid n\}$ of $\mathbf{Q}$ for some "generalized natural number" n which, by hypothesis, will contain a "factor" 2^∞. The identity of A corresponds to $1 \in G(n)$ and, since $\Phi \circ *$ is unital, it follows that $(\Phi \circ *)_* = \mathrm{id}$ on $K_0(A)$. As shown in Corollary 9.2 of [8], $K_1(A) = 0$ and hence, by Theorem 4.2 (ii), the complexification maps $c\colon K_0(R) \to K_0(A)$ and $r\colon K_0(A) \to K_0(R)$ are injective. From the equation $c \circ r = 2\,\mathrm{id}$ and the hypothesis that A contains infinitely many 2×2 matrix factors, $c \circ r$ (and hence c) is surjective. Thus c is an isomorphism from $K_0(R)$ onto $K_0(A)$ and so $K_0(R)$ is isomorphic to $G(n)$. Then, as for c, the realification map r is an isomorphism from $K_0(A)$ onto $K_0(R)$. Since r is positive (and can be identified with a map $a \mapsto \alpha a$ on $G(n)$ for some $\alpha \in \mathbf{Q}^+$) the positive cone of $K_0(R)$ can be identified with $G(n) \cap \mathbf{Q}^+$, which immediately yields the required result.

REMARK. It is possible that Lemma 6.3 holds even if A does not have infinitely many 2×2 matrix factors. The difficulty is to show that possibilities such as $G(3^\infty)$ with positive cone $\{a + (2b/3^d): a, b, d \in \mathbf{N} \cup \{0\}\}$ cannot occur for $K_0(R)$; they are not eliminated by the positivity of r and c and the equations $r \circ c = 2\,\mathrm{id}$, $c \circ r = 2\,\mathrm{id}$.

THEOREM 6.4. *Let Φ be an involutory $*$-antiautomorphism of a* UHF *algebra A with infinitely many 2×2 matrix factors and let u be a unitary in A with $\Phi(u) = u$. Then there exists a unitary v in A such that $u = v\Phi(v)$.*

Proof. Let $B = M_2(A)$, let Ψ be the involutory $*$-antiautomorphism of B defined in Lemma 6.1 and let $R = \{b \in B: \Psi(b) = b^*\}$. Then B is a UHF algebra (with infinitely many 2×2 matrix factors) and, as in the proof of Lemma 6.3, the conditions of Theorem 4.2 (ii) apply to Ψ on B. Hence the projections e_{11} and e_{22} have equal classes in $K_0(R)$.

The algebra B is a simple unital stably finite C^*-algebra for which $\mathrm{tsr}(pM_n(B)p) = 1$ for all n and all projections p in $M_n(B)$ (by Proposition 3.5 of [17]). By Lemma 6.3, $K_0(R)$ contains arbitrarily small positive elements and so, by Theorem 5.2, R has cancellation.

From the previous two paragraphs it follows that e_{11} and e_{22} are equivalent in the real algebra R and the result then follows by Lemma 6.2.

REMARK. It is clear from the proof of Theorem 6.4 that it applies to all algebras for which $M_2(A)$ satisfies the conditions of Theorems 4.2 (ii) and 5.2 (for every involutory $*$-antiautomorphism Ψ of $M_2(A)$).

7. INVARIANT MATRIX ALGEBRAS IN UHF ALGEBRAS

In this section we will apply Theorem 6.4 to construct Φ-invariant matrix subalgebras for an involutory $*$-antiautomorphism Φ on a UHF algebra A with infinitely many 2×2 matrix factors. The proofs use techniques adapted from [11] and [12].

