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ON COMMUTING ISOMETRIES

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1. INTRODUCTION

Let $(U, V; \mathscr{H})$ be a pair of isometries with domains D_U, D_V and ranges R_U, R_V, respectively, closed subspaces of the Hilbert space $\mathscr{H}$. We say that $(U', V'; \mathscr{F})$ is a *commuting unitary extension* of the pair $(U, V; \mathscr{H})$ if $\mathscr{H} \subset \mathscr{F}$ and U', V' are two commuting unitary operators in $\mathscr{F}$ which extend U and V, respectively. We say that $(U', V'; \mathscr{F})$ is a *minimal extension* of $(U, V; \mathscr{H})$ if in addition $\mathscr{F} = \bigvee_{(n,m) \in \mathbf{Z} \times \mathbf{Z}} U'^n V'^m(\mathscr{H})$, the closure of the linear hull of $\bigcup_{(n,m) \in \mathbf{Z} \times \mathbf{Z}} U'^n V'^m(\mathscr{H})$. Assume that the given pair $(U, V; \mathscr{H})$ satisfies $U^n D_V \subset D_U$ and $U^n R_V \subset D_U$ for $n = 0,1,\ldots$, then

$$(1) \qquad \langle U^n Vf, Vf' \rangle = \langle U^n f, f' \rangle, \quad \forall f, f' \in D_V, \ n = 1,2, \ldots$$

is a necessary and sufficient for the existence of a minimal commuting unitary extension $(U', V'; \mathscr{F})$ of $(U, V; \mathscr{H})$ ([1]). In turn this gives a two dimensional extension of the Sz.-Nagy – Foiaş lifting theorem.

Thus a basic problem is to describe the set $\mathscr{U}_{U,V}$ of all equivalence classes of minimal unitary extensions of $(U, V; \mathscr{H})$, where two minimal unitary extensions $(U', V', \mathscr{F}')$ and $(U'', V''; \mathscr{F}'')$ are said equivalent if there exists an unitary isomorphism φ of $\mathscr{F}'$ onto $\mathscr{F}''$ which leaves invariant the elements of $\mathscr{H}$ and satisfies $\varphi U'^n V'^m = U''^n V''^m \varphi$, $\forall n, m \in \mathbf{Z}$.

The problem was solved by Arov-Grossman for the case where U is the identity of $\mathscr{H}$ (cf. [2]). We use some ideas of Sz.-Nagy – Foiaş [5].

The results of this work may be applied to give a description of all positive Toeplitz extensions of a positive Toeplitz-Krein-Cotlar form, defined in [1], and also to the work of M. Cotlar and C. Sadosky [4].

§ 2

We start with the easiest case of the problem: Let us describe $\mathcal{U}_{U,V}$ for $U:\mathcal{H}\to\mathcal{H}$ a unitary operator, D_V and R_V closed subspaces of $\mathcal{H}$, with $U(D_V) = D_V$, $U(R_V) = R_V$ and $V: D_V \to R_V$ an isometry such that $UV = VU|D_V$.

In this part we set $\mathcal{N} = \mathcal{N}_V = \mathcal{H} \ominus D_V$, $\mathcal{M} = \mathcal{M}_V = \mathcal{H} \ominus R_V$. It is easy to see that $U(\mathcal{N}) = \mathcal{N}$, $U(\mathcal{M}) = \mathcal{M}$ and that

PROPOSITION 1. *There exists a minimal commuting unitary extension $(U', V'; \mathcal{F})$ of the above pair $(U, V; \mathcal{H})$, if and only if there exist a minimal unitary extension $(V'; \mathcal{F})$ of V and a unitary extension $(U'; \mathcal{F})$ of U such that U' commutes with V'. Furthermore: $(U', V'; \mathcal{F}')$ is equivalent with $(U'', V''; \mathcal{F}'')$ if and only if the two minimal unitary extensions V' and V'', of V, are equivalent in the sense of* [2].

