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# A CHARACTERIZATION OF UHF $C^*$-ALGEBRAS

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## 1. INTRODUCTION

A $C^*$-algebra $\mathscr{A}$ (with unit 1) is defined to be approximately finite-dimensional (AF $C^*$-algebra) if there is an ascending sequence $\{\mathscr{A}_n\}_{n\geqslant 0}$ of finite-dimensional $C^*$-algebras in $\mathscr{A}$, each of them containing 1, such that $\bigcup\limits_{n\geqslant 0} \mathscr{A}_n$ is dense in $\mathscr{A}$. $\mathscr{A}$ is said to be uniformly hyperfinite (UHF $C^*$-algebra) if the $C^*$-subalgebras $\mathscr{A}_n$ can be chosen to be $*$-isomorphic to full complex matrix algebras. In [9, Theorem I.1.10] Strătilă and Voiculescu showed that given an AF $C^*$-algebra $\mathscr{A}$ and an ascending sequence $\{\mathscr{A}_n\}_{n\geqslant 0}$ with union dense in $\mathscr{A}$, there is a maximal abelian subalgebra (m.a.s.a.) $\mathscr{C}$ of $\mathscr{A}$, a conditional expectation $P$ of $\mathscr{A}$ with respect to $\mathscr{C}$ and a subgroup $\mathscr{U}$ of the unitary group of $\mathscr{A}$ such that

i) $u\mathscr{C}u^* = \mathscr{C}$ for all $u \in \mathscr{U}$ and

ii) $P(uxu^*) = uP(x)u^*$ for all $u \in \mathscr{U}$ and $x \in \mathscr{A}$,

iii) $\mathscr{A}$ = closed linear span of $(\mathscr{U}\mathscr{C})$ = c.l.s.$(\mathscr{C}\mathscr{U})$.

This generalized a similar result for UHF $C^*$-algebras, obtained by Garding and Wightman [4]. The group $\mathscr{U}$ is a countable discrete proper subgroup of the normalizer of $\mathscr{C}$ and can be viewed as a group of $*$-automorphisms of $\mathscr{C}$, by defining for each $u \in \mathscr{U}$, $\alpha_u(x) = uxu^*$, for $x \in \mathscr{C}$. In fact $\mathscr{U}$ is chosen so that the correspondence $u \leftrightarrow \alpha_u$ is one-to-one.

It follows from the work of Garding and Wightman [4], that a UHF $C^*$-algebra $\mathscr{A}$, when diagonalized relative to an ascending sequence $\{\mathscr{A}_n\}_{n\geqslant 0}$ of full matrix a gebras, has the following property

($*$) There is an abelian subgroup $\mathscr{G}$ of $\mathscr{U}$ such that $\mathscr{A}$ is $*$-isomorphic to $\mathscr{G}\times_\alpha \mathscr{C}$, the crossed product of $\mathscr{C}$ by $\mathscr{G}$, where $\alpha$ is the action of $\mathscr{U}$ on $\mathscr{C}$ restricted to $\mathscr{G}$.

(See also [8, 7.10.8] for a proof of this fact.) Thus, it is natural to ask whether any simple AF $C^*$-algebra satisfying ($*$) is necessarily UHF. The purpose of this paper is to provide an answer to this question.

The main result is:

THEOREM 1.1. *Let $\mathcal{A}$ be a simple* AF *$C^*$-algebra, $\{\mathcal{A}_n\}_{n\geq 0}$ an ascending sequence of finite-dimensional $C^*$-subalgebras of $\mathcal{A}$, with $\bigcup_{n\geq 0} \mathcal{A}_n$ dense in $\mathcal{A}$. Let $\mathcal{C}$ be the diagonal of $\mathcal{A}$, $\mathcal{U}$, the unitary group related to the diagonalization of $\mathcal{A}$ by $\mathcal{C}$, $\mathcal{C}_n = \mathcal{C} \cap \mathcal{A}_n$ and for $\mathcal{G} \subseteq \mathcal{U}$ let $\mathcal{G}_n = \mathcal{G} \cap \mathcal{A}_n$. Then $\mathcal{A}$ is a* UHF *$C^*$-algebra iff the following conditions are satisfied*

(a) *There is an abelian subgroup $\mathcal{G}$ of $\mathcal{U}$ such that $\mathcal{A}$ is $*$-isomorphic to $\mathcal{G} \times_\alpha \mathcal{C}$, where $\alpha$ is the action of $\mathcal{U}$ on $\mathcal{C}$ restricted to $\mathcal{G}$. The isomorphism carries $\mathcal{G}$ and $\mathcal{C}$ into their canonical images in $\mathcal{G} \times_\alpha \mathcal{C}$.*

(b) *Given $n$, there is $m \geq n$ such that each minimal projection of $\mathcal{C}_n = \mathcal{C} \cap \mathcal{A}_n$ embeds into every $\mathcal{G}_m$-orbit in $\mathcal{C}_m$ with the same multiplicity.*

Theorem 1.1 is proved in Section 3 of this paper. Section 4 contains an example of a simple non-UHF $C^*$-algebrat hat satisfies condition (a) but does not satisfy condition (b), thus showing that there are simple non-UHF AF $C^*$-algebras satisfying ($*$). Section 2 contains some preliminary results.

