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# LOCAL SPECTRAL PROPERTIES OF CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS IN $L^p(\mathbf{R}^N)$

ERNST ALBRECHT and WERNER J. RICKER

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## INTRODUCTION

Let $N \geqslant 1$ and $\Delta_p$ denote the Laplace operator in $L^p(\mathbf{R}^N)$, $1 < p < \infty$. Then $\Delta_2$ is selfadjoint and so, via the spectral theorem, admits a functional calculus based on the bounded measurable functions defined on $\mathbf{C}$. For $p \neq 2$, $\Delta_p$ is formally selfadjoint in the sense that the dual operator to $\Delta_p$ is $\Delta_q$ where $p^{-1} + q^{-1} = 1$. There arises the question of whether $\Delta_p$ is also a scalar type spectral operator if $p \neq 2$? In fact, the same question can be asked for any constant coefficient differential operator in $L^p(\mathbf{R}^N)$. One of the aims of this note is to answer this question (in the negative); see Section 2. The idea is to consider such differential operators as unbounded Fourier multiplier operators; it then follows that if the operator is spectral it is necessarily of scalar type. Accordingly, it admits an $L^\infty$-functional calculus all of whose elements are necessarily $p$-multipliers. Then multiplier theorems for $L^p(\mathbf{R}^N)$ produce the desired contradiction (not only for differential operators but also for more general Fourier multiplier operators).

This negative answer suggests an alternative question, namely whether there exist such differential operators which are (unbounded) decomposable operators in the sense of C. Foiaş? Operators in this class are not required to split the underlying space in such a strong way as that required by spectral operators. In Section 3 it is shown that all constant coefficient elliptic operators in $L^p(\mathbf{R}^N)$ are decomposable; we use the fact that such operators admit a sufficiently rich functional calculus. An example is given of a (non-elliptic) differential operator which is not decomposable for every $p \neq 2$.

In the first section we establish the basic notations, facts, and definitions concerning spectral and related operators needed in the main text. In particular, a Fuglede type theorem (which is used in § 2 and may be of independent interest) is established;

see Theorem 1.2 and its corollaries. For ease of reading, the proofs of these results are placed in an appendix at the end of the paper.

## 1. A FUGLEDE TYPE THEOREM FOR QUASISPECTRAL OPERATORS

Let $\mathscr{X}$ be a Banach space and $\mathscr{L}(\mathscr{X})$ the Banach algebra of all bounded linear operators on $\mathscr{X}$. The set of all closed linear operators with domain and range contained in $\mathscr{X}$ is denoted by $\mathscr{C}(\mathscr{X})$. If $S$, $T$ are linear mappings with domains of definition $D(S) \subseteq \mathscr{X}$ and $D(T) \subseteq \mathscr{X}$, respectively, and values in $\mathscr{Y}$, then we write $S \subseteq T$ if $D(S) \subseteq D(T)$ and $Tx = Sx$, for all $x \in D(S)$. A point $z \in \mathbf{C}$ is in the resolvent set $\rho(S)$ of $S \in \mathscr{C}(\mathscr{X})$ if $z - S$ is bijective. By definition, the point $\infty$ is in $\rho(S)$ if and only if $D(S) = \mathscr{X}$. The spectrum $\sigma(S) := \bar{\mathbf{C}} \setminus \rho(S)$ is then a compact subset of the one point compactification $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ of the complex plane. For a closed subset $F$ of $\bar{\mathbf{C}}$, we write $\mathscr{X}_S(F)$ for the set of all those $x \in \mathscr{X}$ for which there exists an analytic $\mathscr{X}$-valued function $f: \bar{\mathbf{C}} \setminus F \to \mathscr{X}$ such that $(z - S)f(z) \equiv x$ on $\mathbf{C} \setminus F$. See the monographs [5], [9], [28] for the theory of these spectral manifolds.

Let $\mathscr{B}$ denote the $\sigma$-algebra of Borel subsets in $\mathbf{C}$. The notion of a spectral measure $P: \mathscr{B} \to \mathscr{L}(\mathscr{X})$ of class $\Gamma$ (where $\Gamma$ is a total subspace of the dual space $\mathscr{X}^*$ of $\mathscr{X}$) is standard [7]. Given $x \in \mathscr{X}$ and $\gamma \in \Gamma$, the $\mathbf{C}$-valued measure $E \mapsto \langle P(E)x, \gamma \rangle$ is denoted by $\langle Px, \gamma \rangle$. An operator $S \in \mathscr{C}(\mathscr{X})$ is called *quasispectral of class* $\Gamma$ (see [3] for bounded operators), if there exists a spectral measure of class $\Gamma$, say $P$, such that, for every closed set $E \subseteq \bar{\mathbf{C}}$,

$$(1) \qquad SP(E \cap \mathbf{C}) = P(E \cap \mathbf{C})SP(E \cap \mathbf{C}) \quad \text{and} \quad \sigma(S|P(E \cap \mathbf{C})\mathscr{X}) \subseteq E.$$

Here $S|P(E \cap \mathbf{C})\mathscr{X}$ is the closed operator with domain $D(S|P(E \cap \mathbf{C})\mathscr{X}) := D(S) \cap P(E \cap \mathbf{C})\mathscr{X}$ defined by $(S|P(E \cap \mathbf{C})\mathscr{X})x := Sx$, for all $x \in D(S|P(E \cap \mathbf{C})\mathscr{X})$. In this case $P$ is called a *resolution of the identity of class* $\Gamma$ *for* $S$. By (1) we have $P(K)\mathscr{X} \subseteq D(S)$ and $S\,P(K)\mathscr{X} \in \mathscr{L}(P(K)\mathscr{X})$ for every compact set $K \subseteq \mathbf{C}$. If, instead of (1), we have

$$(2) \qquad P(E)S \subseteq SP(E) \text{ and } \sigma(S|P(E)\mathscr{X}) \subseteq \bar{E}, \quad \text{for every Borel set } E \subseteq \mathbf{C},$$

then $S$ will be called a *prespectral operator of class* $\Gamma$ [7]. Here $\bar{E}$ is the closure of $E$ in $\bar{\mathbf{C}}$. A prespectral operator of class $\mathscr{X}^*$ is *spectral* in the sense of N. Dunford [9]. It is known, even in the bounded case, that the dual of a spectral operator need not be spectral.

1.1. LEMMA. *If $T \in \mathscr{C}(\mathscr{X})$ is a spectral operator, then the dual operator $T^* \in \mathscr{C}(\mathscr{X}^*)$ is prespectral of class $\mathscr{X}$. In particular, if $\mathscr{X}$ is reflexive, then $T^*$ is spectral.*

If $S \in \mathscr{C}(\mathscr{X})$ is a quasispectral operator with a resolution of the identity $P$ of class $\Gamma$ then, for any bounded Borel measurable function $f: \mathbf{C} \to \mathbf{C}$, define

$$\Phi_S(f) := \int_{\mathbf{C}} f(z)\, \mathrm{d}P(z).$$

Then $\Phi_S(f)$ is a scalar type prespectral operator with a resolution of the identity of class $\Gamma$ given by

$$P_f(E) := P(f^{-1}(E)), \quad \text{for } E \in \mathscr{B}; \tag{3}$$

(see [7], Proposition 5.8). The mapping $f \mapsto \Phi_S(f)$ defines a continuous unital homomorphism from the Banach algebra of all bounded Borel measurable functions to $\mathscr{L}(\mathscr{X})$.

1.2. THEOREM. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces and $A \in \mathscr{C}(\mathscr{X})$ (resp. $B \in \mathscr{C}(\mathscr{Y})$) be a quasispectral operator with resolution $P_A$ (resp. $P_B$) of the identity of class $\Gamma_A$ (resp. $\Gamma_B$).*

(a) *If $T: \mathscr{X} \to \mathscr{Y}$ is a bounded linear operator satisfying $TA \subseteq BT$, then for all bounded continuous functions $f: \mathbf{C} \to \mathbf{C}$ we have*

$$T\Phi_A(f) = \Phi_B(f)T. \tag{4}$$

(b) *If, in addition, $T^*(\Gamma_B) \subseteq \Gamma_A$, then*

$$TP_A(E) = P_B(E)T, \quad \textit{for all } E \in \mathscr{B}, \tag{5}$$

*and* (4) *holds for all bounded Borel measurable functions on* $\mathbf{C}$.

This theorem generalizes Trditev 2.6 and Izrek 2.9 (iii) in [18], formulated for bounded operators. From the special case $\mathscr{X} = \mathscr{Y}$, $A = B$, $\Gamma_A = \Gamma_B$, and $T = 1$ (the identity operator in $\mathscr{X}$) in the theorem, we obtain the following

1.3. COROLLARY. *The resolution of the identity of class $\Gamma$ of a quasispectral operator of class $\Gamma$ is uniquely determined.*

Let $S \in \mathscr{C}(\mathscr{X})$ be a quasispectral operator of class $\Gamma$ with resolution $P_S$ of the identity of class $\Gamma$. For a Borel measurable function $f: \mathbf{C} \to \mathbf{C}$ define (with $\mathbf{D}(n) := \{z \in \mathbf{C}\,;\ |z| \leqslant n\}$),

$$\mathrm{D}(\Phi_S(f)) := \left\{x \in \mathscr{X}\,;\ \lim_{n\to\infty} \int_{\mathbf{C}} \chi_{f^{-1}(\mathbf{D}(n))} f(z)\mathrm{d}P_S(z)x \text{ exists}\right\}$$

and, for $x \in \mathrm{D}(\Phi_S(f))$,

$$\Phi_S(f)x := \lim_{n\to\infty} \int_{\mathbf{C}} \chi_{f^{-1}(\mathbf{D}(n))} f(z) \mathrm{d}P_S(z)x.$$

Then $S_0 := \Phi_S(\mathrm{id})$ is called the *scalar part* of $S$ where "id" is the identity function on **C**. With these notations and definitions we obtain the following result from Theorem 1.2. Part (b) is the generalization of Putnam [26] of the original Fuglede theorem [16].