Next we have:

PROPOSITION 2. *Let $(S; \mathcal{F})$ be a minimal unitary extension of V. There exists a unitary extension $(R; \mathcal{F})$ of U, such that R commutes with S, if and only if*

$$\langle S^n Uh, Uh'\rangle = \langle S^n h, h'\rangle \quad \forall h, h' \in \mathcal{H},\ n = 1,2, \dots .$$

Proof. If there exists such R, then for every $h, h' \in \mathcal{H}$, $n = 1, 2, \dots,$

$$\langle S^n Uh, Uh'\rangle = \langle S^n Rh, Rh'\rangle = \langle RS^n h, Rh'\rangle = \langle S^n h, h'\rangle .$$

Conversely using that $\langle S^n Uh, Uh'\rangle = \langle S^n h, h'\rangle$ $\forall h, h' \in \mathcal{H}$, $n = 0,1,\dots$ we have that $R_n \sum_{n=-N}^{N} S^n h_n = \sum_{n=-N}^{N} S^n Uh_n$ defines a unitary operator from the linear hull of the elements of the form $\sum_{n=-N}^{N} S^n h_n$ onto the linear hull of the elements of the form $\sum_{n=-N}^{N} S^n h_n$ and so we can extend R to a unitary operator in $\mathcal{F}$ which we still denote by R such that extends U and commutes with S.

Let us introduce some more notation. Let $\mathcal{N}$, $\mathcal{M}$ be Hilbert spaces. The class of all holomorphic functions θ in $|z| < 1$ such that $\theta(z) \in L(\mathcal{N}, \mathcal{M})$ and $\theta(z)$ is a contraction for each z will be denoted by $\mathcal{B}(\mathcal{N}, \mathcal{M})$.

If $(S; \mathcal{F})$ is a minimal unitary extension of V, and if P_M, and $P_{\mathcal{F}\ominus\mathcal{H}}$ are the orthogonal projection of $\mathcal{F}$ onto $\mathcal{M}$ and $\mathcal{F}\ominus\mathcal{H}$ respectively, then the function

$$\theta_S(z) = P_M\{S(I - zP_{\mathcal{F}\ominus\mathcal{H}}S)^{-1}\}|\mathcal{N} \tag{2}$$

will be called the *holomorphic function associated to the minimal extension* $(S; \mathcal{F})$.

PROPOSITION 3. *Let S be a minimal unitary extension of V, and $\theta_S(z)$ its holomorphic function then:*

$$P_{\mathscr{H}}S^nU\,|\,\mathscr{H} = UP_{\mathscr{H}}S^n\,|\,\mathscr{H} \quad \text{for each } n \geq 1$$

if and only if

$$U\theta_S(z) = \theta_S(z)U\,|\,\mathscr{N}.$$

Proof. Let us assume that $P_{\mathscr{H}}S^nU\,|\,\mathscr{H} = UP_{\mathscr{H}}S^n\,|\,\mathscr{H}$, $n = 1,2, \ldots$; since $\theta_S(z) = \sum_{n\geq 0} z^n\hat{\theta}_S(n)$ with

$$\theta_S(m+2) = \left\{P_{\mathscr{H}}S^{m+3} - \sum_{k=0}^{m+1} P_{\mathscr{H}}S^{m+2-k}\hat{\theta}(k)\right\}\,\Big|\,\mathscr{N}$$

we can prove (by induction) that

$$P_{\mathscr{H}}S^nU\,|\,\mathscr{H} = UP_{\mathscr{H}}S^n\,|\,\mathscr{H} \quad n = 1, \ldots, r+2 \Rightarrow$$

$$\Rightarrow \theta_S(k)U|\mathscr{H} = U\theta_S(k) \quad k = 0, \ldots, r+1,$$

$\forall r \geq 0$, from which it is easy to conclude that $U\theta_S(z) = \theta_S(z)U\,|\,\mathscr{N}$. Now if we have that $U\theta_S(z) = \theta_S(z)U\,|\,\mathscr{N}$, using that $UVP_{D_V} = VP_{D_V}U\,|\,D_V$ we can conclude that

$$U(VP_{D_V} + \theta_S(z)P_{\mathscr{N}})[I - z(VP_{D_V} + \theta_S(z)P_{\mathscr{N}})]^{-1}\,|\,\mathscr{H} =$$

$$= (VP_{D_V} + \theta_S(z)P_{\mathscr{N}})[I - z(VP_{D_V} + \theta_S(z)P_{\mathscr{N}})]^{-1}U$$

which is the same as

$$UP_{\mathscr{H}}S(I - zS)^{-1}\,|\,\mathscr{H} = P_{\mathscr{H}}S(I - zS)^{-1}U$$

and so

$$UP_{\mathscr{H}}S^n\,|\,\mathscr{H} = P_{\mathscr{H}}S^nU \quad \forall\, n \geq 1.$$

The next statement is a corollary of Propositions 2 and 3.