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## 2. PRELIMINARIES

Given an AF $C^*$-algebra $\mathcal{A}$ with approximating sequence $\{\mathcal{A}_n\}_{n\geq 0}$, we will always assume that $\mathcal{A}_0 = \mathbf{C}1$, where $\mathbf{C}$ denote the complex numbers, and we will denote by $\mathcal{C}$, the diagonal algebra of $\mathcal{A}$, and by $\mathcal{C}_n$, the $C^*$-algebra $\mathcal{A}_n \cap \mathcal{C}$, which is maximal abelian in $\mathcal{A}_n$ [9, Lemma I.1.1.]. $\mathcal{U}$ will denote the unitary group related to the diagonalization of $\mathcal{A}$ by $\mathcal{C}$, and $\mathcal{U}_n$ will be $\mathcal{U} \cap \mathcal{A}_n$.

The group $\mathcal{U}$, constructed in [9, I.1.9], is a proper subgroup of the normalizer of $\mathcal{C}$. $\mathcal{U} = \bigcup_{n\geq 0} \mathcal{U}_n$, where $\mathcal{U}_n$ is the finite group corresponding to the permutations of the minimal projections of $\mathcal{C}_n$ that fix the central projections of $\mathcal{A}_n$. Thus $\mathcal{U}$ is a countable discrete group.

We have also noted that $\mathcal{U}$ acts as a group of automorphisms of $\mathcal{C}$, by

$$\alpha_u(x) = uxu^* \quad \text{for } u \in \mathcal{U}, x \in \mathcal{C}.$$

This action is faithful [9, I.1.9].

Let $\mathcal{Z}$ denote the Gelfand spectrum of $\mathcal{C}$, and let $\sigma_u$ be the homeomorphism of $\mathcal{Z}$ associated with $\alpha_u$, that is

$$\hat{x}(\sigma_u(z)) = (\alpha_u(x))^\wedge(z) \quad \text{for } z \in \mathcal{Z},\ x \in \mathcal{C},$$

where ^ denotes the Gelfand transform of $\mathcal{C}$.

Let us begin by obtaining some consequences of condition (a) in Theorem 1.1. If $\mathscr{A}$ is a simple AF $C^*$-algebra with $\mathscr{G}$ and $\mathscr{C}$ as in (a) of Theorem 1.1, then it follows from [8, Proposition 7.6.4] that $\mathscr{A}$ is generated by $\mathscr{G}$ and $\mathscr{C}$, and since $\mathscr{G}$ acts on $\mathscr{C}$ as a group of automorphisms, we have that

$$\mathscr{A} = \text{closed linear span of } (\mathscr{G}\mathscr{C}) = \text{closed linear span}(\mathscr{C}\mathscr{G}).$$

We first establish the following variations of [9, Lemma I.2.2 and Theorem I.2.4].

LEMMA 2.1. *Let $\mathscr{I}$ be a $\mathscr{G}$-invariant closed ideal of $\mathscr{C}$. Then*

$$\mathscr{J}(\mathscr{I}) = \{x \in \mathscr{A} : P(x^*x) \in \mathscr{I}\}$$

*where $P$ is the conditional expectation of $\mathscr{A}$ with respect to $\mathscr{C}$ constructed in* [9], *is a closed two-sided ideal of $\mathscr{A}$.*

*Proof.* The argument used in [9, Lemma I.2.2] can be used here to show that $\mathscr{J}(\mathscr{I})$ is a closed left ideal of $\mathscr{A}$. In order to show that $\mathscr{J}(\mathscr{I})$ is also a right ideal, let $a \in \mathscr{A}$ and $x \in \mathscr{J}(\mathscr{I})$. By the remark previous to this lemma, it suffices to take $a = gc$, $g \in \mathscr{G}$, $c \in \mathscr{C}$. Then:

$$P((xa)^*(xa)) = P(c^*g^*x^*xgc) = c^*cg^*P(x^*x)g.$$

$\mathscr{I}$ is $\mathscr{G}$-invariant and $P(x^*x) \in \mathscr{I}$, so $P((xa)^*(xa)) \in \mathscr{I}$, and therefore $\mathscr{J}(\mathscr{I})$ is also a right ideal. ▨

LEMMA 2.2. *For any closed two-sided ideal $\mathscr{K}$ of $\mathscr{A}$ we have that*

$$\mathscr{K} = \mathscr{J}(\mathscr{K} \cap \mathscr{C}) = \{x \in \mathscr{A} : P(x^*x) \in \mathscr{K} \cap \mathscr{C}\}.$$

*For any closed $\mathscr{G}$-invariant ideal $\mathscr{I}$ of $\mathscr{C}$ we have that*

$$\mathscr{I} = \mathscr{J}(\mathscr{I}) \cap \mathscr{C}.$$

*Thus the correspondences $\mathscr{K} \mapsto \mathscr{K} \cap \mathscr{C}$ and $\mathscr{I} \mapsto \mathscr{J}(\mathscr{I})$ are inverses of each other and they define a bijective correspondence between the set of all closed two-sided ideals of $\mathscr{A}$ and the set of all closed $\mathscr{G}$-invariant ideals of $\mathscr{C}$.*

*Proof.* The same argument given in [9, Theorem I.2.4] will show that $\mathscr{K} = \mathscr{J}(\mathscr{K} \cap \mathscr{C})$, since that equality only depends on $\mathscr{K}$ being a closed two-sided ideal of $\mathscr{A}$.

