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INTERSECTIONS OF K-SPECTRAL SETS

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INTRODUCTION

Spectral sets were invented by J. von Neumann in [4] and there he showed that the closed unit disk is a spectral set for every contractive linear operator on a Hilbert space. This result seems to be the only useful example of easily verified assumptions leading to the conclusion that a set is a spectral set. However, the situation for K-spectral sets is better. If G is any bounded open set containing the spectrum of the bounded linear operator T, then an easy estimate using the Riesz functional calculus shows the closure of G is a complete K-spectral set for T.

Also, if T is similar to an operator for which X is a spectral set, then a simple estimate shows X is a K-spectral set for T with $K = \|S\|\,\|S^{-1}\|$, where S is the operator effecting the similarity. It is not known whether the converse of this is true, but if X is a complete K-spectral set for T then a result of V. Paulsen [5] shows that T is similar to an operator for which X is a spectral set.

It is known that the intersection of two spectral sets is not in general a spectral set [2], [4] but the same question for K-spectral sets remains open. Of course, the counterexample for spectral sets shows that the same constant cannot be used for the intersection. Some cases of the K-spectral set problem have been solved. If two K-spectral sets have disjoint boundaries, then by a result of R. G. Douglas and V. Paulsen [2] the intersection is a K'-spectral set for some K'.

J. Stampfli has considered the case when the boundaries meet. In particular he has shown [7] that the intersection of a simply connected spectral set whose interior has finitely many components with the closure of a simply connected open set containing the spectrum of the bounded linear operator T is a K-spectral set for T and moreover, the operator is similar to an operator for which the intersection is a spectral set. More recently he has obtained results for the finitely connected case [8].

In this paper we are able to remove Stampfli's topological conditions and the necessity of starting with a spectral set and show that the intersection of a (complete) K-spectral set for the bounded linear operator T with the closure of any open set containing the spectrum of T is a (complete) K-spectral set for T. As a corollary of this we will show that the intersection of any finitely connected spectral set for T with the closure of any open set containing the spectrum of T is a complete K'-spectral set for T so by Paulsen's result, T is similar to an operator for which the intersection is a spectral set.

PRELIMINARIES

In this section we collect some basic facts and definitions we will need in the rest of the paper. A reference for the operator theory is [1] and for uniform algebras is [3].

We will denote the space of all bounded linear operators on the Hilbert space $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$. The space of trace class operators is the closure of the finite rank operators in the norm $\|F\|_1 = \operatorname{tr}(F^*F)^{1/2}$ and is denoted $\mathscr{C}_1(\mathscr{H})$. Recall that $\mathscr{B}(\mathscr{H})$ can be identified with the dual of $\mathscr{C}_1(\mathscr{H})$ under the dual pairing $\langle T, L\rangle = \operatorname{tr} TL$. We also have $\|TL\|_1 \leqslant \|T\| \, \|L\|_1$.

For a compact subset of the plane, X, we let $M(X)$ be the space of all complex Borel measures supported on X, $C(X)$ the space of all continuous function on X, and $R(X)$ the uniform closure in $C(X)$ of $\operatorname{Rat}(X)$, the space of all rational functions with poles off X.

A compact subset of the plane X is called a K-spectral set for T if X contain the spectrum of T and $\|f(T)\| \leqslant K\|f\|_X$ for all $f \in \operatorname{Rat}(X)$. That is, the natural functional calculus from $\operatorname{Rat}(X)$ to $\mathscr{B}(\mathscr{H})$ given by $f \mapsto f(X)$ has norm at most K and therefore extends to a linear map $\Phi\colon R(X) \to \mathscr{B}(\mathscr{H})$ with the same norm. As is customary we will also denote the image of f under this extension by $f(T)$. It is not hard to see that this map also preserves multiplication since the map $(f, g) \mapsto fg(T) - f(T)g(T)$ is continuous and identically zero on the dense set $\operatorname{Rat}(X) \times \operatorname{Rat}(X)$. If the functional calculus is contractive we say that X is a spectral set for T.