PROPOSITION 4. *Let S be a minimal unitary extension of V, defined in $\mathscr{F}$ and $\theta_S(z)$ its analytic function defined by* (2). *There exists R defined in $\mathscr{F}$, such that $(R, S; \mathscr{F})$ is a minimal commuting unitary extension of (U, V) if and only $U\theta_S(z) = \theta_S(z)U\,|\,\mathscr{N}$.*

Now we recall the Arov-Grossman result: *The correspondence given by* (2) *is a bijection from all minimal unitary extensions of V onto $\mathscr{B}(\mathscr{N}, \mathscr{M})$.*

PROPOSITION 5. *The set* $\mathcal{U}_{U,V}$ *is not empty.*

Proof. Using Proposition 4 and the Arov-Grossman result we have that $\mathcal{U}_{U,V} \neq \emptyset$ if and only if there exists θ in $\mathcal{B}(\mathcal{N}, \mathcal{M})$ such that $\theta(z)U \mid \mathcal{N} = U\theta(z)$. Let $\theta : D \to L(\mathcal{N}, \mathcal{M})$ defined as $\theta(z)(n) = 0 \in \mathcal{M}$, $\forall z \in D$, $n \in \mathbf{N}$.

PROPOSITION 6. *Let* $F : \mathcal{U}_{U,V} \to \{\theta \in \mathcal{B}(\mathcal{N}, \mathcal{M}) : U\theta(z) = \theta(z)U \mid \mathcal{N}\}$ *defined by*

$$F(R, S) = \{P_{\mathrm{M}} S(I - zP_{[\bigvee_{n \in \mathbf{Z}} S^n(\mathcal{H})] \ominus \mathcal{H}} S)^{-1}\} \mid \mathcal{N}.$$

Then F is a well-defined bijection.

Using the last proposition we have that $\mathcal{U}_{U,V}$ has only one element if and only if

$$\begin{aligned} &\{\theta \in \mathcal{B}(\mathcal{N}, \mathcal{M}) : \theta(z)U \mid \mathcal{N} = U\theta(z)\} = \\ (3) \qquad &= \{\theta \in \mathcal{B}(\mathcal{N}, \mathcal{M}) : \theta(z)(n) = 0, \quad n \in \mathbf{N},\ z \in \mathbf{D}\}. \end{aligned}$$

But (3) is the same as

$$\begin{aligned} &\{T \in L(\mathcal{N}, \mathcal{M}) : \|T\| \leqslant 1 \quad TU \mid \mathcal{N} = UT\} = \\ (4) \qquad &= \{T \in L(\mathcal{N}, \mathcal{M}) : T(n) = 0, \quad \forall n \in \mathrm{N}\}. \end{aligned}$$

Fixing $U_1 = U \mid \mathcal{N}$ and $U_2 = U \mid \mathcal{M}$ we have that for $i = 1, 2$, $\Gamma_i = \{U_i^n\}_{n \in \mathbf{Z}}$ is a unitary representation of $\mathbf{Z}$ and in [3] it is proved that (4) is the same as

$$(5) \qquad \mathcal{N} = \{0\} \quad \text{or} \quad \mathcal{M} = \{0\}$$

or Γ_1 and Γ_2 are two disjoint unitary representations of $\mathbf{Z}$.