$\mathscr{I} = \mathscr{J}(\mathscr{I}) \cap \mathscr{C}$ can be easily verified for any closed ideal $\mathscr{I}$ of $\mathscr{C}$. If $\mathscr{I}$ is $\mathscr{G}$-invariant, Lemma 2.1 shows that $\mathscr{J}(\mathscr{I})$ is a closed two-sided ideal of $\mathscr{A}$, so the correspondences establish a bijection between closed two-sided ideals of $\mathscr{A}$ and closed $\mathscr{G}$-invariant ideals of $\mathscr{C}$. ▨

COROLLARY 2.3. *A closed subset $F$ of $\mathscr{X}$ is $\mathscr{G}$-invariant iff it is $\mathscr{U}$-invariant.*

*Proof.* $\mathscr{G} \subseteq \mathscr{U}$, so any $\mathscr{U}$-invariant set is also $\mathscr{G}$-invariant.

Let $F \subseteq \mathscr{X}$ be $\mathscr{G}$-invariant. Then $\mathscr{I}_F = \{x \in \mathscr{C} : \hat{x}|F = 0\}$ is a $\mathscr{G}$-invariant closed ideal of $\mathscr{C}$. Thus, by Lemma 2.1, $\mathscr{J}(\mathscr{I}_F)$ is a closed two-sided ideal of $\mathscr{A}$, so [9, Theorem I.2.4] implies that $\mathscr{I}_F = \mathscr{J}(\mathscr{I}_F) \cap \mathscr{C}$ is $\mathscr{U}$-invariant. It follows that $F$ is $\mathscr{U}$-invariant. ▨

LEMMA 2.4. *$\mathscr{G}$ acts freely and minimally on $\mathscr{X}$.*

*Proof.* Let $F$ be a closed $\mathscr{G}$-invariant subset of $\mathscr{X}$. Then by Corollary 2.3, $\mathscr{J}(\mathscr{I}_F)$ is a two-sided ideal of $\mathscr{A}$. The simplicity of $\mathscr{A}$ gives that $\mathscr{J}(\mathscr{I}_F)$ is either $\{0\}$ or $\mathscr{A}$. Hence $\mathscr{I}_F = \mathscr{J}(\mathscr{I}_F) \cap \mathscr{C}$ is either $\{0\}$ or $\mathscr{C}$, and therefore $F$ is either $\emptyset$ or $\mathscr{X}$.

If $\sigma_t(z) = z$ for some $t \in \mathscr{G}$, $z \in \mathscr{X}$ then

$$\sigma_t(\sigma_s(z)) = \sigma_s(\sigma_t(z)) = \sigma_s(z).$$

Since $\{\sigma_s(z) : s \in \mathscr{G}\}$ is dense in $\mathscr{X}$, it follows that $\sigma_t$ is the identity map on $\mathscr{X}$. But $\mathscr{G}$ acts faithfully on $\mathscr{X}$ (since $\mathscr{U}$ does), so $t = e$, the identity in $\mathscr{G}$. ▨

Let $\mathscr{G}_n$ be the finite group $\mathscr{G} \cap \mathscr{A}_n$ and let $\mathscr{X}_n$ be the set of minimal projections of $\mathscr{C}_n$. Note that $\mathscr{X}$ is homeomorphic to the inverse limit of the sequence $\{\mathscr{X}_n\}_{n>0}$.

LEMMA 2.5. *$\mathscr{G}_n$ acts freely on $\mathscr{X}_n$.*

*Proof.* Let $t \in \mathscr{G}_n$ and $p_0 \in \mathscr{X}_n$ satisfying $\alpha_t(p_0) = p_0$. By the definition of $\mathscr{U}_n$ [9, I.1.9]

$$t = \sum_{p,q \in \mathscr{X}_n} \lambda_{pq} e_{pq}$$

where $e_{pq}$ are matrix units for $\mathscr{A}_n$ with respect to $\mathscr{C}_n$, and $\lambda_{pq}$ is either 0 or 1. Since $e_{p_0 p_0} = p_0$, $\alpha_t(p_0) = p_0$ implies that

$$\Big(\sum_{p,q \in \mathscr{X}_n} \lambda_{pq} e_{pq}\Big) e_{p_0 p_0} \Big(\sum_{p,q \in \mathscr{X}_n} \lambda_{pq} e_{pq}^*\Big) = e_{p_0 p_0}.$$

Thus

$$e_{kp_0} e_{p_0 p_0} e_{p_0 k} = e_{p_0 p_0}$$

where $e_{kp_0}$ is the only matrix unit in the expansion of $t$ with initial space the range of $e_{p_0 p_0}$ and $\lambda_{kp_0} = 1$. Thus $k = p_0$, so $e_{p_0 p_0}$ appears in the expansion of $t$ with coefficient one, which shows that $\sigma_t$ fixes every point in the subset of $\mathscr{X}$ corresponding to $p_0$, contradicting Lemma 2.4. ▨

It follows that $\mathscr{X}_n$ can be written as a union of disjoint $\mathscr{G}_n$-orbits, each of them with $|\mathscr{G}_n|$ elements (where $|\mathscr{G}_n|$ is the order of $\mathscr{G}_n$).