PROPOSITION 7.1. *Let Φ be an involutory $*$-antiautomorphism of a* UHF *algebra $A = \overline{\bigcup A_n}$ and let e_{ij}^n be a set of matrix units for A_n. Then, for each $n \in \mathbf{N}$, there exists a unitary t such that $\Phi(e_{ij}^n) = te_{ji}^n t^*$ and such that $t^*\Phi(t) \in A_n^c$, where A_n^c denotes the relative commutant of A_n in A.*

Proof. This is essentially the same as the proof of Lemma 7.7 of [8]. Since $\Phi_* = \mathrm{id}$ on $K_0(A)$ it follows by cancellation that there exists a partial isometry v in A such that $\Phi(e_{11}^n) = vv^*$ and $e_{11}^n = v^*v$. Let $t = \sum_i \Phi(e_{1i}^n)ve_{1i}^n$. Then, for each matrix unit e_{ij}^n,

$$te_{ji}^n t^* = \Phi(e_{1j}^n)vv^*\Phi(e_{1i}^n)^* = \Phi(e_{i1}^n e_{11}^n e_{1j}^n) = \Phi(e_{ij}^n).$$

It follows that $tt^* = 1$ and then, from finiteness, $t^*t = 1$ as well.

For each matrix unit e^n_{ij},

$$\Phi(t^*)te^n_{ij}t^*\Phi(t) = \Phi(t^*)\Phi(e^n_{ji})\Phi(t) = \Phi(te^n_{ji}t^*) = e^n_{ij},$$

so that $t^*\Phi(t) \in A^c_n$, as required.

LEMMA 7.2. *Let Φ be an involutory $*$-antiautomorphism of a* UHF *algebra* $A = \overline{\bigcup A_n}$, *let t be a unitary given by Proposition* 7.1 *and let A^c_n contain a set f_{ij} of 2×2 matrix units. Then A^c_n contains a projection f and a unitary v such that $t^*\Phi(f)t = v^*(1-f)v$.*

Proof. Let $\gamma = \mathrm{Ad}\, t^* \circ \Phi$. By Proposition 7.1 γ leaves A_n, and hence the UHF algebra A^c_n, globally invariant. Then, since $\gamma_* = \mathrm{id}$ on $K_0(A^c_n)$, $\gamma(f_{11})$ is equivalent to f_{11} and hence to $1 - f_{11}$ in A^c_n; let $f = f_{11}$ and let $s \in A^c_n$ satisfy $\gamma(f) = ss^*$, $1 - f = s^*s$. Then

$$(\Phi(t^*)\Phi(s^*)t)(\Phi(t^*)\Phi(s^*)t)^* = \Phi(t^*)\Phi(s^*)\Phi(s)\Phi(t) =$$

$$= \Phi(tss^*t^*) = \Phi(t\gamma(f)t^*) = f,$$

so that $(1-f)\Phi(t^*)\Phi(s^*)t = 0$. Hence, letting $v = s^* + \Phi(t^*)\Phi(s^*)t$,

$$v^*(1-f)v = s(1-f)s^* = ss^*ss^* = \gamma(f) = t^*\Phi(f)t,$$

as required.

THEOREM 7.3. *Let Φ be an involutory $*$-antiautomorphism of the* UHF *algebra* $A = \overline{\bigcup A_n}$, *let e^n_{ij} be a set of matrix units in A_n and suppose that the relative commutant A^c_n of A_n in A contains a set of 2×2 matrix units. Then there exists a unitary u in A with $\Phi(u) = u$ such that $\Phi(e^n_{ij}) = ue^n_{ji}u^*$ for each matrix unit e^n_{ij} in A_n.*

Proof. Let t be as in Proposition 7.1, let v, f, γ be as in Lemma 7.2 and let $u = tv^*f + \Phi(tv^*f)$. Clearly $\Phi(u) = u$. To see that $\Phi(e^n_{ij}) = ue^n_{ji}u^*$ notice that the following calculation shows that $t^*u \in A^c_n$:

$$t^*u = v^*f + t^*\Phi(v^*f)\Phi(t) = v^*f + \gamma(v^*f)t^*\Phi(t).$$

Hence $\mathrm{Ad}\, u = \mathrm{Ad}\, t$ on A_n.

To check that u is unitary note that

$$\Phi(tv^*f) = \Phi(tv^* - tv^*(1-f)) =$$

$$= \Phi(tv^* - \Phi(f)tv^*) = \Phi(tv^*)(1-f).$$

Hence

$$uu^* = (tv^*)f(tv^*) + \Phi(tv^*f)\Phi(tv^*f)^* =$$

$$= 1 - \Phi(f) + \Phi(f(tv^*)^*(tv^*)f) = 1 - \Phi(f) + \Phi(f) = 1.$$

From finiteness we also have $u^*u = 1$, as required.