1.4. COROLLARY. (a) *With the hypothesis of Theorem* 1.2 (b) *we have*

$$T\Phi_A(f) \subseteq \Phi_B(f)T,$$

*for all Borel measurable functions* $f : \mathbf{C} \to \mathbf{C}$.

(b) *If* $\mathscr{X}$ *and* $\mathscr{Y}$ *are Hilbert spaces, $A$ and $B$ are normal operators (possibly unbounded) in* $\mathscr{X}$ *and* $\mathscr{Y}$, *respectively, and* $T : \mathscr{X} \to \mathscr{Y}$ *is an operator as in Theorem* 1.2 (a), *then* $TA^* \subseteq B^*T$, *where* $^*$ *denotes the Hilbert space adjoint.*

1.5. COROLLARY. *Suppose that, in the setting of Theorem* 1.2 *the operator $T$ is injective, $A$ is a spectral operator and $B$ is a scalar type spectral operator. Then the operator $A$ is actually of scalar type and* $\sigma(\Phi_A(f)) \subseteq \sigma(\Phi_B(f))$, *for all Borel measurable functions* $f : \mathbf{C} \to \mathbf{C}$.

An example of U. Fixman [14] shows that the class of quasispectral operators is strictly larger than the class of prespectral operators; there are situations when both classes of operators coincide.

1.6. PROPOSITION. *Every quasispectral operator $T$ on a weakly sequentially complete Banach space* $\mathscr{X}$ *is spectral.*

## 2. NON-SPECTRALITY OF DIFFERENTIAL AND MULTIPLIER OPERATORS

Let $G$ be a locally compact abelian group, let $\Gamma$ denote the dual group of $G$ and write $\mu$ for the Haar measure of $\Gamma$. Suppose that $1 \leqslant p \leqslant 2$. Then the Fourier transform $\mathscr{F}$ (also denoted by $\hat{\ }$) is a continuous injective linear mapping $\mathscr{F} : L^p(G) \to L^q(\Gamma)$ where $p^{-1} + q^{-1} = 1$. For a Borel measurable function $\varphi : \Gamma \to \mathbf{C}$ define

$$\mathrm{D}_p(\varphi) := \{f \in L^p(G)\ ;\ \varphi\hat{f} \in \mathscr{F}(L^p(G))\}$$

and

$$T^p_\varphi f := \mathscr{F}^{-1}(\varphi\hat{f}), \quad \text{for every } f \in \mathrm{D}_p(\varphi).$$

This defines a closed linear operator $T^p_\varphi$ with domain $D_p(\varphi)$. As usual, $\varphi$ will be called a $p$-multiplier for $G$ if $D_p(\varphi) = L^p(G)$, that is, if $T^p_\varphi \in \mathscr{L}(L^p(G))$. The space of all $p$-multipliers for $G$ is denoted by $\mathscr{M}_p(G)$.

For $1 \leqslant q < \infty$ let $M^q_\varphi$ denote the multiplication operator in $L^q(\Gamma)$ given by $M^q_\varphi g := \varphi g$, for every $g$ in the domain $D(M^q_\varphi) := \{h \in L^q(\Gamma)\,;\ \varphi h \in L^q(\Gamma)\}$ of $M^q_\varphi$. Then $M^q_\varphi$ is a scalar type spectral operator with resolution of the identity $Q_\varphi$ given by $Q_\varphi(E) := M^q_{\chi_E \circ \varphi}$, $E \in \mathscr{B}$; its functional calculus is specified via the spectral integral

$$\Phi^M_\varphi(\psi) = \int_{\mathbf{C}} \psi \, dQ_\varphi = M^q_{\psi \circ \varphi},$$

for every Borel measurable function $\psi$ on $\mathbf{C}$. Since $\mathscr{F} T^p_\varphi \subseteq M^q_\varphi \mathscr{F}$ (for $1 < p \leqslant 2$, $p^{-1} + q^{-1} = 1$), the following result is a consequence of Theorem 1.2, Corollary 1.5 and [9], Theorem XVII.2.10. For the statement concerning spectra we have used the fact that the $\mu$-essential range of any $m \in \mathscr{M}_p(G)$ coincides with the spectrum of $M^q_m \in \mathscr{L}(L^q(\Gamma))$ and is a subset of the spectrum of $T^p_m \in \mathscr{L}(L^p(G))$.

2.1. Proposition. *Let $1 < p \leqslant 2$ and $A \subseteq T^p_\varphi$. If $A$ is a spectral operator in $L^p(G)$, then $A$ is necessarily of scalar type and the resolution of the identity $P_A$ of $A$ is given by*

$$P_A(E) = \mathscr{F}^{-1} Q_\varphi(E) \mathscr{F} = T^p_{\chi_E \circ \varphi}, \quad \textit{for every } E \in \mathscr{B}.$$

*Moreover, for every bounded Borel measurable function $\psi$ on $\mathbf{C}$ we have*

$$\Phi_A(\psi) = \int_{\mathbf{C}} \psi(z) \, dP_A(z) = \mathscr{F}^{-1} \Phi^M_\varphi(\psi) \mathscr{F} = T^p_{\psi \circ \varphi} \in \mathscr{L}(L^p(G))$$

*and there is some constant $K_p > 0$ (independent of $\psi$) such that*

$$\|T^p_{\psi \circ \varphi}\| \leqslant K_p \|\psi\|_{L^\infty(P_A)}. \tag{6}$$

*In particular, $\psi \circ \varphi \in \mathscr{M}_p(G)$. In addition, the spectrum of $T^p_{\psi \circ \varphi} \in \mathscr{L}(L^p(G))$ is the $\mu$-essential range of $\psi \circ \varphi \in L^\infty(\Gamma, \mu)$.*

The above proposition provides various means for establishing the non-spectrality of $T^p_\varphi$, $1 < p < 2$. Indeed, it suffices to show that either,

(i) there is a $\psi$ such that the spectrum of $T^p_{\psi \circ \varphi} \in \mathscr{L}(L^p(G))$ properly contains the $\mu$-essential range of $\psi \circ \varphi \in L^\infty(\Gamma, \mu)$, or

(ii) the estimates (6) lead to a contradiction, or

(iii) there exists $\psi$ such that $\psi \circ \varphi$ is not a $p$-multiplier for $G$.

Our main interest is in the case when $G = \mathbf{R}^N$. For an application of (i) we refer to § 3. To exploit (ii) it is often possible to use a result of Hörmander [19], Theorem 1.14, stating that if $f \in \mathscr{C}^2(\mathbf{R}^N, \mathbf{R})$ and there exists a sequence of real numbers $t_n \to \infty$ such that each function $\rho_n(x) = \exp(it_n f(x))$, $x \in \mathbf{R}^N$, is a $p$-multiplier and $\sup_{n\in\mathbf{N}} \|T^p_{\rho_n}\| < \infty$, then $f(x) = \alpha_0 + \sum_{j=1}^{N} \alpha_j x_j$, $x \in \mathbf{R}^N$, for suitable $\alpha_j \in \mathbf{R}$, $0 \leqslant j \leqslant N$. We give an immediate application.

2.2. Proposition. *Let* $1 < p < 2$ *and* $\varphi \in \mathscr{C}^2(\mathbf{R}^N, \mathbf{C})$ *be a non-constant function. Then* $T^p_\varphi$ *is not a spectral operator in* $L^p(\mathbf{R}^N)$.

*Proof.* Suppose that $T^p_\varphi$ is a spectral operator in $L^p(\mathbf{R}^N)$. Consider the bounded continuous functions $\psi_n, \eta_n, \xi_n : \mathbf{C} \to \mathbf{C}$ given by $\psi_n(z) := \exp(in|z|^2)$, $\eta_n(z) := \exp(in \cdot \mathrm{Re}(z))$, and $\xi_n(z) := \exp(in \cdot \mathrm{Im}(z))$, for $z \in \mathbf{C}$. Then Proposition 2.1 (cf. (6)) implies that $\psi_n \circ \varphi, \eta_n \circ \varphi, \xi_n \circ \varphi \in \mathscr{M}_p(\mathbf{R}^N)$, for each $n \in \mathbf{N}$, and $\sup_{n\in\mathbf{N}} \|T^p_{\psi_n\circ\varphi}\| < \infty$, $\sup_{n\in\mathbf{N}} \|T^p_{\eta_n\circ\varphi}\| < \infty$, and $\sup_{n\in\mathbf{N}} \|T^p_{\xi_n\circ\varphi}\| < \infty$. Hörmander's result now implies that the functions $|\varphi|^2$, $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$ must be affine. This contradicts $\varphi$ being non-constant. ▨

For example, if $0 \neq y \in \mathbf{R}^N$ is fixed and $\varphi_y(x) := \exp(i\langle x, y\rangle)$, $x \in \mathbf{R}^N$, then the operator $T^p_{\varphi_y}$, which is the translation operator by amount $y$ in $L^p(\mathbf{R}^N)$, cannot be spectral. This gives an alternative proof (in the special case $G = \mathbf{R}^N$) of the fact that translation operators in $L^p(G)$ are rarely spectral operators [17].