Let us label the minimal projections of $\mathscr{Z}_n$ as $p_{jt}$, for $j = 1, \ldots, N$ and $t \in \mathscr{G}_n$, where $N$ is the number of $\mathscr{G}$-orbits in $\mathscr{Z}_n$, and

$$\alpha_s(p_{jt}) = p_{j(st)}$$

so that $\{p_{jt} : t \in \mathscr{G}_n\}$ forms a $\mathscr{G}_n$-orbit in $\mathscr{Z}_n$. Let

$$P_j = \sum_{t \in \mathscr{G}_n} p_{jt}.$$

The next lemma follows from general theorems about ergodic actions of abelian groups (see, for example [3, Corollary 5.16]). However, we outline a proof for completeness.

Let $\mathscr{B}_n$ be the $C^*$-algebra generated by $\mathscr{G}_n$ and $\mathscr{C}_n$, and let

$$P_j\mathscr{G}_n = \{P_j t : t \in \mathscr{G}_n\}.$$

LEMMA 2.6. *Fix $j$. Let $\mathscr{M}$ be the $C^*$-subalgebra of $\mathscr{B}_n$ generated by $P_j\mathscr{G}_n$ and $\{p_{jt} : t \in \mathscr{G}_n\}$. Then $\mathscr{M}$ is $*$-isomorphic to the algebra $M_{|\mathscr{G}_n|}$ of $|\mathscr{G}_n| \times |\mathscr{G}_n|$ complex matrices.*

*Proof.* For $s, t \in \mathscr{G}_n$ define:

$$e_{st} = p_{js}(st^{-1})p_{jt};$$

clearly

$$e_{tt} = p_{jt}, \quad \sum_{t \in \mathscr{G}_n} p_{jt} = P_j$$

and for $r, s, t, u \in \mathscr{G}_n$

$$e_{sr}e_{ut} = \begin{cases} 0 & \text{if } r \neq u \\ e_{st} & \text{if } r = u. \end{cases}$$

Then setting $\Phi(e_{st})$ equal to the $|\mathscr{G}_n| \times |\mathscr{G}_n|$ matrix with $(s, t)$ entry equal to 1 and all others equal to 0, standard arguments show that $\Phi$ extends to a $*$-isomorphism from $\mathscr{M}$ to $M_{|\mathscr{G}_n|}$ (see for example [6, Lemma 6.6.3]). ▨

It follows that $\mathscr{B}_n$ is the direct sum of $N$ $*$-subalgebras, each of them isomorphic to $M_{|\mathscr{G}_n|}$. Let $Q_i$, $i = 1, \ldots, |\mathscr{G}_n|$ be a family of self-adjoint projections in $\mathscr{C}_n$ such that

$$\sum_{i=1}^{|\mathscr{G}_n|} Q_i = 1$$

and $Q_i P_j$ is minimal in $\mathscr{C}_n$. Then if $u$ is a unitary in $\mathscr{B}_n$ such that

$$uQ_i u^* = Q_{i+1} \quad i = 1, 2, \ldots, |\mathscr{G}_n| - 1$$

$$uQ_{|\mathscr{G}_n|} u^* = Q_1$$

we have that an easy adaptation of the proof of Lemma 2.6 shows

LEMMA 2.7. *The unital $*$-subalgebra of $\mathscr{B}_n$ generated by $u$ and $\{Q_i : i = 1, \ldots \ldots, |\mathscr{G}_n|\}$ is $*$-isomorphic to $M_{|\mathscr{G}_n|}$.*

Given an AF $C^*$-algebra $\mathscr{A}$, and its approximating sequence $\{\mathscr{A}_n\}_{n\geq 0}$ of finite-dimensional $*$-subalgebras, we will denote by $\mathscr{A}(j, k)$ the finite-dimensional factors such that

$$\mathscr{A}_j = \bigoplus_{k=1}^{N(j)} \mathscr{A}(j, k).$$

THEOREM 2.8. *Let $\mathscr{A}$ be an AF $C^*$-algebra and $\mathscr{A}_j$ and $\mathscr{A}(j, k)$ as above. If given $n$, there exists $m \geq n$ such that for each $k = 1, \ldots, N(n)$, $r(j, k)$ the multiplicity of the partial embedding of $\mathscr{A}(n, k)$ into $\mathscr{A}(m, j)$ is non-zero and independent of $j$, then $\mathscr{A}$ is a UHF $C^*$-algebra.*

*Proof.* For each $k = 1, \ldots, N(n)$ pick a complete set $\{p_{kl} : l = 1, \ldots, N(n, k)\}$ of minimal projections in $\mathscr{A}(n, k)$. Let $u_k$ be the partial isometry in $\mathscr{A}(n, k)$ corresponding to a cyclic permutation of the $p_{kl}$'s as defined in Lemma 2.7 for the $Q_i$'s. The proof of that lemma shows that $u_k$ and $\{p_{kl} : l = 1, \ldots, N(n, k)\}$ generate $\mathscr{A}(n, k)$.

Now for each $p_{kl}$, let us construct projections $Q(k, l, i)$ in $\mathscr{A}_m$, $i = 1, \ldots \ldots, r(j, k)$ such that

$$\sum_{i=1}^{r(j,k)} Q(k, l, i) = p_{kl} \tag{2.8.1}$$

and $Q(k, l, i)$ is a sum of $N(m)$ minimal projections of $\mathscr{A}_m$, one in each of $\mathscr{A}(m, j)$, $j = 1, \ldots, N(m)$.