§ 3

In this part we will modify slightly the hypothesis of Part 2. We suppose that, $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator, D_V, R_V closed subspaces of $\mathcal{H}$, and $V : \mathrm{D}_V \to \mathrm{R}_V$ an isometry such that condition (1) holds, i.e.:

$$\langle U^n Vf, Vf' \rangle = \langle U^n f, f' \rangle \quad \forall f, f' \in \mathrm{D}_V,\ n = 1, 2 \ \ldots \ .$$

In this case we define (as we did in the proof of Proposition 2) $V' : \bigvee_{n \in \mathbf{Z}} U^n \mathrm{D}_V \to \bigvee_{n \in \mathbf{Z}} U^n \mathrm{R}_V$ to be the isometric operator which satisfies $V' U^n f = U^n V f \quad \forall n \in \mathbf{Z}$,

$f \in D_V$. Condition (1) insures that V' is a well defined isometric operator and it is easy to prove that V' is an extension of V, which commutes with U and such that the domain $D_{V'}$ and the range $R_{V'}$ of V' satisfy $U(D_{V'}) = D_{V'}$ and $U(R_{V'}) = R_{V'}$. Thus U and V' satisfy the conditions of Part 2 and we can describe $\mathscr{U}_{U,V'}$. But from the definition of V' it is easy to see:

PROPOSITION 7. $\mathscr{U}_{U,V} = \mathscr{U}_{U,V'}$.

The proof of the next result follows directly from Propositions 6 and 7.

PROPOSITION 8. *There is a bijection between* $\mathscr{U}_{U,V}$ *and the set* $\{\theta \in \mathscr{B}(\mathscr{H} \ominus \bigvee_{n\in\mathbf{Z}} U^n D_V, \mathscr{H} \ominus \bigvee_{n\in\mathbf{Z}} U^n R_V) : \theta U | \mathscr{H} \ominus \bigvee_{n\in\mathbf{Z}} U^n D_V = U\theta\}$. *Therefore* $\mathscr{U}_{U,V} \neq \emptyset$ *and* $\mathscr{U}_{U,V}$ *has only one element if and only if one of the next condition hold:*

(i) $\mathscr{H} = \bigvee_{n\in\mathbf{Z}} U^n D_V$,

(ii) $\mathscr{H} = \bigvee_{n\in\mathbf{Z}} U^n R_V$,

(iii) *If we set* $U_1 = U | \mathscr{H} \ominus \bigvee_{n\in\mathbf{Z}} U^n D_V$, $U_2 = U | \mathscr{H} \ominus \bigvee_{n\in\mathbf{Z}} U^n R_V$ *and for* $i = 1, 2$, $\Gamma_i = \{U_i^n\}_{n\in\mathbf{Z}}$ *then* Γ_1 *and* Γ_2 *are two disjoint unitary representations of* $\mathbf{Z}$.

§ 4

In this section we will describe $\mathscr{U}_{U,V}$ for $U: D_U \to R_U$, $V: D_V \to R_V$ a couple of isometries with D_U, R_U, D_V, and R_V, closed subspaces of a Hilbert space $\mathscr{H}$ with $U^n D_V \subset D_U$, $U^n R_V \subset D_U$ for $n = 0,1, \ldots$ and such that condition (1) holds.

Set W be a minimal unitary extension of U and let us define $V_W : \bigvee_{n\in\mathbf{Z}} W^n D_V \to \bigvee_{n\in\mathbf{Z}} W^n R_V$ as the unique isometric operator which verify $V_W W^n f = W^n V f$, $\forall f \in D_V$, $n \in \mathbf{Z}$ (as we did in the proof of Proposition 2). Condition (1) insures that V_W is a well-defined isometric operator and it is easy to prove that V_W extends V and commutes with W. Furthermore, the range and the domain of V_W are invariants by W. Thus W, V_W satisfy the conditions of Part 2 and we can describe $\mathscr{U}_{W,V_W}$.