Recall that if $Q : \mathbf{R}^N \to \mathbf{C}$ is a polynomial and $D_j = -i\frac{\partial}{\partial x_j}$, $1 \leqslant j \leqslant N$ then the differential operator $Q_p(D)$ given by $Q_p(D)f := Q(D)f$, for every $f \in D(Q_p(D)) := \{h \in L^p(\mathbf{R}^N) ; Q(D)h \in L^p(\mathbf{R}^N)\}$, where $Q(D)f$ is defined in the distributional sense, coincides with $T^p_Q$. Hence, Proposition 2.2 implies the following

2.3. Corollary. *Let* $1 < p < 2$ *and* $Q : \mathbf{R}^N \to \mathbf{C}$ *be a non-constant polynomial. Then the differential operator* $Q_p(D)$ *is not a spectral operator in* $L^p(\mathbf{R}^N)$.

We remark that a similar argument as in the proof of Proposition 2.2 shows, whenever $\varphi : \mathbf{R}^N \to \mathbf{C}$ is a function for which there exists $\psi : \mathbf{C} \to \mathbf{R}$ such that $\psi \circ \varphi \in \mathscr{C}^2(\mathbf{R}^N, \mathbf{R})$ and $\psi \circ \varphi$ is not affine, that $T^p_\varphi$ is not a spectral operator for any $1 < p < 2$.

We now indicate how (iii) can be applied in showing non-spectrality of $T^p_\varphi$.

2.4. Proposition. *Let* $\varphi : \mathbf{R}^N \to \mathbf{R}$ *be a Borel measurable function and suppose there exists* $u \in \mathbf{R}^N$ *such that* $\varphi$ *is continuous in a neighbourhood* $U$ *of* $u$ *and, for some non-zero* $v \in \mathbf{R}^N$, *the function* $t \mapsto \varphi(u + tv)$ *is strictly monotonic in a neighbourhood of* 0. *Then no restriction* $A \subseteq T^p_\varphi$, $1 < p < 2$, *considered in* $L^p(\mathbf{R}^N)$, *is a spectral operator.*

*Proof.* Choose a neighbourhood $V$ of $u$ and $c > 0$ such that $|\varphi(x)| \leqslant c$, for every $x \in V$. Then $\psi$, defined by $\psi(z) := \operatorname{sign}(\operatorname{Re}(z)) \cdot \min\{c, |\operatorname{Re}(z)|\}$, $z \in \mathbf{C}$, is bounded, continuous and satisfies $\psi \circ \varphi \equiv \varphi$ on $V$. Assume that $A$ is spectral. Then Proposition 2.1 and [9], Theorem XVII.2.10, imply that $T^p_{\psi \circ \varphi} = \Phi_A(\psi) = \int_{\mathbf{C}} \psi(z) \mathrm{d}P_A(z)$ is a scalar type spectral operator in $\mathscr{L}(L^p(\mathbf{R}^N))$. Hence, we may replace $\varphi$ by $\psi \circ \varphi$ and so assume that $\varphi \in \mathscr{M}_p(\mathbf{R}^N)$ and $A = T^p_\varphi$. By means of an affine transformation [19], Theorem 1.13, it may also be assumed that $u = 0$ and $v = (1, 0, \ldots, 0)$.

Let $\rho \in \mathscr{C}^\infty_c(\mathbf{R}^N)$ be identically 1 in a neighbourhood of 0 and with support small enough so that $\varphi\rho \in \mathscr{C}_c(\mathbf{R}^N)$. Fix an arbitrary bounded continuous function $\psi$ on $\mathbf{C}$. Proposition 2.1 implies that $\Phi_A(\psi) = T^p_{\psi \circ \varphi} \in \mathscr{L}(L^p(\mathbf{R}^N))$, that is, $\psi \circ \varphi \in \mathscr{M}_p(\mathbf{R}^N)$. In addition, $\psi \circ \varphi$ is continuous on $\operatorname{supp}(\rho)$ and, by (6), we have

$$\|T^p_{\rho \cdot (\psi \circ \varphi)}\| \leqslant \|T^p_\rho\| \cdot \|T^p_{\psi \circ \varphi}\| \leqslant K_p \|T^p_\rho\| \cdot \|\psi\|_{L^\infty(P)},$$

where $P$ is the resolution of the identity of $T^p_\varphi$. It follows from a result of de Leeuw [6] (see especially the proof given in [20]) that the restricton $\rho \cdot (\psi \circ \varphi)|\mathbf{R} \times \{0\} = \tilde{\rho} \cdot (\psi \circ \tilde{\varphi})$, where $\tilde{\rho} = \rho|\mathbf{R} \times \{0\}$ and $\tilde{\varphi} = \varphi|\mathbf{R} \times \{0\}$, is an element of $\mathscr{M}_p(\mathbf{R})$ with the norm (in $\mathscr{L}(L^p(\mathbf{R}))$) of the corresponding multiplier operator not exceeding $K_p \|T^p_\rho\| \cdot \|\psi\|_{L^\infty(\Omega)}$. Here $\Omega$ is the essential range of $\varphi$ and we have used the fact that $P(E) = 0$ if and only if $\varphi^{-1}(E)$ is a null set in $\mathbf{R}^N$. Moreover, $\tilde{\rho}$ is identically 1 and $\tilde{\varphi}$ is strictly monotonic in some neighbourhood $W$ of 0.

Fix any interval $I = [a, b] \subseteq W$ with $a < b$ and define

$$\mathscr{X} := \{f \in L^p(\mathbf{R})\,;\, \operatorname{supp}(\hat{f}) \subseteq I\}.$$

Since $\mathscr{X}$ is reflexive (being a closed subspace of $L^p(\mathbf{R})$), the restriction $\tilde{T}$ of $T^p_{\tilde{\rho} \cdot \tilde{\varphi}}$ to $\mathscr{X}$ is spectral of scalar type; this follows from [9], Theorem XVII.2.5, by considering the continuous homomorphism $\Phi : \mathscr{C}(\Omega) \to \mathscr{L}(\mathscr{X})$ given by $\Phi(\psi) := T^p_{\tilde{\rho} \cdot (\psi \circ \tilde{\varphi})}|\mathscr{X}$, $\psi \in \mathscr{C}(\Omega)$. Let $Q$ be the resolution of the identity of $\tilde{T}$ and $T : \mathscr{X} \to L^q(I)$ be the restriction of $\mathscr{F}$ from $L^p(\mathbf{R})$ to $\mathscr{X}$. It follows from Proposition 1.2 and Corollary 1.5 that

$$Q(E)f = \mathscr{F}^{-1}(\chi_E \circ \tilde{\varphi}|I \cdot \hat{f}), \quad f \in \mathscr{X},\ E \in \mathscr{B}.$$

Let $a < a_1 < b_1 < b$ and choose a Borel set $E \subseteq I_1 := [a_1, b_1]$ such that $\chi_E$ is not a $p$-multiplier for $\mathbf{R}$ (see for example [22], p. 111). Since $\tilde{\varphi}|I$ is continuous and strictly monotonic we have $E = (\tilde{\varphi}|I)^{-1}(\tilde{\varphi}(E))$. Fix $\alpha \in \mathscr{C}^\infty_c(\mathbf{R})$ with $\operatorname{supp}(\alpha) \subseteq I$ and $\alpha \equiv 1$ in a neighbourhood of $I_1$. Then $T^p_\alpha$ may be considered as a continuous linear operator from $L^p(\mathbf{R})$ to $\mathscr{X}$. It follows that $JQ(\tilde{\varphi}(E))T_\alpha^p = T_{\chi_E}^p$ belongs to $\mathscr{L}(L^p(\mathbf{R}))$, where $J:\mathscr{X} \to L^p(\mathbf{R})$ is the canonical injection. This contradicts $\chi_E \notin \mathscr{M}_p(\mathbf{R})$. ▨

Proposition 2.4 can be used to show the non-spectrality of certain familiar operators. For example the Riesz transforms in $\mathbf{R}^N$, $N \geqslant 2$, corresponding to the multipliers $x \mapsto x_j \cdot |x|^{-1}$, $x \neq 0$, for $1 \leqslant j \leqslant N$, are not spectral whenever $1 < p < 2$. The same is true of the operators $T_{\varphi_\alpha}^p$ corresponding to $\varphi_\alpha(x) = |x|^\alpha$, $x \in \mathbf{R}^N$, for $\alpha > 0$. The operator $T_{\varphi_1}^p$ is just the infinitesimal generator of the Poisson semigroup. For $\lambda > -1$, the Bochner-Riesz mean $\mu_\lambda$ of order $\lambda$ defined by $\mu_\lambda(x) := (1 - |x|^2)^\lambda \cdot \chi_B(x)$ for $x \in \mathbf{R}^N$ (where $B$ is the closed unit ball of $\mathbf{R}^N$), is an element of $\mathscr{M}_p(\mathbf{R}^N)$ for all $1 < p < \infty$ whenever $2\lambda > n - 1$. Proposition 2.4 shows that $T_{\mu_\lambda}^p$ is non-spectral whenever $1 < p < 2$.

Proposition 2.4 has extensions to other groups. For example, by considering the real and the imaginary parts of the functions $t \mapsto \varphi(x + te_j, z)$ and $t \mapsto \varphi(x, z + (e^{it} - z_k)f_k)$, $t \in \mathbf{R}$, where $x \in \mathbf{R}^N$, $z \in \mathbf{T}^M$ and $e_j$ (resp. $f_k$) is the $j^{\text{th}}$ (resp. $k^{\text{th}}$) coordinate direction in $\mathbf{R}^N$ (resp. $\mathbf{T}^M$), a similar argument as that for the proof of Proposition 2.4 can be combined with Jodeit's proof [20] of the de Leeuw theorem and [22], p. 111, to establish the following result.