One of the main tools for studying $R(X)$ is the Cauchy transform. If μ is a complex measure with compact support in the plane, define $\hat{\mu}(z) = \int (z - w)^{-1} d\mu(w)$ whenever the integral is absolutely convergent. It is well known that $\hat{\mu}$ is locally integrable, holomorphic off its support, and vanishes at infinity. Its importance in the study of $R(X)$ comes from the fact that $\mu \in R(X)^{\perp}$ f and only if $\hat{\mu} = 0$ off X [3, p. 46].

We will make use of the following two facts. The first is a version of Green's Theorem and the second can be established using Fubini's Theorem.

(i) If h is a continuously differentiable function with compact support in the plane and λ denotes Lebesgue area measure, then

$$(\bar{\partial} h \lambda)\hat{} = -\pi h.$$

(ii) If μ and ν are two complex Borel measures with compact support in the plane, then

$$(\mu\hat{\nu} + \hat{\mu}\nu)\hat{} = \hat{\mu}\hat{\nu}$$

whenever all the integrals exist.

SURGERY ON K-SPECTRAL SETS

The next lemma is simple but it enables us to reduce the construction of a K-spectral set to a more amenable problem about measures. This idea is not entirely new. If $T \in \mathcal{B}(\mathcal{H})$ and X is a spectral set for T, a standard construction in the theory of spectral sets gives measures, $\mu_{x,y}$, for $x, y \in \mathcal{H}$ such that $(f(T)x, y) = \int f \, d\mu_{x,y}$ for all $f \in R(X)$. The equivalence class modulo $R(X)^{\perp}$ of this measure corresponds to image of $x \otimes y$ under the map in the lemma below.

If $\Phi\colon R(X) \to \mathcal{B}(\mathcal{H})$ is bounded, then so is its adjoint $\Phi^*\colon \mathcal{B}(\mathcal{H})^* \to M(X)/R(X)^{\perp}$. Let $\varphi\colon \mathcal{C}_1(\mathcal{H}) \to M(X)/R(X)^{\perp}$ be the composition of Φ^* with the canonical imbedding of $\mathcal{C}_1(\mathcal{H})$ into its double dual $\mathcal{B}(\mathcal{H})^*$. For $z \notin X$ we have $\operatorname{tr}(z - T)^{-1}L = \langle (z - T)^{-1}, L\rangle = \int (z - w)^{-1} d\mu_L(w) = \hat{\mu}_L(z)$ where μ_L is any measure from the equivalence class of $\varphi(L)$. The converse also holds.

LEMMA 1. *Suppose $T \in \mathcal{B}(\mathcal{H})$, X is a compact subset of the plane, and $\varphi\colon \mathcal{C}_1(\mathcal{H}) \to M(X)/R(X)^{\perp}$ is a bounded linear map such that $\hat{\mu}_L(z) = \operatorname{tr}(z - T)^{-1}L$ for $z \notin X$ whenever $\mu_L + R(X)^{\perp} = \varphi(L)$, then X is a K-spectral set for T with $K = \|\varphi\|$.*

Proof. Taking the adjoint of φ and composing with the canonical imbedding of $R(X)$ into its double dual gives a map $\Phi\colon R(X) \to \mathcal{B}(\mathcal{H})$ with $\|\Phi\| \leqslant \|\varphi\|$. The condition involving the Cauchy transform is used to show this map agrees with the natural Rat(X) functional calculus. If f is a rational function with poles off X and γ is a path surrounding X but no poles of f, then by the Riesz functional calculus

$$\operatorname{tr}\Phi(f)L = \langle f, \varphi(L)\rangle = \int f \, d\mu_L = \frac{1}{2\pi i}\int f(z)\hat{\mu}_L(z)dz = \operatorname{tr} f(T)L$$

for each $L \in \mathcal{C}_1(\mathcal{H})$. Hence $\|f(T)\| = \|\Phi(f)\| \leqslant K\|f\|_X$ where $K = \|\varphi\|$.