Now if we consider for $i = 1, 2$, R_i, S_i two commuting minimal unitary operators which extends W and V_W, respectively (so they extend U and V respectively) then (R_1, S_1) and (R_2, S_2) represent the same element of $\mathscr{U}_{W,V_W}$ if there exists a unitary isomorphism $\varphi : \bigvee_{n\in\mathbf{Z}} \bigvee_{m\in\mathbf{Z}} S_1^n R_1^m(\mathscr{H}) \to \bigvee_{n\in\mathbf{Z}} \bigvee_{m\in\mathbf{Z}} S_2^n R_2^m(\mathscr{H})$ which leaves invariant the elements of $\bigvee_{n\in\mathbf{Z}} W^n(\mathscr{H})$ (and so φ leaves invariant the elements of $\mathscr{H}$) and such that $\varphi S_1^n R_1^m = S_2^n R_2^m \varphi$. From the last observation we can conclude:

PROPOSITION 9. *Let* W, V_W *be defined as above.* $I_W : \mathscr{U}_{W,V_W} \to \mathscr{U}_{U,V}$ *defined by* $I_W(R, S) = (R, S)$ *is a well-defined injective function.*

Using that W, V_W are two operators which satisfy the conditions of Part 2 we have that $\mathcal{U}_{W,V_W}$ can be parametrized by

$$\left\{\begin{array}{l} \theta \in \mathcal{B}((\bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})) \ominus (\bigvee_{n\in\mathbf{Z}} W^n(\mathrm{D}_V)), (\bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})) \ominus (\bigvee_{n\in\mathbf{Z}} W^n(\mathrm{R}_V))): \\ \\ W\theta(z) = \theta(z)W_{|}(\bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})) \ominus (\bigvee_{n\in\mathbf{Z}} W^n(\mathrm{D}_V)) \end{array}\right\}.$$

The last set gives a description of a part of $\mathcal{U}_{U,V}$; the next proposition will tell us which part.

PROPOSITION 10. *Let W, V_W be as in Proposition 8. Let (R, S) be an element of $\mathcal{U}_{U,V}$. (R, S) belongs to the range of I_W if and only if there is a unitary isomorphism $\varphi\colon \bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H}) \to \bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})$ which leaves invariant the elements of $\mathcal{H}$ such that $W = \varphi R_{|} \bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H})$.*

Proof. Let us assume that (R, S) belongs to the range of I_W, then there exists (R', S') a couple of minimal commuting unitary operators which extends W, V_W, respectively and a unitary isomorphism $\Phi\colon \bigvee_{n\in\mathbf{Z}}\bigvee_{m\in\mathbf{Z}} R^nS^m(\mathcal{H}) \to \bigvee_{n\in\mathbf{Z}}\bigvee_{m\in\mathbf{Z}} R'^nS'^m(\mathcal{H})$ which leaves invariant the elements of $\mathcal{H}$ and such that $\Phi R^nS^m = R'^nS'^m\Phi$. Let φ be the restriction of Φ to $\bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H})$ then φ is an unitary isomorphism $\varphi\colon \bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H}) \to \bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})$ which leaves invariant the elements of $\mathcal{H}$ and such that $\varphi R = R'\varphi = W\varphi$.

Conversely, let us assume that there is a unitary isomorphism $\varphi\colon \bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H}) \to \bigvee_{n\in\mathbf{Z}} W^n(\mathcal{H})$ and we remark that $(R, S) \in \mathcal{U}_{U,V}$. Let us denote by R' the restriction of R to $\bigvee_{n\in\mathbf{Z}} R^n(\mathcal{H})$, and by $V_{R'}$, the restriction of S to $\bigvee_{n\in\mathbf{Z}} R'^n(\mathcal{H})$. Then $V_{R'}$ is an isometric extension of V, which commutes with R' and such that the domain and the range of $V_{R'}$ are invariant under R'. Since R', $V_{R'}$ verify the conditions of Part 2 and (R, S) is an element of $\mathcal{U}_{R',V_{R'}}$, then there exists an holomorphic function $\theta \in \mathcal{B}(\bigvee_{n\in\mathbf{Z}} R'^n(\mathcal{H}) \ominus \bigvee_{n\in\mathbf{Z}} R'^n(\mathrm{D}_V), \bigvee_{n\in\mathbf{Z}} R'^n(\mathcal{H}) \ominus \bigvee_{n\in\mathbf{Z}} R'^n(\mathrm{R}_V))$ such that $R'\theta = \theta R'_{|} \bigvee_{n\in\mathbf{Z}} R'^n(\mathcal{H}) \ominus \bigvee_{n\in\mathbf{Z}} R'^n(\mathrm{D}_V)$. Now we observe that:

$$\varphi(\bigvee_{n\in\mathbf{Z}} R'^n\mathrm{D}_V) = \varphi(\bigvee_{n\in\mathbf{Z}} R^n\mathrm{D}_V) = \bigvee_{n\in\mathbf{Z}} \varphi R^n\mathrm{D}_V = \bigvee_{n\in\mathbf{Z}} W^n\varphi\mathrm{D}_V = \bigvee_{n\in\mathbf{Z}} W^n\mathrm{D}_V$$

and

$$\varphi(\bigvee_{n\in\mathbf{Z}} R'^n\mathrm{R}_V) = \varphi(\bigvee_{n\in\mathbf{Z}} R^n\mathrm{R}_V) = \bigvee_{n\in\mathbf{Z}} \varphi R^n\mathrm{R}_V) = \bigvee_{n\in\mathbf{Z}} W^n\varphi\mathrm{R}_V = \bigvee_{n\in\mathbf{Z}} W^n\mathrm{R}_V.$$

Let us define $\beta = \varphi\theta\varphi^{-1} \big| (\bigvee_{n\in\mathbf{Z}} W^n(\mathscr{H})) \ominus (\bigvee_{n\in\mathbf{Z}} W^n \mathrm{D}_V)$, clearly

$$\beta \in \mathscr{B}(\bigvee_{n\in\mathbf{Z}} W^n(\mathscr{H}) \ominus \bigvee_{n\in\mathbf{Z}} W^n(\mathrm{D}_V), \bigvee_{n\in\mathbf{Z}} W^n(\mathscr{H}) \ominus \bigvee_{n\in\mathbf{Z}} W^n(\mathrm{R}_V))$$

and $\beta W \big| \bigvee_{n\in\mathbf{Z}} W^n(\mathscr{H}) \ominus \bigvee_{n\in\mathbf{Z}} W^n(\mathrm{R}_V) = W\beta$. It is a consequence of Proposition 6 that there exists $(L, M) \in \mathscr{U}_{W,V_W}$ such that $I_W(L, M) = (R, S)$. Q.E.D.

For the next theorem we need to fix some notation:

$$\mathscr{N}_U = \mathscr{H} \ominus \mathrm{D}_U, \quad \mathscr{M}_U = \mathscr{H} \ominus \mathrm{R}_U,$$

$$H^2(\mathscr{M}_U) = \{\sum_{n\geqslant 0} \mathrm{e}^{int} m_n : \{m_n\}_{n\geqslant 0} C \mathscr{M}_U \sum_{n\geqslant 0} \|m_n\|^2 < \infty\}$$

$$H^2_-(\mathscr{N}_U) = \{\sum_{n\leqslant -1} \mathrm{e}^{int} c_n : \{c_n\}_{n\leqslant -1} C \mathscr{N}_U \sum_{n\leqslant -1} \|c_n\|^2 < \infty\}.$$

If $\mathscr{H} = \mathrm{D}_U$ then $U' : \mathscr{H} \oplus H^2(\mathscr{M}_U) \to \mathscr{H} \oplus H^2(\mathscr{M}_U)$ defined by $U'(f \oplus (m_n)_{n\geqslant 0}) = Uf + m_0 \oplus (m_{n+1})_{n\geqslant 0}$ is a unitary operator and we set:

$$A_1 = \bigvee_{n\geqslant 0} U^n \mathrm{D}_V \vee \bigvee_{n\leqslant -1} \left\{(U^{-1}P_{\mathrm{R}_U})^{-n} f + \sum_{k=0}^{-n-1} \mathrm{e}^{ikt} P_{\mathscr{M}_U} (U^{-1}P_{\mathrm{R}_U})^{-n-1-k} f : f \in \mathrm{D}_V\right\}$$

$$A_2 = \bigvee_{n\geqslant 0} U^n \mathrm{R}_V \vee \bigvee_{n\leqslant -1} \left\{(U^{-1}P_{\mathrm{R}_U})^{-n} f + \sum_{k=0}^{-n-1} \mathrm{e}^{ikt} P_{\mathscr{M}_U} (U^{-1}P_{\mathrm{R}_U})^{-n-1-k} f : f \in \mathrm{R}_V\right\}$$

and for $i = 1, 2$

$$U'_i = U' \big| \mathscr{H} \oplus H^2(\mathscr{M}_U) \ominus A_i, \quad \Gamma_i = \{U_i'^n\}_{n\in\mathbf{Z}}.$$