2.5. PROPOSITION. *Let $G := \mathbf{R}^N \times \mathbf{Z}^M$ and suppose that $\varphi: \Gamma \to \mathbf{C}$ is a Borel measurable function which is non-constant and of class $\mathscr{C}^1$ on some open subset of $\Gamma = \mathbf{R}^N \times \mathbf{T}^M$. Then no restriction $A \subseteq T_\varphi^p$, $1 < p < 2$, is a spectral operator in $L^p(G)$.*

For $p > 2$ the Fourier transform of an $L^p$-function on $G$ need not be a function on $\Gamma$. Accordingly, $T_\varphi^p$ cannot be defined as above. For the sake of simplicity we restrict ourselves to the case $G = \mathbf{R}^N$ (or $G = \mathbf{R}^N \times \mathbf{Z}^M$). For any $1 \leqslant p < \infty$, let $\mathscr{U}_p(G)$ denote the set of those Borel measurable functions $\varphi: \Gamma \to \mathbf{C}$ with the property that $\rho\varphi \in \mathscr{M}_p(G)$ whenever $\rho \in \mathscr{C}_c^\infty(\Gamma)$. Let $\mathscr{D}_p \subseteq L^p(G)$ be the subspace defined by

$$\mathscr{D}_p := \{f \in L^p(G)\ ;\ \operatorname{supp}(\hat{f}) \text{ is compact}\},$$

where $\hat{f}$ is considered in the sense of distributions. If $f \in \mathscr{D}_p$, then there exists $\rho \in \mathscr{C}_c^\infty(\Gamma)$ which is 1 in a neighbourhood of $\operatorname{supp}(\hat{f})$. Since $\varphi \in \mathscr{U}_p(G)$ it follows that $g := \mathscr{F}^{-1}(\rho\varphi\hat{f}) = \mathscr{F}^{-1}(\varphi\hat{f}) \in L^p(G)$ is independent of $\rho$ (where $\mathscr{F}^{-1}$ has to be taken in the sense of tempered distributions). The operator $S_\varphi$ defined by $S_\varphi f := \mathscr{F}^{-1}(\varphi\hat{f})$, for $f \in \mathscr{D}_p$, is closable; its closure is denoted by $S_\varphi^p$. For $1 \leqslant p \leqslant 2$, we have $S_\varphi^p \subseteq T_\varphi^p$. Since $S_\varphi^p$ is densely defined, the dual operator $(S_\varphi^p)^*$ exists; it satisfies $(S_\varphi^p)^* \subseteq T_\varphi^q$ whenever $2 \leqslant p < \infty$ (where $p^{-1} + q^{-1} = 1$). The reflexivity of $L^p(G)$ implies in this case that $S_\varphi^p$ is spectral if and only if $(S_\varphi^p)^*$ is spectral in $L^q(G)$. By considering $A := (S_\varphi^p)^*$ it follows that *the analogues of the statements* 2.1 – 2.5 *are also valid for the operators $S_\varphi^p$, $\varphi \in \mathscr{U}_p(G)$, whenever $2 < p < \infty$.*

We conclude this section with some remarks. It has been shown that $Q(D)$ is not a spectral operator in $L^p(\mathbf{R}^N)$, $1 < p < \infty$, $p \neq 2$, whenever $Q: \mathbf{R}^N \to \mathbf{C}$ is a non-constant polynomial. For the special case $Q(D) = D_j = -\mathrm{i}\dfrac{\partial}{\partial x_j}$, $1 \leqslant j \leqslant N$, we refer to [12]. We also mention a result of B. S. Mitjagin [24], [25], (obtained by means of an extension of C. Fefferman's negative solution [13] of he multiplier problem for the unit ball) stating that if $Q(D, x)$ is an elliptic differential operator on $\mathscr{C}_c^\infty(\Omega)$ for an open set $\Omega \subseteq \mathbf{R}^N$ (or certain $N$-dimensional differentiable manifolds) having a semi-bounded selfadjoint extension in $L^2(\Omega)$ (with resolution of the identity $P$) then, for $N \geqslant 2$, the family $Q_K P((-\infty, t))Q_K$, $t \in \mathbf{R}$, $K \subseteq \Omega$ compact, is unbounded in $L^p(\Omega)$. Here $Q_K$ is the operator of multiplication with $\chi_K$. Consequently, it is unlikely that such operators can be spectral in $L^p(\Omega)$ (for bounded domains this follows from Mitjagin's result, Theorem 1.2 and Corollary 1.5).

On $L^1(G)$ (resp. $L^\infty(G)$) with $G = \mathbf{R}^N \times \mathbf{Z}^M$ no non-trivial spectral (resp. quasi--spectral) operators exist. To see this, we observe that the values of the corresponding resolution of the identity would be translation invariant (by Theorem 1.2 with $A = B$ and $T$ running through all translations) and hence, would be multiplier operators. However, the only idempotent multipliers in $\mathscr{M}_1(G) = \mathscr{M}_\infty(G)$ are 0 and 1.

The results of this section show that translation invariant spectral operators in $L^p(G)$, $p \neq 2$, seem to be rather scarce. Nevertheless, since $\Gamma = \mathbf{R}^N \times \mathbf{T}^M$ admits Littlewood-Paley decompositions, it follows (from [10], Proposition 1.2.9, for example) that given any compact set $K \subset \mathbf{C}$, there exists $\varphi \in \mathscr{M}_p(G)$, $1 < p < \infty$, such that $T_\varphi^p$ is a scalar type spectral operator with $\sigma(T_\varphi^p) = K$.

## 3. DECOMPOSABILITY AND FUNCTIONAL CALCULUS FOR CONSTANT COEFFICIENT ELLIPTIC OPERATORS ON $L^p(\mathbf{R}^N)$

Let $\psi$ be in $\mathscr{U}_p(\mathbf{R}^N)$, in which case $S_\psi^p$ is defined for $1 < p < \infty$ and let $\mathscr{A}_p(\psi)$ be the set of all (equivalence classes of) Borel measurable functions $\varphi$ on $\mathbf{C}$ with $\varphi \circ \psi \in \mathscr{M}_p(\mathbf{R}^N)$, where $\varphi_1$ and $\varphi_2$ are identified whenever $\varphi_1 \circ \psi = \varphi_2 \circ \psi$ almost everywhere in $\mathbf{R}^N$. Then $\mathscr{A}_p(\psi) \circ \psi := \{\varphi \circ \psi;\ \varphi \in \mathscr{A}_p(\psi)\}$ is a closed full subalgebra of $\mathscr{M}_p(\mathbf{R}^N)$. Endowed with the norm $\|\varphi\|_\psi := \|\varphi \circ \psi\|_{\mathscr{M}_p(\mathbf{R}^N)}$, $\varphi \in \mathscr{A}_p(\psi)$, the algebra $\mathscr{A}_p(\psi)$ is a Banach algebra and the mapping defined by

$$\Phi_p: \mathscr{A}_p(\psi) \to \mathscr{L}(L^p(\mathbf{R}^N)), \quad \varphi \mapsto T^p_{\varphi \circ \psi} \tag{7}$$

is a continuous unital homomorphism. The question arises, how rich is this algebra $\mathscr{A}_p(\psi)$ and how is $\Phi_p$ related to $S_\psi^p$?

In general there is no adequate solution to this problem. Already for $\psi \in \mathscr{M}_p(\mathbf{R}^N)$ he operator $S_\psi^p = T_\psi^p$ may not be decomposable in the sense of C. Foiaş [15] (see

[4], [11] for some examples). This shows that the functional calculus (7) for $S^p_\Psi$ may be very poor. However, for elliptic differential operators $\Psi(D)$ on $\mathbf{R}^N$ with constant coefficients, the situation turns out to be satisfactory. Here

$$\Psi(\xi) = \sum_{|\alpha| \leqslant m} c_\alpha \xi^\alpha, \quad \xi \in \mathbf{R}^N,$$

is a polynomial with complex coefficients whose principal part

$$\Psi_0(\xi) = \sum_{|\alpha| = m} c_\alpha \xi^\alpha, \quad \xi \in \mathbf{R}^N,$$

has no zero in $\mathbf{R}^N \setminus \{0\}$.

Consider the set $\mathscr{A}^k_m$ of all functions $\varphi \in \mathscr{C}^k(\mathbf{C})$ satisfying (with $z = x + iy$)

$$\|\varphi\|_{m,k} := \sum_{\alpha_1 + \alpha_2 \leqslant k} \frac{1}{\alpha_1! \cdot \alpha_2!} \cdot \sup_{z \in \mathbf{C}} (1 + |z|)^{m(\alpha_1 + \alpha_2)} \left| \frac{\partial^{\alpha_1 + \alpha_2} \varphi}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(z) \right| < \infty.$$

Endowed with this norm $\mathscr{A}^k_m$ is a Banach algebra. Let $\mathscr{B}^k_m$ be the closed subalgebra of all $\varphi \in \mathscr{A}^k_m$ having continuous extensions at $\infty$. Then $\mathscr{A}^k_m$ and $\mathscr{B}^k_m$ have the following two properties.