We now use this lemma to prove our first theorem.

THEOREM 1. *Suppose X is a K-spectral set for T, G is an open set containing $\sigma(T)$, and every component of the complement of $\bar{G}$ meets the complement of X, then $X \cap \bar{G}$ is a K'-spectral set for T for some K'.*

Note that the condition of this theorem are satisfied when $\bar{G}$ is bounded and simply connected.

Proof. Choose a positive continuously differentiable function h with $\|h\|_\infty = 1$, $h = 1$ on a neighbourhood of $\sigma(T)$, and whose support is contained in G. For $L \in \mathscr{C}_1(\mathscr{H})$ pick $\mu_L \in M(X)$ that is a representative of $\varphi(L)$. Put $\nu_L = h\mu_L -$ $- (1/\pi)\bar{\partial}h(\hat{\mu}_L - F_L)\lambda$, where $F_L(z) = \operatorname{tr}(z - T)^{-1}L$ for $z \notin \sigma(T)$ and λ is Lebesgue area measure. Note that ν_L is supported on $X \cap \bar{G}$.

Since $h = 1$ on a neighbourhood of $\sigma(T)$ we can find a measure σ such that $\hat{\sigma} = F_L$ off the set where $h = 1$. Using facts (i) and (ii) we have $((1/\pi)\bar{\partial}hF_L\lambda)\hat{} =$ $= ((1/\pi)\bar{\partial}h\hat{\sigma}\lambda)\hat{} = -h\hat{\sigma} + (h\sigma)\hat{} = (1 - h)F_L$. Another computation using these two facts gives $\hat{\nu}_L = h\hat{\mu}_L + (1 - h)F_L$ and so $\hat{\nu}_L = F_L$ off $X \cap \bar{G}$ since $\hat{\mu}_L = F_L$ off X and $h = 0$ off G.

To see that the map $\tilde{\varphi}: \mathscr{C}_1(\mathscr{H}) \to M(X \cap \bar{G})/R(X \cap \bar{G})^\perp$ by $L \mapsto \nu_L +$ $+ R(X \cap \bar{G})^\perp$ is well-defined we must show that $\nu_L \in R(X \cap \bar{G})^\perp$ whenever $\mu_L \in R(X)^\perp$. So suppose $\mu_L \in R(X)^\perp$, then $\hat{\mu}_L = 0$ off X. Since $\hat{\nu}_L = F_L$ off $X \cap \bar{G}$, $\hat{\mu}_L = F_L$ off X, and every component of the complement of $\bar{G}$ meets the complement of X we must have $\hat{\nu}_L = 0$ off $X \cap \bar{G}$. Hence $\nu_L \in R(X \cap \bar{G})^\perp$.

We now must show that the map is bounded. Suppose $f \in R(X \cap \bar{G})$ with $\|f\|_{X \cap \bar{G}} \leqslant 1$. Put $f = 0$ off $X \cap \bar{G}$. We have

$$\left|\int f\mathrm{d}\nu_L\right| \leqslant \left|\int fh\mathrm{d}\mu_L\right| + \left|\frac{1}{\pi}\int f\bar{\partial}h\hat{\mu}_L\mathrm{d}\lambda\right| + \left|\frac{1}{\pi}\int f\bar{\partial}hF_L\mathrm{d}\lambda.\right.$$

Estimating the first term gives $\left|\int fh\mathrm{d}\mu_L\right| \leqslant \|\mu_L\|$. Since, by the Hahn-Banach Theorem, we may choose μ_L such that $\|\mu_L\| = \|\mu_L + R(X)^\perp\|$ we have $\left|\int fh\mathrm{d}\mu_L\right| \leqslant$ $\leqslant \|\mu_L + R(X)^\perp\| = \|\varphi(L)\| \leqslant K\|L\|_1$.