Order the set:

$$\{(k, l, i) : i = 1, \ldots, r(j, k); \quad l = 1, \ldots, N(n, k); \quad k = 1, \ldots, N(n)\}$$

lexicographically. Let $v$ be the unitary on $\mathscr{A}_m$ corresponding to the cyclic permutation of

$$\{Q(k, l, i) : i = 1, \ldots, r(j, k); \quad l = 1, \ldots, N(n, k); \quad k = 1, \ldots, N(n)\}$$

that sends $Q(N(n), N(n, k), r(j, k))$ to $Q(1, 1, 1)$, and all others to their succesors in the order just defined.

Thus the proof of Lemma 2.6 shows that $v$ and the $Q(k, l, i)$'s generate a subalgebra $\mathscr{M}$ of $\mathscr{A}_m$ $*$-isomorphic to a matrix algebra. In this case, $\mathscr{M}$ is unital. By (2.8.1) $\{p_{kl} : l = 1, \ldots, N(n, k), k = 1, \ldots, N(n)\}$ is contained in $\mathscr{M}$. Also, it can be verified that if

$$M_k = \sum_{\substack{s=1 \\ s \neq k}}^{N(n)} N(n, s)\, r(j, s)$$

then

$$u_k = v^{r(j,k)} \left( \sum_{l=1}^{N(n,k)-1} \sum_{i=1}^{r(j,k)} Q(k, l, i)] \right) + v^{M_k} \sum_{i=1}^{r(j,k)} Q(k, N(n, k), i)$$

so that $u_k \in \mathscr{M}$ and hence $\mathscr{A}_n \subseteq \mathscr{M}$.

Now, if $x_1, \ldots, x_n \in \mathscr{A}$ and $\varepsilon > 0$ are given, we can choose $j$ and $y_1, \ldots$ $\ldots, y_n \in \mathscr{A}_j$ such that

$$\|x_i - y_i\| < \varepsilon, \quad \text{for } i = 1, \ldots, n.$$

Above argument shows that $\mathscr{A}_j$ is contained in a matrix subalgebra of $\mathscr{A}$, so [5, Theorem 1.13] implies that $\mathscr{A}$ is UHF. ▨

## 3. PROOF OF THEOREM 1.1

Let us assume first that $\mathscr{A}$ is a simple AF $C^*$-algebra satisfying conditions (a) and (b) of Theorem 1.1. Let $\mathscr{B}_n$ be as defined in Section 2. Note that

$$\bigcup_{n \geqslant 0} \mathscr{B}_n$$

is dense in $\mathscr{A}$.

We will show that $\{\mathscr{B}_n\}_{n \geqslant 0}$ satisfies the hypothesis of Theorem 2.8. As a consequence we get that $\mathscr{A}$ is a UHF $C^*$-algebra.

Let

$$\mathscr{B}_j = \bigoplus_{k=1}^{N(j)} \mathscr{B}(j, k)$$

be the central decomposition of $\mathscr{B}_j$ into finite-dimensional factors. Recall that $\mathscr{B}(j, k)$ is $*$-isomorphic to $M_{|\mathscr{P}_j|}$.

LEMMA 3.1. *Given $n$, there exists $m \geqslant n$ such that for each $k$, $r(j, k)$, the multiplicity of the partial embedding of $\mathscr{B}(n, k)$ into $\mathscr{B}(m, j)$ is non-zero and independent of $j$.*

*Proof.* Notice that any two minimal projections of $\mathscr{C}_n$ that are in the same $\mathscr{G}_n$-orbit will embed in exactly the same way into every $\mathscr{G}_j$-orbit in $\mathscr{C}_j$, for $j \geqslant n$. This is because a projection of $\mathscr{C}_j$ corresponding to a $\mathscr{G}_j$-orbit is $\mathscr{G}_j$-invariant. Thus if $P$ is the projection corresponding to a $\mathscr{G}_j$-orbit in $\mathscr{C}_j$ and $p$, $q$ are minimal projections of $\mathscr{C}_n$ such that $\alpha_t(p) = q$, for $t \in \mathscr{G}_n$, then

$$\alpha_t(pP) = \alpha_t(p)P = qP$$

and hence $p$ and $q$ embed with the same multiplicity into the orbit corresponding to $P$.

We also remark that the minimal central projections of $\mathscr{B}_n$ correspond to the $\mathscr{G}_n$-orbits in $\mathscr{C}_n$ and are precisely the projections $P_j = \sum_{t \in \mathscr{G}_n} \alpha_t(p)$, where $p$ is a minimal projection of $\mathscr{C}_n$, defined before Lemma 2.6.