(A) For every finite open cover $\{U_1, \ldots, U_r\}$ of $\overline{\mathbf{C}}$ there are $\varphi_1, \ldots, \varphi_r$ in $\mathscr{B}^k_m$ such that $\mathrm{supp}(\varphi_j) \subset U_j$, for $1 \leqslant j \leqslant r$, and $\varphi_1 + \ldots + \varphi_r \equiv 1$ on $\mathbf{C}$.

(B) For every $\varphi \in \mathscr{A}^k_m$ and every $\lambda \in \mathbf{C} \setminus \mathrm{supp}(\varphi)$ the function $\varphi_\lambda$ given by $\varphi_\lambda(z) := \varphi(z)(\lambda - z)^{-1}\chi_{\mathrm{supp}(\varphi)}(z)$, $z \in \mathbf{C}$, is in $\mathscr{B}^k_m$.

3.1. THEOREM. *Let $K \leqslant N$ and $\Psi : \mathbf{R}^K \to \mathbf{C}$ be an elliptic polynomial of degree $m$. Consider $\Psi$ as being defined on all of $\mathbf{R}^N$ (i.e. $\Psi$ is constant with respect to the variables $\xi_j$ for $K < j \leqslant N$). Let $k > N/2$. Then, for every $\varphi \in \mathscr{A}^k_m$, the function $\varphi \circ \Psi$ is in $\mathscr{M}_p(\mathbf{R}^N)$ and the homomorphism $\varphi \mapsto T^p_{\varphi \circ \Psi} =: \Phi(\varphi)$ from $\mathscr{A}^k_m$ to $\mathscr{L}(L^p(\mathbf{R}^N))$ is continuous with the following properties.*

(a) *For all $\varphi \in \mathscr{A}^k_m$ with compact support*

$$\Phi(\varphi)\Psi_p(D) \subseteq \Psi_p(D)\Phi(\varphi) = \Phi(\varphi \cdot \mathrm{id}_{\mathbf{C}}).$$

(b) *There exists a sequence $(\rho_n)_{n=1}^\infty$ of functions in $\mathscr{A}^k_m$ with compact supports such that $\lim_{n\to\infty} \Phi(\rho_n)f = f$, for all $f \in L^p(\mathbf{R}^N)$. Moreover,*

$$\mathrm{D}(\Psi_p(D)) = D_p := \{f \in L^p(\mathbf{R}^N)\, ;\, \lim_{n\to\infty} \Phi(\rho_n \cdot \mathrm{id}_{\mathbf{C}})f \textit{ exists in } L^p(\mathbf{R}^N)\}$$

*and*

$$\Psi_p(D)f = \lim_{n\to\infty} \Phi(\rho_n \cdot \mathrm{id}_{\mathbf{C}})f, \quad \textit{for all } f \in \mathrm{D}(\Psi_p(D)).$$

*Proof.* We shall need the perturbed polynomials $\Psi_n$, $n \in \mathbf{N}$, given by

$$\Psi_n(\xi) := \sum_{|\alpha| \leqslant m} n^{|\alpha|-m} c_\alpha \xi^\alpha, \quad \xi \in \mathbf{R}^N,\ n \in \mathbf{N},$$

where $c_\alpha$, $|\alpha| \leqslant m$, are the coefficients of $\Psi$. Notice, that $\Psi_1 = \Psi$ and that all the polynomials $\Psi_n$, $n \in \mathbf{N}$, have the same principal part $\Psi_0$. Since $\Psi_0$ has no zero in $\mathbf{R}^K \setminus \{0\}$ and the lower order coefficients of $\Psi_n$ tend to 0, for $n \to \infty$, there are constants, $c$, $C$, $M > 0$ (independent of $n$) such that

$$\text{(8)} \qquad c|\xi|^m \leqslant |\Psi_n(\xi)| \leqslant C|\xi|^m, \quad \text{for all } \xi \in \mathbf{R}^K \text{ with } |\xi| \geqslant M.$$

Let $\varphi$ be an arbitrary function in $\mathscr{A}^k_m$. By means of induction, for $\beta \in \mathbf{N}_0^N$ we have $\mathrm{D}^\beta(\varphi \circ \Psi_n) \equiv 0$ if $\beta_j \neq 0$ for some $j > K$ and for $0 \neq \beta \in \mathbf{N}_0^K$ we have

$$\mathrm{D}^\beta(\varphi \circ \Psi_n)(\xi) = \sum_{\alpha_1+\alpha_2 \leqslant |\beta|} p^{(n)}_{\alpha_1,\alpha_2}(\xi) \cdot \frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x^{\alpha_1}\partial y^{\alpha_2}}(\Psi_n(\xi))$$

for some polynomials $p^{(n)}_{\alpha_1,\alpha_2}$ of degree not exceeding $(\alpha_1 + \alpha_2)m - |\beta|$ and depending only on $\Psi$ and $n$. Moreover, the coefficients of $p^{(n)}_{\alpha_1,\alpha_2}$, as functions of $n$, are bounded. Since $\varphi \in \mathscr{A}^k_m$, it follows (see (8)), for $|\xi| > M = M(K)$ and all $n \in \mathbf{N}$, that

$$\left| \frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x^{\alpha_1}\partial y^{\alpha_2}}(\Psi_n(\xi)) \right| \leqslant \frac{C'\|\varphi\|_{m,k}}{c|\xi|^{m(\alpha_1+\alpha_2)}}$$

for some constant $C' > 0$. Hence, for all $\xi \in \mathbf{R}^N$ and $\beta \in \mathbf{N}_0^N$,

$$|\xi|^{|\beta|} \cdot |\mathrm{D}^\beta(\varphi \circ \Psi_n)(\xi)| \leqslant C'' \cdot \|\varphi\|_{m,k}$$

for some constant $C'' > 0$. By the Mihlin multiplier theorem (cf. [27], p. 96) $\varphi \circ \Psi_n$ is a $p$-multiplier on $\mathbf{R}^N$ and

$$\|\varphi \circ \Psi_n\|_{\mathscr{M}_p(\mathbf{R}^N)} \leqslant C_0\|\varphi\|_{m,k}$$

where $C_0 > 0$ depends only on $p$, $N$, and $\Psi$. Putting $n = 1$ it follows that $\Phi: \mathscr{A}^k_m \to \mathscr{L}(L^p(\mathbf{R}^N))$ is a continuous unital homomorphism. (a) is now obvious. To prove (b), fix any $\rho \in \mathscr{C}^\infty_c(\mathbf{C})$ with $\rho(0)=1$ and define $\rho_n(z) := \rho(n^{-m}z)$ for $z \in \mathbf{C}$ and $n \in \mathbf{N}$. Then

$$\|\Phi(\rho_n)\| = \|\rho_n \circ \Psi\|_{\mathscr{M}_p(\mathbf{R}^N)} = \|\delta_n(\rho \circ \Psi_n)\|_{\mathscr{M}_p(\mathbf{R}^N)} \leqslant \|\rho \circ \Psi_n\|_{\mathscr{M}_p(\mathbf{R}^N)} \leqslant C_0\|\rho\|_{m,k},$$

for all $n \in \mathbf{N}$, where $\delta_n$ is the dilation operator defined by $(\delta_n f)(\xi) := f(n^{-1}\xi)$. Since, for all $\varphi \in \mathscr{C}_c^\infty(\mathbf{R}^N)$, we have

$$(\rho_n \circ \Psi)(\xi)\varphi(\xi) = \rho(n^{-1}\Psi(\xi))\varphi(\xi) \to \varphi(\xi) \quad \text{for } n \to \infty$$

in the topology of $\mathscr{S}(\mathbf{R}^N)$ (the Schwartz space of rapidly decreasing functions) we see that $T^p_{\rho_n \circ \varphi} f \to f$ in $L^p(\mathbf{R}^N)$ for $f$ in a dense subset and thus $T^p_{\rho_n \circ \varphi} \to 1$ in the strong operator topology. Obviously $D_p \subseteq D(\Psi_p(D))$ since $\Psi_p(D)$ is closed and $\Phi(\rho_n \cdot \mathrm{id}_{\mathbf{C}}) = \Psi_p(D)\Phi(\rho_n)$ by (a). Conversely, if $f \in D(\Psi_p(D))$, then $\Phi(\rho_n)f \to f$ for $n \to \infty$ and, by (a), $\Psi_p(D)\Phi(\rho_n)f = \Phi(\rho_n)\Psi_p(D)f \to \Psi_p(D)f$. This shows that $D_p = D(\Psi_p(D))$. ∎

If $T \in \mathscr{C}(\mathscr{X})$ and $\mathscr{Y}$ is a closed subspace of $\mathscr{X}$, then $\mathscr{Y}$ is invariant for $T$ if $T(D(T) \cap \mathscr{Y}) \subseteq \mathscr{Y}$. The operator $T|\mathscr{Y}$ with $D(T|\mathscr{Y}) = D(T) \cap \mathscr{Y}$ and $(T|\mathscr{Y})y := Ty$ for $y \in D(T|\mathscr{Y})$ is an element of $\mathscr{C}(\mathscr{Y})$. Recall that $T$ is decomposable in the sense of C. Foiaş if, for every finite open cover $U_1, \ldots, U_r$ of $\bar{\mathbf{C}}$, there are closed invariant subspaces $\mathscr{X}_1, \ldots, \mathscr{X}_r$ for $T$ such that $\sigma(T|\mathscr{X}_j) \subset U_j$, for $j = 1, \ldots, r$, and $\mathscr{X} = \mathscr{X}_1 + \ldots + \mathscr{X}_r$; see [15], [5] for the theory of bounded decomposable operators and [28] for unbounded decomposable operators. An operator $T \in \mathscr{C}(\mathscr{X})$ has the Ljubic-Macaev property [23] if, for every compact set $F \subset \mathbf{C}$, the manifold $\mathscr{X}_T(F)$ is closed in $\mathscr{X}$ and if, for every locally finite open cover $(U_n)_{n=1}^\infty$ of $\mathbf{C}$ by bounded open sets, the space $\mathscr{X}$ is the closed linear span of $\mathscr{X}_T(\overline{U_n})$, $n \in \mathbf{N}$. In the case of bounded operators this class is strictly larger than the class of decomposable operators (see [2] for an example). In the unbounded case these two classes are no longer comparable. For example, the operator $T$ in the Banach space $\mathscr{C}_b(\mathbf{C})$ on all bounded continuous functions on $\mathbf{C}$, defined by $D(T) := \{f \in \mathscr{C}_b(\mathbf{C});\ \sup_{z \in \mathbf{C}} |zf(z)| < \infty\}$ and $(Tf)(z) := zf(z)$, for $f \in D(T)$, $z \in \mathbf{C}$, is decomposable in the sense of Foiaş but does not have the Ljubic-Macaev property.