If $\mathscr{H} = \mathrm{R}_U$ then:

$$U'' : \mathscr{H} \oplus H^2_-(\mathscr{N}_U) \to \mathscr{H} \oplus H^2_-(\mathscr{N}_U)$$

defined by

$$U''(f \oplus \sum_{s\leqslant -1} \mathrm{e}^{ist} n_s) = UP_{\mathrm{D}_U} f \oplus (\mathrm{e}^{-it}(P_{\mathscr{N}_U} f) + \sum_{s\leqslant -1} \mathrm{e}^{i(s-1)t} n_s)$$

is a unitary operator and we set

$$U''_1 = U'' \big| \mathscr{H} \oplus H^2_-(\mathscr{N}_U) \ominus (\bigvee_{n\in\mathbf{Z}} U^n(\mathrm{D}_U)),$$

$$U''_2 = U'' \big| \mathscr{H} \oplus H^2_-(\mathscr{N}_U) \ominus (\bigvee_{n\in\mathbf{Z}} U^n(\mathrm{R}_V)),$$

and for $i = 1, 2$, $\Gamma_i' = \{U_i'^n\}_{n \in \mathbf{Z}}$. Set $\theta \in \mathscr{B}(\mathscr{N}_U, \mathscr{M}_U)$, we define U^θ the minimal unitary extension of U, having θ as its associated holomorphic function (see (2)).

The proof of the next theorem is easy using Proposition 9.

THEOREM 1. *There is a bijection between* $\mathscr{U}_{U,V}$ *and*

$$\bigcup_{\theta \in \mathscr{B}(\mathscr{N}_U, \mathscr{M}_U)} \left\{ \begin{array}{l} \beta \in \mathscr{B}(\bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathscr{H}) \ominus \bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathrm{D}_V), \bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathscr{H}) \ominus \bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathrm{R}_V)): \\ U^\theta \beta = \beta U^\theta \mid \bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathscr{H}) \ominus \bigvee_{n \in \mathbf{Z}} U^{\theta n}(\mathrm{R}_V) \end{array} \right\}$$

Furthermore:

(i) $\mathscr{U}_{U,V} \neq \emptyset$,

(ii) $\mathscr{U}_{U,V}$ *has only one element if and only if one of the following statements hold:*

A) $\mathscr{H} = \mathrm{D}_U$ *and* $\mathscr{H} \oplus H^2(\mathscr{M}_U) = A_1$,

B) $\mathscr{H} = \mathrm{D}_U$ *and* $\mathscr{H} \oplus H^2(\mathscr{M}_U) = A_2$,

C) $\mathscr{H} = \mathrm{D}_U$ *and* Γ_1 *and* Γ_2 *are two disjoint unitary representations of* $\mathbf{Z}$,

D) $\mathscr{H} = \mathrm{R}_U$ *and* Γ_1' *and* Γ_2' *are two disjoint unitary representations of* $\mathbf{Z}$.

The last result is: if $\dim \mathscr{N}_U = 0$ or $\dim \mathscr{M}_U = 0$ then Theorem 1 simplifies to: There is a bijection between $\mathscr{U}_{U,V}$ and

$$\left\{ \begin{array}{l} \theta \in \mathscr{B}(\mathscr{H} \ominus \mathrm{D}_V, \mathscr{H} \ominus \mathrm{R}_V): P_{\mathrm{R}_U}(VP_{\mathrm{D}_V} + \theta(z)P_{\mathscr{N}_V})[I - z(VP_{\mathrm{D}_V} + \theta(z)P_{\mathscr{N}_V})]^{-1}U = \\ = UP_{\mathrm{D}_U}(VP_{\mathrm{D}_V} + \theta(z)P_{\mathscr{N}_V})[I - z(VP_{\mathrm{D}_V} + \theta(z)P_{\mathscr{N}_V})]^{-1} \mid \mathscr{H} \end{array} \right\}.$$

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