Thus, combining the above, we see that (b) in Theorem 1.1 implies that given $n$, there is $m \geqslant n$ such that each $\mathscr{B}(n, k)$ embeds into every $\mathscr{B}(m, j)$ with the same multiplicity. ▨

In order to prove the remaining implication in Theorem 1.1, let us assume first that $\mathscr{A}$ is a UHF $C^*$-algebra with $\{\mathscr{A}_n\}$, a sequence of complex matrix algebras. Then, by [8, 7.10.8], condition (a) is satisfied if we choose $\mathscr{C}_n$ to be the commutative $C^*$-subalgebra of $\mathscr{A}_n$, corresponding to the diagonal matrices, and $\mathscr{G}_n$, the unitary group generated by a cyclic permutation of the minimal projections of $\mathscr{C}_n$. In [8, 7.10.8], $\mathscr{G}_n$ acts minimally on $\mathscr{C}_n$, hence condition (b) is also satisfied for any $m \geqslant n$.

Now, let $\mathscr{A}$ be a UHF $C^*$-algebra, but assume $\{\mathscr{A}_n\}_{n \geqslant 0}$ is an arbitrary sequence of finite-dimensional $C^*$-subalgebras of $\mathscr{A}$ with $\bigcup_{n \geqslant 0} \mathscr{A}_n$ dense in $\mathscr{A}$. Let $\mathscr{C}$ and $\mathscr{U}$ be the diagonal and unitary group, respectively, obtained through the Strătilă-Voiculescu diagonalization, when applied to $\mathscr{A}$ and $\{\mathscr{A}_n\}$. Let $\mathscr{P}$ be the set of projections of $\mathscr{C}$. Then if $\{p_i : i = 1, \ldots, N\}$ is a collection of pairwise disjoint projections with $\sum_{i=1}^{N} p_i = 1$, and if $\{g_i : i = 1, \ldots, N\} \subseteq \mathscr{U}$ satisfies that $\{\alpha_{g_i}(p_i) : i = 1, \ldots, N\}$ is also a partition, then it can be verified that

$$u = \sum_{i=1}^{N} \alpha_{g_i}(p_i) g_i p_i$$

is an element of $\mathscr{U}$ such that $\alpha_u(p_i) = \alpha_{g_i}(p_i)$. Thus, any homeomorphism of $\mathscr{X}$ that can be patched together from elements of $\mathscr{U}$ is already $\mathscr{U}$, and hence $(\mathscr{P}, \mathscr{U})$ forms a unit system, in the terminology of [7].

Let $\{\mathcal{M}_n\}$ be an ascending sequence of full matrix algebras with union dense in $\mathcal{A}$. Let $\mathcal{D}$ and $\mathcal{V}$ be the diagonal and unitary group, respectively, obtained from $\mathcal{A}$ and $\{\mathcal{M}_n\}$. Then, if $\mathcal{Q}$ denotes the set of projections in $\mathcal{D}$, we also have that $(\mathcal{Q}, \mathcal{V})$ forms a unit system.

LEMMA 3.2. *There exists an automorphism $\Phi$ of $\mathcal{A}$ such that $\Phi(\mathcal{D}) = \mathcal{C}$ and $\Phi(\mathcal{V}) = \mathcal{U}$.*

*Proof.* By [7, Lemma 3.2] we obtain that the scale of the dimension group of $\mathcal{A}$ is isomorphic to the dimension range of $(\mathcal{P}, \mathcal{U})$ (resp. $(\mathcal{Q}, \mathcal{V})$). Thus, the dimension range of $(\mathcal{P}, \mathcal{U})$ is isomorphic to the dimension range of $(\mathcal{Q}, \mathcal{V})$, and [7, Theorem 3.5] implies that there is an automorphism of $\mathcal{A}$ mapping $\mathcal{D}$ onto $\mathcal{C}$ and $\mathcal{V}$ onto $\mathcal{U}$. ▨

REMARK 3.3. Lemma 3.2 holds for any AF $C^*$-algebra.

Let $\mathcal{H} \subseteq \mathcal{V}$ be the abelian group constructed in [8, 7.10.8] satisfying that $\mathcal{A}$ is $*$-isomorphic to $\mathcal{H} \times_\beta \mathcal{D}$, where $\beta$ denotes the action of $\mathcal{V}$ on $\mathcal{D}$ restricted to $\mathcal{H}$. Define $\mathcal{G} = \Phi(\mathcal{H})$. Then $\Phi$ extends to a $*$-isomorphism between $\mathcal{H} \times_\beta \mathcal{D}$ and $\mathcal{G} \times_\alpha \mathcal{C}$, where $\alpha$ denotes the action of $\mathcal{G}$ on $\mathcal{C}$. Therefore $\mathcal{A}$ is $*$-isomorphic to $\mathcal{G} \times_\alpha \mathcal{C}$ and hence $\mathcal{A}$ and $\{\mathcal{A}_n\}$ satisfy condition (a) in Theorem 1.1.

Let $\mathcal{B}_n$ be the $C^*$-algebra generated by $\mathcal{G}_n$ and $\mathcal{C}_n$, as defined in Section 2. Since we know that $\mathcal{A}$ is $*$-isomorphic to $\mathcal{G} \times_\alpha \mathcal{C}$, from the results of Section 2 we have that $\mathcal{A} = (\bigcup_{n \geq 0} \mathcal{B}_n)^-$. As $\mathcal{A} = (\bigcup_{n \geq 0} \mathcal{M}_n)^-$ as well, [2 Lemma 2.6] yields an automorphism $\Psi$ of $\mathcal{A}$ such that given $n$, there are positive integers $k$ and $m$ with $\mathcal{B}_n \subseteq \Psi(\mathcal{M}_k) \subseteq \mathcal{B}_m$. Hence, given $n$ we can find $m$ such that $\mathcal{B}_n$ is contained in a matrix subalgebra of $\mathcal{B}_m$.