3.2. LEMMA. *Let $A \in \mathscr{C}(\mathscr{X})$ and $B \in \mathscr{C}(\mathscr{Y})$ be decomposable operators and suppose that $J \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is an injective operator satisfying $JA \subseteq BJ$. Then $\sigma(A) \subseteq \sigma(B)$.*

*Proof.* Let $\lambda \in \bar{\mathbf{C}} \setminus \sigma(B)$ and fix open sets $U, W \subseteq \bar{\mathbf{C}}$ with $\lambda \in W$, $\bar{U} \cap \sigma(B) = \emptyset$ and $\bar{W} \subset U$. From the definition of $\mathscr{X}_A(F)$ and $\mathscr{Y}_B(F)$ it follows that $J\mathscr{X}_A(F) \subseteq \mathscr{Y}_B(F)$, for all closed $F \subseteq \bar{\mathbf{C}}$. Hence, using the fact that $\mathscr{Y} = \mathscr{Y}_B(\sigma(B))$ and [28], Lemma IV.4.20, we obtain $J\mathscr{X}_A(\bar{U}) \subseteq \mathscr{Y}_B(\bar{U}) \cap \mathscr{Y}_B(\sigma(B)) = \mathscr{Y}_B(\emptyset) = \{0\}$. It follows that $\mathscr{X} = \mathscr{X}_A(\bar{U}) + \mathscr{X}_A(\overline{\bar{\mathbf{C}} \setminus W}) = \mathscr{X}_A(\overline{\bar{\mathbf{C}} \setminus W}) \subseteq \mathscr{X}$ and thus $\sigma(A) = \sigma(A|\mathscr{X}_A(\overline{\bar{\mathbf{C}} \setminus W})) \subseteq \bar{\mathbf{C}} \setminus W$ (cf. [28], Proposition IV.3.8.). ∎

3.3. COROLLARY. *Let $1 \leqslant p \leqslant 2$. Suppose $\varphi$ is a Borel measurable function on the dual group $\Gamma$ of a locally compact abelian group $G$ and $A \subseteq T^p_\varphi$ is decomposable in $L^p(G)$. Then $\sigma(A) \subseteq (\mathrm{ess\,range}(\varphi))^-$ where the closure is taken in $\bar{\mathbf{C}}$.*

*Proof.* We use the facts that $\mathcal{F}A \subseteq M_\varphi^q \mathcal{F}$ (where $\mathcal{F} : L^p(G) \to L^q(\Gamma)$ is the Fourier transform, $p^{-1} + q^{-1} = 1$) and $M_\varphi^q$ (the operator of multiplication with $\varphi$ on $L^q(\Gamma)$) is spectral and hence decomposable. Since $\sigma(M_\varphi^q) = (\text{ess range}(\varphi))^-$, the statement follows from Lemma 3.2. ▨

3.4. COROLLARY. *Let* $1 \leqslant p < \infty$ *and* $\varphi \in \mathcal{U}_p(\mathbf{R}^N)$. *If* $S_\varphi^p$ *is decomposable, then* $\sigma(S_\varphi^p) \subseteq (\text{ess range}(\varphi))^-$ *where the closure is in* $\overline{\mathbf{C}}$.

*Proof.* Only the case $p > 2$ remains to be proved. Since $S_\varphi^p$ is densely defined the dual operator $(S_\varphi^p)^*$ exists and is decomposable by Proposition IV.5.6 in [28]. But, $(S_\varphi^p)^* \subseteq T_\varphi^q$ (with $p^{-1} + q^{-1} = 1$) and so the statement follows from Corollary 3.3. ▨

So, if $\varphi \in \mathcal{U}_p(\mathbf{R}^N)$ is any bounded function which is not in $\mathcal{M}_p(\mathbf{R}^N)$, then $S_\varphi^p$ cannot be decomposable (and hence not spectral). For example, the function

$$\varphi(\xi) := (\xi_1 - \xi_2^2 - \ldots - \xi_N^2 + \mathrm{i})^{-1}, \quad \text{for } \xi = (\xi_1, \ldots, \xi_N) \in \mathbf{R}^N, \tag{9}$$

is known not to be a $p$-multiplier on $\mathbf{R}^N$ for $p \neq 2$, [21]. Hence, $S_\varphi^p$ cannot be decomposable for $p \neq 2$. This result of Kenig and Tomas can also be used to prove the following

3.5. COROLLARY. *Let* $N \geqslant 2$ *and* $\Psi(\xi) := \xi_1 - \xi_2^2 - \ldots - \xi_N^2$, *for* $\xi \in \mathbf{R}^N$. *Then the differential operator*

$$\Psi_p(D) = \mathrm{i}\frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$$

*is not decomposable in* $L^p(\mathbf{R}^N)$ *whenever* $p \neq 2$.

*Proof.* If $(\mathrm{i} - \Psi_p(D))^{-1}$ would exist in $\mathcal{L}(L^p(\mathbf{R}^N))$, then (by restriction to $\mathcal{S}(\mathbf{R}^N)$) it can be seen that $(\mathrm{i} - \Psi_p(D))^{-1}$ would have to coincide with $S_\varphi^p$ ($\varphi$ as in (9)). But, $\varphi \notin \mathcal{M}_p(\mathbf{R}^N)$ and hence, $\mathrm{i} \in \sigma(\Psi_p(D)) \setminus (\text{ess range}(\varphi))^-$. Corollary 3.4. implies that $\Psi_p(D)$ cannot be decomposable. ▨

Thus, for $N \geqslant 2$, not all linear partial differential operators with constant coefficients are decomposable in $L^p(\mathbf{R}^N)$ if $p \neq 2$.

3.6. THEOREM. *Let* $K \leqslant N$ *and* $\Psi : \mathbf{R}^K \to \mathbf{C}$ *be an elliptic polynomial of degree* $m \geqslant 1$. *Considering* $\Psi$ *as being defined on all of* $\mathbf{R}^N$ *(cf. Theorem* 3.1*) the operator* $\Psi_p(D)$ *is decomposable in* $L^p(\mathbf{R}^N)$ *and has the Ljubic-Macaev property. Moreover,*

(a) $\sigma(\Psi_p(D)) = \Psi(\mathbf{R}^N) \cup \{\infty\} = \text{supp}(\Phi)$, *where* $\Phi$ *is the functional calculus of Theorem* 3.1, *and*

(b) *for all closed* $F \subseteq \bar{\mathbf{C}}$ *we have*

$$L^p(\mathbf{R}^N)_{\Psi_p(D)}(F) = \mathscr{E}(F) := \bigcap \{\ker(\Phi(\varphi))\,;\ \varphi \in \mathscr{B}^k_m,\ \operatorname{supp}(\varphi) \cap F = \emptyset\}.$$

*Proof.* To simplify the notation, write $\mathscr{X} := L^p(\mathbf{R}^N)$ and $T := \Psi_p(D)$. Since $\mathscr{B}^k_m$ is quasi-admissible in the sense of [28], Definition IV.9.2, the operator $T = \Psi_p(D)$ is decomposable by Theorem 3.1 and [28], Corollary IV.9.8. Corollary 3.3 implies the first equality in (a). In particular, the decomposability of $\Psi_p(D)$ implies that the spaces $\mathscr{X}_T(F)$ are closed invariant subspaces for $T$ with $\sigma(T|\mathscr{X}_T(F)) \subseteq F \cap \sigma(T)$ (see the proof of Lemma IV.4.19 in [28]). By the proof of Theorem IV.9.6 in [28], the mapping $F \mapsto \mathscr{E}(F)$ is a spectral capacity for $T$ and $\sigma(T|\mathscr{E}(F)) \subseteq F$, for all closed $F \subseteq \bar{\mathbf{C}}$. In particular, this proves $\mathscr{E}(F) \subseteq \mathscr{X}_T(F)$. Since $\Phi(\varphi)\mathscr{X}_T(F) \subseteq \mathscr{X}_T(F)$ and $\Phi(\varphi)(L^p(\mathbf{R}^N)) \subseteq \mathscr{E}(\operatorname{supp}(\varphi))$ it follows, for all $x \in \mathscr{X}_T(F)$ and $\varphi \in \mathscr{B}^k_m$ with $\operatorname{supp}(\varphi) \cap F = \emptyset$, that

$$\Phi(\varphi)x \in \mathscr{X}_T(F) \cap \mathscr{E}(\operatorname{supp}(\varphi)) \subseteq \mathscr{X}_T(F) \cap \mathscr{X}_T(\operatorname{supp}(\varphi)) =$$

$$= \mathscr{X}_T(F \cap \operatorname{supp}(\varphi)) = \mathscr{X}_T(\emptyset) = \{0\},$$

where we used Lemma IV.4.20, Corollary IV.4.17, and Proposition IV.3.6 in [28]. Hence, $\mathscr{X}_T(F) = \mathscr{E}(F)$ holds for all closed subsets of $\bar{\mathbf{C}}$.