THEOREM 7.4. *Let Φ be an involutory $*$-antiautomorphism of a* UHF *algebra* $A = \overline{\bigcup A_n}$ *with infinitely many* 2×2 *matrix factors. Then* A *possesses an increasing sequence* B_n *of* Φ*-invariant subalgebras such that* B_n *is* $*$*-isomorphic to* A_n *and such that* B_n *has a set of matrix units* f^n_{ij} *with* $\Phi(f^n_{ij}) = f^n_{ji}$ *for each* i, j, n.

Proof. We apply Theorem 7.3 and 6.4 inductively. Notice first that, choosing u such that $\Phi(u) = u$ and $\Phi(e^1_{ij}) = ue^1_{ji}u^*$ and then choosing v_1 such that $u = \Phi(v_1)v_1$, we can define $f^1_{ij} = v_1 e^1_{ij} v_1^*$ and obtain

$$\Phi(f^1_{ij}) = \Phi(v_1^*)\Phi(e^1_{ij})\Phi(v_1) =$$

$$= \Phi(v_1^*)ue^1_{ji}u^*\Phi(v_1) = f^1_{ji} .$$

Hence we can let B_1 be the algebra spanned by the matrix units f^1_{ij}.

To perform the inductive step, firstly choose inductively a self-adjoint set of matrix units e^{n+1}_{ij} in A_{n+1} to be of the form $e^n_{pq}e'^n_{rs}$ where e^n_{pq} is a self-adjoint set of matrix units in A_n and e'^n_{rs} is a set in $A^c_n \cap A_{n+1}$. Assume that $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n$ have been constructed, together with unitaries $v_p \in B^c_{p-1}$ such that B_p is spanned by the matrix units $f^p_{ij} = w_p e^p_{ij} w_p^*$, where $w_p = v_p \ldots v_1$. For each r let $C_r = w_n A_r w_n^*$ so that, by construction, $C_r = B_r$ for $1 \leqslant r \leqslant n$ and $w_n e'^n_{rs} w_n^*$ is a set of matrix units in $C_{n+1} \cap C^c_n$. Applying Theorems 7.3 and 6.4 to the UHF algebra $B^c_n = C^c_n = \overline{\bigcup (C_r \cap C^c_n)}$ there exists a unitary v_{n+1} in B^c_n such that

$$\Phi(w_n e'^n_{rs} w_n^*) = \Phi(v_{n+1})v_{n+1}w_n e'^n_{sr} w_n^* v^*_{n+1}\Phi(v^*_{n+1})$$

i.e. such that

$$\Phi(w_{n+1}e'^n_{rs}w^*_{n+1}) = w_{n+1}e'^n_{sr}w^*_{n+1} .$$

Let B_{n+1} be the algebra spanned by the matrix units $f^{n+1}_{ij} = w_{n+1}e^{n+1}_{ij}w^*_{n+1} = w_{n+1}e^n_{pq}e'^n_{rs}w^*_{n+1}$. Clearly B_{n+1} is $*$-isomorphic to A_{n+1} and, since $v_{n+1} \in B^c_n$, $B_{n+1} \supseteq B_n$. Also

$$\Phi(w_{n+1}e^n_{pq}e'^n_{rs}w^*_{n+1}) = \Phi(w_{n+1}e'^n_{rs}w^*_{n+1})\Phi(w_{n+1}e^n_{pq}w^*_{n+1}) =$$

$$= w_{n+1}e'^n_{sr}w^*_{n+1}\Phi(w_n e^n_{pq} w^*_n) = \qquad \text{(since } v_{n+1} \in B^c_n)$$

$$= w_{n+1}e'^n_{sr}w^*_{n+1}w_n e^n_{pq} w^*_n = w_{n+1}e'^n_{sr}e^n_{qp}w^*_{n+1} = w_{n+1}e^n_{qp}e'^n_{sr}w^*_{n+1} ,$$

as required.

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