LEMMA 3.4. *Let $\mathcal{M}$ be a $C^*$-subalgebra of $\mathcal{B}_m$ $*$-isomorphic to $M_k$, the algebra of $k \times k$ complex matrices. If $p$ is a projection in $\mathcal{M} \cap \mathcal{C}_m$ and $P$ is a minimal central projection in $\mathcal{B}_m$ (and thus corresponding to a $\mathcal{G}_m$-orbit), then $pP$ is the sum of $\tau(p)|\mathcal{G}_m|/k$ minimal projections of $\mathcal{C}_m$, where $\tau$ is the canonical trace on $\mathcal{M}$.*

*Proof.* From Section 2 we know that $P\mathcal{B}_m$ is $*$-isomorphic to $M_{|\mathcal{G}_n|}$. $f(x) = xP$ maps $\mathcal{M}$ into $P\mathcal{B}_m$ in a 1-1 way. Thus if $\tau$ (resp. $\tau'$) is the canonical trace on $\mathcal{M}$ (resp. $P\mathcal{B}_m$) we have that

$$\tau'(f(p)) = |\mathcal{G}_m|\tau(p)/k.$$ ▨

Thus if $p$ is a minimal projection of $\mathcal{C}_n$, $p$ will embed with the same multiplicity into every orbit of $\mathcal{B}_m$, as shown in the previous lemma. It follows that $\mathcal{G}$ and $\mathcal{C}$ satisfy condition (b) of Theorem 1.1.

REMARK 3.5. Define, for a countable locally finite abelian group $\mathcal{G}$, its supernatural number as

$$\prod \{p^{n_p} : p \text{ a prime}\}$$

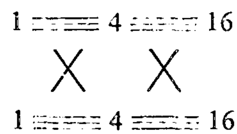
where

$$n_p = \max\{m : \mathscr{G} \text{ has an element of order } p^m\}$$

(with the usual conventions). It follows from the proof of Theorem 1.1 that if a UHF $C^*$-algebra $\mathscr{A}$ is expressed as $\mathscr{G} \times_\alpha \mathscr{C}$, as in (b) of Theorem 1.1, then the supernatural number of $\mathscr{A}$ coincides with the supernatural number of $\mathscr{G}$.

## 4. A NON-UHF EXAMPLE

Let $\mathscr{A}$ be the AF-algebra with Bratteli diagram:



etc.

Thus $\mathscr{A}_n$ is $*$-isomorphic to $M_{4^{n-1}} \oplus M_{4^{n-1}}$ and the embeddings are all given by the matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

If follows from [1, 7.6 and 7.7.1] that $\mathscr{A}$ is simple and that $K_0(\mathscr{A}) = D^2$, where $D$ is the group of dyadic rationals. Thus $\mathscr{A}$ is not UHF. We will show that $\mathscr{A}$ is $*$-isomorphic to $\mathscr{G} \times_\alpha \mathscr{C}$, where $\mathscr{C}$ is a diagonal for $\mathscr{A}$ and $\mathscr{G}$ is an abelian subgroup of $\mathscr{U}$, as in condition (a) of Theorem 1.1, but that the action of $\mathscr{G}$ on $\mathscr{C}$ does not satisfy condition (b) in Theorem 1.1.

Let $\mathscr{C}_n$ be the subalgebra of $\mathscr{A}_n$ corresponding to the diagonal matrices in $M_{4^{n-1}} \oplus M_{4^{n-1}}$, so $\mathscr{U}_n$ is the group of permutation matrices in $M_{4^{n-1}} \oplus M_{4^{n-1}}$. Let $\mathscr{G}_n$ be the subgroup of $\mathscr{U}_n$ generated by two copies of a $4^{n-1}$-cyclic permutation matrix. Let

$$\mathscr{G} = \bigcup_{n \geqslant 1} \mathscr{G}_n \quad \text{and} \quad \mathscr{C} = \Big(\bigcup_{n \geqslant 1} \mathscr{C}_n\Big)^-.$$

Notice that $\mathscr{G}_n$ has two orbits in $\mathscr{C}_n$, each of them corresponding to the set of minimal projections of $\mathscr{C}_n$ dominated by one of the central projections in $\mathscr{A}_n$. Thus, if $\mathscr{G}$ and $\mathscr{C}$ satisfy condition (b) in Theorem 1.1 it will follow that given $n$, there is $m \geqslant n$ such that each central factor of $\mathscr{A}_n$ embeds into each of the two central factors of $\mathscr{A}_m$ with the same multiplicity. (This is because two minimal projections in the same $\mathscr{G}_n$-orbit of $\mathscr{C}_n$ will embed in identical ways into each $\mathscr{G}_k$ for $k \geqslant n$.) Thus:

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{m-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix} \quad \text{for some } b \in \mathbf{Z}_+.$$

But this is impossible because it would imply that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}^{n-m} \begin{pmatrix} b \\ b \end{pmatrix}$$

so $1 = 4^{n-m}b = 0$.