Since $\mathscr{X}_T(\sigma(T)) = \mathscr{X}$ we have $\Phi(\varphi) = 0$ for all $\varphi \in \mathscr{B}^k_m$ with $\operatorname{supp}(\varphi) \cap \sigma(T) = \emptyset$. Hence, $\Phi$ vanishes on $\bar{\mathbf{C}} \setminus \sigma(T)$. Conversely, if $F \subseteq \bar{\mathbf{C}}$ is a closed set such that $\Phi(\varphi) = 0$ for all $\varphi \in \mathscr{B}^k_m$ with $\operatorname{supp}(\varphi) \cap \sigma(T) = \emptyset$, then $\mathscr{X} = \mathscr{E}(F) = \mathscr{X}_T(F)$ and $\sigma(T) = \sigma(T|\mathscr{X}_T(F)) \subseteq F$. This establishes (a).

The Ljubic-Macaev property for $T$ follows from Theorem 3.1 and [1], Satz III.2.17. For the sake of completeness we include the argument. Let $(U_n)_{n=1}^{\infty}$ be a locally finite cover of $\mathbf{C}$ by bounded open sets. For a given $n \in \mathbf{N}$ there is some $r \in \mathbf{N}$ such that (with $\rho_n$ as in the proof of Theorem 3.1) $\operatorname{supp}(\rho_n) \subseteq U_1 \cup \ldots \cup U_r$. By property (A) there are $\varphi_1, \ldots, \varphi_r \in \mathscr{B}^k_m$ such that $\operatorname{supp}(\varphi_j) \subseteq U_j$ $(1 \leq j \leq r)$ and $\varphi_1 + \ldots + \varphi_r \equiv 1$ in a neighbourhood of $\operatorname{supp}(\rho_n)$. If $x \in \mathscr{X}$ is given, then

$$\Phi(\rho_n)x = \sum_{j=1}^{r} \Phi(\varphi_j\rho_n)x \in \sum_{j=1}^{r} \mathscr{E}(\overline{U_j}) = \sum_{j=1}^{r} \mathscr{X}_T(\overline{U_j}).$$

Hence, $x = \lim_{n\to\infty} \Phi(\rho_n)x$ is contained in the closed linear span of $\mathscr{X}_T(\overline{U_j})$, $j \in \mathbf{N}$. ∎

In $\mathbf{R}^1$ every polynomial of positive degree is elliptic. Since constant multiples of the identity operator are decomposable, we obtain the following

3.7. Corollary. *Every linear differential operator with constant coefficients is decomposable in* $L^p(\mathbf{R})$, $1 < p < \infty$, *and has the Ljubic-Macaev property.*

## 4. APPENDIX: PROOFS OF THE STATEMENTS IN SECTION 1

*Proof of Lemma 1.1.* Since $T|P(E)\mathscr{X}$, for $E \in \mathscr{B}$, is densely defined the dual operators $(T|P(E)\mathscr{X})^*$ exist and can be identified with $T^*|P^*(E)\mathscr{X}^*$ (after noting that $P^*(E)\mathscr{X}^*$ is isomorphic with $(P(E)\mathscr{X})^*$. The argument then proceeds analogously to the case when $T$ is bounded ([8], p. 250). ▨

For the proof of Theorem 1.2 we need the following

4.1. LEMMA. *Let $S \in \mathscr{C}(\mathscr{X})$ be a quasispectral operator of class $\Gamma$ and let $P : \mathscr{B} \to \mathscr{L}(\mathscr{X})$ be a resolution of the identity of class $\Gamma$ for $S$.*

(a) *$P(U)S = P(U)SP(U)$ for every bounded open set $U \subseteq \mathbf{C}$.*

(b) *For a bounded open set $U \subseteq \mathbf{C}$ define $S_U \in \mathscr{L}(P(U)\mathscr{X})$ by $S_U x := P(U)Sx$ for all $x \in \mathscr{X}$. Then $\sigma(S_U) \subseteq \overline{U}$.*

(c) *For every closed set $F \subseteq \mathbf{C}$ we have $P(F)\mathscr{X} = \mathscr{X}_S(\bar{F})$ where $\bar{F}$ is the closure of $F$ in $\bar{\mathbf{C}}$.*

*Proof.* (a) follows from (1).

(b) Since $U$ is bounded we have $P(U)\mathscr{X} \subseteq P(\overline{U})\mathscr{X} \subseteq \mathrm{D}(S)$ and therefore $\infty \in \rho(S|P(U)\mathscr{X})$. Fix $z \in \mathbf{C}\setminus\overline{U}$ and let $x \in P(U)\mathscr{X}$. By (1) we have $z \in \rho(S|P(\overline{U})\mathscr{X})$. Choose $y \in P(\overline{U})\mathscr{X}$ with $(z - S)y = x$. Then (a) implies that

$$(z - S_U)P(U)y = P(U)(z - S)P(U)y = P(U)(z - S)y = P(U)x = x$$

and hence, $z - S_U$ is surjective. If now $x \in P(U)\mathscr{X}$ satisfies $S_U x = 0$, then

$$(z - S)P(U)x = (z - S)P(\overline{U})P(U)x = P(\overline{U})(z - S)P(\overline{U})P(U)x =$$

$$= P(\overline{U})(z - S)P(U)x = P(\partial U)(z - S)P(U)x + (z - S_U)x = P(\partial U)(z - S)P(U)x,$$

where $\partial U$ denotes the boundary of $U$. Because $z \in \rho(S|P(\partial U)\mathscr{X})$ there is $v = P(\partial U)v \in P(\partial U)\mathscr{X}$ with $(z - S)P(\partial U)v = (z - S)P(U)x$. Since $P(\partial U)v - P(U)x \in P(\overline{U})\mathscr{X}$ and $z \in \rho(S|P(\overline{U})\mathscr{X})$, this implies $P(U)x = P(\partial U)v$ and thus $x = P(U)x = 0$. Therefore, $z - S_U$ is also injective.

(c) If $x \in P(F)\mathscr{X}$, then $f(z) := (z - S|P(F))^{-1}x$, for every $z \in \mathbf{C}\setminus F$, is an analytic $\mathscr{X}$-valued function on $\mathbf{C}\setminus F$ (vanishing at $\infty$ if $F$ is bounded) satisfying $f(\mathbf{C}\setminus F) \subseteq \mathrm{D}(S)$ and $(z - S)f(z) \equiv x$ on $\mathbf{C}\setminus F$. Hence, $P(F)\mathscr{X} \subseteq \mathscr{X}_S(\bar{F})$. Conversely, suppose that $x \in \mathscr{X}_S(\bar{F})$. Then $(z - S)f(z) \equiv x$ on $\mathbf{C}\setminus F$ for some analytic $\mathscr{X}$-valued function $f$ on $\bar{\mathbf{C}}\setminus\bar{F}$ with $f(\mathbf{C}\setminus F) \subseteq \mathrm{D}(S)$. Let $U$ be a bounded open set with $\overline{U} \subseteq \mathbf{C}\setminus F$ and define the analytic function

$$h(z) := \begin{cases} P(U)f(z), & \text{for } z \in \mathbf{C}\setminus F \\ (z - S_U)^{-1}P(U)x, & \text{for } z \in \mathbf{C}\setminus\overline{U}. \end{cases}$$

By means of (a) and (b) it follows that $h$ is well-defined. Since $S_U$ is bounded $h$ vanishes at $\infty$ and so Liouville's theorem implies that $h \equiv 0$. Accordingly, $P(U)x = (z - S_U)(z - S_U)^{-1}P(U)x = 0$, for every $z \in \mathbf{C}\setminus\bar{U}$. Now $\mathbf{C}\setminus F$ is a countable union of increasing bounded open sets $U_n$ in $\mathbf{C}$ with $\bar{U}_n \subseteq \mathbf{C}\setminus F$. For all $\gamma \in \Gamma$, it follows that

$$\langle P(\mathbf{C}\setminus F)x, \gamma\rangle = \lim_{n\to\infty} \langle P(U_n)x, \gamma\rangle = 0$$

and hence $P(\mathbf{C}\setminus F)x = 0$, that is, $x = P(F)x \in P(F)\mathscr{X}$. ▨

*Proof of Theorem 1.2.* (a). Fix a bounded continuous function $f: \mathbf{C} \to \mathbf{C}$. By (3) and Lemma 1.1 it follows, for all closed sets $F \subseteq \mathbf{C}$, that

$$\mathscr{X}_{\Phi_A(f)}(F) = \mathscr{X}_A(f^{-1}(F)) \text{ and } \mathscr{Y}_{\Phi_B(f)}(F) = \mathscr{Y}_B(f^{-1}(F)).$$

From the definition of the spaces $\mathscr{X}_A(F)$ we have

$$T(\mathscr{X}_{\Phi_A(f)}(F)) = T(\mathscr{X}_A(f^{-1}(F))) = T(\mathscr{Y}_B(f^{-1}(F))) = T(\mathscr{Y}_{\Phi_B(f)}(F)). \tag{10}$$

Since $\Phi_A(f)$ and $\Phi_B(f)$ admit continuous functional calculi based on the continuous functions on $\mathbf{C}$ via the formulae $g \mapsto \Phi_A(g\circ f)$ and $g \mapsto \Phi_B(g\circ f)$, it follows (cf. [3], Remark 1(b)) that $\Phi_A(f)$ and $\Phi_B(f)$ are normal equivalent operators. By the proof of Theorem 1 in [3], Equation (10) implies (4).

(b) For all bounded continuous functions $f$ on $\mathbf{C}$, (4) implies that $\langle \Phi_A(f)x, T^*\gamma\rangle = \langle \Phi_B(f)Tx, \gamma\rangle$, for every $x \in \mathscr{X}$ and $\gamma \in \Gamma_B$, and hence

$$\int_{\mathbf{C}} f(z)\, d\langle P_A(z)x, T^*\gamma\rangle = \int_{\mathbf{C}} f(z)\, d\langle P_B(z)Tx, \gamma\rangle.$$

Then (5) follows from the Riesz representation theorem and the totality of $\Gamma_B$. It is clear that (5) implies (4) for all bounded Borel measurable functions on $\mathbf{C}$. ▨

*Proof of Corollary 1.5.* We have to show that the operator $A$ coincides with its scalar part $\Phi_A(\mathrm{id})$, where "id" denotes the identity function on $\mathbf{C}$. Since $B$ is of scalar type Corollary 1.4 implies that $T\Phi_A(\mathrm{id}) \subseteq \Phi_B(\mathrm{id})T = BT$. Since $P_A(\mathrm{D}(n))\mathscr{X} \subseteq \mathrm{D}(A) \cap \mathrm{D}(\Phi_A(\mathrm{id}))$, where $\mathrm{D}(n)$ is as in § 1, we obtain

$$TAP_A(\mathrm{D}(n)) = BTP_A(\mathrm{D}(n)) = T\Phi_A(\mathrm{id})P_A(\mathrm{D}(n)).$$

Thus, $AP_A(\mathrm{D}(n)) = \Phi_A(\mathrm{id})P_A(\mathrm{D}(n))$, for all $n \in \mathbf{N}$. Hence, for $x \in \mathrm{D}(A)$, we have $Ax = \lim_{n\to\infty} P_A(\mathrm{D}(n))Ax = \lim_{n\to\infty} AP_A(\mathrm{D}(n))x = \lim_{n\to\infty} \Phi_A(\mathrm{id})P_A(\mathrm{D}(n))x$. Since $\Phi_A(\mathrm{id})$ is closed this implies $A \subseteq \Phi_A(\mathrm{id})$. In the same way one obtains $\Phi_A(\mathrm{id}) \subseteq A$ and hence $A = \Phi_A(\mathrm{id})$. Moreover, Theorem 1.2 implies $0 = P_B(\mathbf{C} \setminus \sigma(B))T = TP_A(\mathbf{C} \setminus \sigma(B))$ and hence, $P_A(\sigma(B)) = 1$. This shows that

$$\sigma(A) \subseteq \sigma(B). \tag{11}$$

If $f$ is a bounded Borel measurable function on $\mathbf{C}$, then $\Phi_A(f)$ and $\Phi_B(f)$ are again spectral of scalar type (see [9], Theorem XVII.2.10) and satisfy $T\Phi_A(f) = \Phi_B(f)T$ (by Theorem 1.2). Therefore, inclusion (11) also holds for $\Phi_A(f)$ and $\Phi_B(f)$ instead of $A$ and $B$. ▨

*Proof of Proposition 1.6.* Let $P$ be a resolution of the identity of class $\Gamma$ for $T$, where $\Gamma$ is a total subspace of $\mathscr{X}^*$. Extend $P$ to the set $\bar{\mathscr{B}}$ of all Borel sets in $\bar{\mathbf{C}}$ by $\bar{P}(E) := P(E \cap \mathbf{C})$, for $E \in \bar{\mathscr{B}}$. Then the mapping $\Phi : \mathscr{C}(\bar{\mathbf{C}}) \to \mathscr{L}(\mathscr{X})$ given by

$$\Phi(f) := \int_{\mathbf{C}} f(z)\, \mathrm{d}P(z), \quad \text{for } f \in \mathscr{C}(\bar{\mathbf{C}}), \tag{12}$$

is a continuous unital homomorphism. Since $\mathscr{X}$ is weakly sequentially complete there exists a unique spectral measure $Q : \bar{\mathscr{B}} \to \mathscr{L}(\mathscr{X})$, which is $\sigma$-additive in the weak and hence in the strong operator topology and satisfies

$$\Phi(f) := \int_{\bar{\mathbf{C}}} f(z)\, \mathrm{d}Q(z), \quad \text{for all } f \in \mathscr{C}(\bar{\mathbf{C}}). \tag{13}$$

(See Theorem XVII.2.5 in [9].) Evaluating (12) and (13) at $x \in \mathscr{X}$ and $\gamma \in \Gamma$, it follows from the Riesz representation theorem and the totality of $\Gamma$ that $\bar{P} = Q$ and hence, $P(E) = Q(E)$, for all $E \in \mathscr{B}$. Thus, $P$ is actually $\sigma$-additive in the strong operator topology. In particular, this implies that $\mathrm{D}(T)$ is dense in $\mathscr{X}$ and so $T^*$ exists.

In order to prove that $T$ is a spectral operator it remains to show that $T$ satisfies condition (2).

Define $P^* : \mathscr{B} \to \mathscr{L}(\mathscr{X}^*)$ by $P^*(E) := P(E)^*$, $E \in \mathscr{B}$. Then $P^*$ is a spectral measure of class $\mathscr{X}$ on $\mathscr{X}^*$. We show that $P(E^*)\mathscr{X}^* \subseteq \mathrm{D}(T^*)$ holds for all bounded $E \in \mathscr{B}$. Let $U$ be an open bounded set containing $E$. For all $x \in \mathrm{D}(T)$ and $x^* \in \mathscr{X}^*$ we have, by Lemma 4.1 (a), that

$$\langle Tx,\ P^*(E)x^* \rangle = \langle Tx,\ P^*(U)P^*(E)x^* \rangle = \langle P(U)TP(U)P(U)x,\ P^*(E)x^* \rangle.$$

Then Lemma 4.1 (b) implies that $P^*(E)x^* \in \mathrm{D}(T^*)$. Since $P^*$ is a spectral measure of class $\mathscr{X}$ the set $A := \{P^*(E)x^*\,;\ x^* \in \mathscr{X}^*,\ E \in \mathscr{B} \text{ bounded}\}$ is weak*-dense in $\mathscr{X}^*$ and therefore total. Fix an arbitrary compact set $K \subseteq \mathbf{C}$, a bounded $E \in \mathscr{B}$, and vectors $x \in \mathrm{D}(T)$, $x^* \in \mathscr{X}^*$. Let $(H_n)_{n=1}^{\infty}$ be an increasing sequence of compact sets with union $\mathbf{C}\setminus K$. By (1), we have $P(K)TP(H_n) = P(K)P(H_n)TP(H_n) = 0$ for all $n \in \mathbf{N}$ and thus,

$$0 = \lim_{n\to\infty} \langle P(K)TP(H_n)x,\ P^*(E)x^*\rangle = \lim_{n\to\infty} \langle P(H_n)x,\ T^*P^*(K)P^*(E)x^*\rangle =$$

$$= \langle P(\mathbf{C}\setminus K)x,\ T^*P^*(K)P^*(E)x^*\rangle = \langle P(K)T(1-P(K))x,\ P^*(E)x^*\rangle.$$

Hence, using (1) again, it follows that

$$\langle P(K)Tx,\ P^*(E)x^*\rangle = \langle P(K)TP(K)x,\ P^*(E)x^*\rangle = \langle TP(K)x,\ P^*(E)x^*\rangle.$$

Since the Borel measures $\langle P(\cdot)Tx,\ P^*(E)x^*\rangle$ and $\langle P(\cdot)x,\ T^*P^*(E)x^*\rangle$ coincide on all compact sets they coincide on $\mathscr{B}$. It follows that $P(E)T \subseteq TP(E)$ for all bounded sets $E \in \mathscr{B}$. If $E$ is now a unbounded Borel set, then $E$ is the union of an increasing sequence of bounded Borel sets $E_n \subseteq \mathbf{C}$. It follows, for all $x \in D(T)$, that $P(E)x = \lim_{n\to\infty} P(E_n)x$ and $\lim_{n\to\infty} TP(E_n)x = \lim_{n\to\infty} P(E_n)Tx = P(E)Tx$. Since $T$ is a closed operator, this shows that $P(E)x \in D(T)$ and $TP(E)x = P(E)Tx$. Hence, $P(E)T \subseteq TP(E)$. For $z \in \mathbf{C}\setminus \bar{E}$ the operator $R := P(E)(z - T\,P(\bar{E})\mathscr{X})^{-1}P(E)\mathscr{X}$ belongs to $\mathscr{L}(P(E)\mathscr{X})$ and satisfies $R = (z - T\,P(E)\mathscr{X})^{-1}$. Therefore, condition (2) is satisfied for $T$. ∎

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*ERNST ALBRECHT*
*Fachbereich Mathematik,*
*Universität des Saarlandes,*
*D-6600 Saarbrücken,*
*Fed. Rep. Germany.*

*WERNER J. RICKER*
*School of Mathematics,*
*University of N.S.W.*
*Kensington, N.S.W, 2033,*
*Australia.*

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