It is clear that $\mathscr{G}_n$ and $\mathscr{C}_n$ generate $\mathscr{A}_n$, and hence $\mathscr{G}$ and $\mathscr{C}$ generate $\mathscr{A}$. Theorem 4.2 will show that under these circumstances $\mathscr{A}$ is $*$-isomorphic to $\mathscr{G} \times_\alpha \mathscr{C}$.

Let $\mathscr{A}$ be a simple AF $\mathscr{C}^*$-algebra with $\mathscr{C}$ and $\mathscr{U}$ the diagonal and unitary group associated to the diagonalization of $\mathscr{A}$ by $\mathscr{C}$, respectively. Let $\mathscr{G}$ be an abelian subgroup of $\mathscr{U}$ such that $\mathscr{G}$ and $\mathscr{C}$ generate $\mathscr{A}$.

LEMMA 4.1. *Let $g_1, \ldots, g_n \in \mathscr{G}$ be such that $g_i \neq g_j$ for $i \neq j$. If $x_1, \ldots, x_n \in \mathscr{C}$ satisfy that $\sum_{i=1}^{n} x_i g_i = 0$ then $x_i = 0$ for all $i = 1, \ldots, n$.*

*Proof.* Let $m$ be big enough so that $g_i \in \mathscr{G}_m$ for $i = 1, \ldots, n$. Let $\{e^l_{jk}\}$ be a system of matrix units for $\mathscr{A}_m$ with respect to $\mathscr{C}_m$, associated with the diagonalization of $\mathscr{A}$ by $\mathscr{C}$ [9, I.1.8]. Here $l$ runs through the factors in the central decomposition of $\mathscr{A}_m$, and $j$ and $k$ run through the minimal projections in the $l$ central factor of $\mathscr{A}_m$.

By construction [9, I.1.9], every $g_i$, $i = 1, \ldots, n$, has the form

$$g_i = \sum_l \sum_{j,k} b^l_{jk}(g_i) e^l_{jk} \quad \text{with } b^l_{jk}(g_i) \in \{0, 1\}.$$

Moreover, if $b^l_{jk}(g_i) = 1$, then $b^l_{jk}(g_{i'}) = 0$ for $i \neq i'$. Otherwise, $g^*_{i'} g_i$ would fix $e^l_{kk}$ and $g^*_{i'} g_i \neq 1$, contradicting Lemma 2.5.

Fix $i_0$, and let $e^{l_0}_{j_0 k_0}$ be such that $b^{l_0}_{j_0 k_0}(g_{i_0}) = 1$. Then by above

$$e^{l_0}_{j_0 j_0}\left(\sum_{i=1}^{n} x_i g_i\right) e^{l_0}_{k_0 k_0} = b^{l_0}_{j_0 k_0}(g_{i_0}) x_{i_0} e^{l_0}_{j_0 k_0}.$$

Thus, if $\sum_{i=1}^{n} x_i g_i = 0$, we get above expression equal to 0. That will be true of any $e^l_{jk}$ for which $b^l_{jk}(g_{i_0}) = 1$. Thus $x_{i_0} g_{i_0} = 0$, and hence $x_{i_0} = 0$. ▨

THEOREM 4.2. *$\mathscr{A}$ is $*$-isomorphic to $\mathscr{G} \times_\alpha \mathscr{C}$.*

*Proof.* Consider $K(\mathscr{G}, \mathscr{C})$, the linear space of continuous functions from $\mathscr{G}$ to $\mathscr{C}$ with compact supports. For $g \in \mathscr{G}$, let $\delta_g \in K(\mathscr{G}, \mathscr{C})$ be such that

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h. \end{cases}$$

From Lemma 4.1 it follows that the map $g \mapsto \delta_g$, $x \mapsto x\delta_e$ for $g \in \mathscr{G}$ and $x \in \mathscr{C}$, where $e$ denotes the identity of $\mathscr{G}$, extends to a $*$-homomorphism from $\mathscr{A}$ onto $\mathscr{G} \times_\alpha \mathscr{C}$. Since $\mathscr{A}$ is simple, we actually have a $*$-isomorphism. ∎

## REFERENCES

1. BLACKADAR, B., *K-theory for operator algebras*, Springer Verlag, New York, 1986.
2. BRATTELI, O., Inductive limits of finite-dimensional $C^*$-algebras, *Trans. Amer. Math. Soc.*, **171**(1972), 195–234.
3. EFFROS, E. G.; HAHN, F., *Locally compact transformation groups and $C^*$-algebras*, Memoirs Amer. Math. Soc., No. **75**, 1967.
4. GARDING, L.; WIGHTMAN, A., Representations of the anticommutation relations, *Proc. Nat. Acad. Sci. USA*, **40**(1954), 617–621.
5. GLIMM, J., On a certain class of operator algebras, *Trans. Amer. Math. Soc.*, **95**(1960), 318–340.
6. KADISON, R.; RINGROSE, J., *Fundamentals of the theory of operator algebras*, Volume II, Academic Press, New York, 1986.
7. KRIEGER, W., On a dimension for a class of homeomorphism groups, *Math. Ann.*, **252**(1980), 87–95.
8. PEDERSEN, G., *$C^*$-algebras and their automorphism groups*, Academic Press, New York, 1979.
9. STRĂTILĂ, Ş.; VOICULESCU, D., *Representations of* AF*-algebras and of the group $U(\infty)$*, Lecture Notes in Mathematics, **486**, Springer Verlag, Berlin, 1975.

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