For the second term we have

$$\left|\int f\bar{\partial}h\hat{\mu}_L\mathrm{d}\lambda\right| \leqslant \|\bar{\partial}h\|\int_G |\hat{\mu}_L|\mathrm{d}\lambda \leqslant$$

$$\leqslant \|\bar{\partial}h\|\iint_G |z - w|^{-1}\mathrm{d}\lambda\mathrm{d}|\mu_L| \leqslant \|\bar{\partial}h\|\sqrt{4\pi\lambda(G)}\|\mu_L\|$$

where in the last step we have used the standard estimate $\int_G |z - w|^{-1}\mathrm{d}\lambda \leqslant \sqrt{4\pi\lambda(G)}$.

Hence $\left|\int f\bar{\partial}h\hat{\mu}_L\mathrm{d}\lambda\right| \leqslant K\|\bar{\partial}h\|\sqrt{4\pi\lambda(G)}\|L\|_1$.

Finally, let $M = \sup\{\|(z - T)^{-1}\| : h(z) \neq 1\}$, then

$$\left|\int f\bar{\partial}hF_L \mathrm{d}\lambda\right| \leqslant M\|L\|_1 \int |\bar{\partial}h| \, \mathrm{d}\lambda.$$

Putting the estimates together and using the previous lemma shows that $X \cap \bar{G}$ is a K'-spectral set where $K' = K + 2K\|\bar{\partial}h\| \sqrt{\lambda(G)/\pi} + (M/\pi)\int |\bar{\partial}h| \mathrm{d}\lambda$.

Now we use a result of Douglas and Paulsen to reduce the general case to the one considered above.

THEOREM 2. *Suppose X is a K-spectral set for T and G is an open set containing $\sigma(T)$, then $X \cap \bar{G}$ is a K'-spectral set for T for some K'.*

Proof. Let $U_1, U_2, \ldots$ be the components of the complement of $\bar{G}$ that do not meet the complement of X. By a theorem of Douglas and Paulsen [2, 6 Proposition 9.4] we can find disks $D_j \subset U_j$ such that $X \setminus \bigcup D_j$ is a K''-spectral set for T for some K''.

Now every component of the complement of $\bar{G}$ meets the complement of $X \setminus \bigcup D_j$ so by the previous theorem we have that $(X \setminus \bigcup D_j) \cap \bar{G} = X \cap \bar{G}$ is a K'-spectral set for T for some K'.

COMPLETE K-SPECTRAL SETS

Now we show how to extend the previous theorems to the complete case.

Define $\mathcal{M}_n(R(X))$ to be the algebra of $n \times n$ matricies with entries in $R(X)$. This is a Banach space with norm $\|(f_{ij})\| = \sup_{x \in X} \|(f_{ij}(x))\|$. Its dual can be identified with $\mathcal{M}_n(M(x)/R(X)^\perp)$ via the dual pairing

$$\langle (f_{ij}), ([\mu_{ij}]) \rangle = \sum_{i,j} \int f_{ij} \mathrm{d}\mu_{ij}.$$

We can define $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ similarly and its predual can be identified with $\mathcal{M}_n(\mathcal{C}_1(\mathcal{H}))$ via the dual pairing

$$\langle (T_{ij}), (L_{ij}) \rangle = \sum_{i,j} \operatorname{tr} T_{ij} L_{ij}.$$

If A and B are Banach algebras and $\varphi: A \to B$ is a linear map, define $\varphi_n : \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ by $\varphi_n((a_{ij})) = (\varphi(a_{ij}))$ for $(a_{ij}) \in \mathcal{M}_n(A)$.

A compact subset of the plane X is called a *complete K-spectral set* for T if $\|(f_{ij}(T))\| \leqslant K\|(f_{ij})\|$ for all $(f_{ij}) \in \mathcal{M}_n(\mathrm{Rat}(X))$ and all positive integers n. If $K = 1$ we say X is a complete spectral set for T.

The preceding results can be reformulated in matrix versions. In Lemma 1 replace φ by φ_n and assume $\sup_n \|\varphi_n\| \leqslant K$, then the conclusion becomes X is a complete K-spectral set. In Theorem 1 we assume X is a complete K-spectral set and the conclusion becomes $X \cap \bar{G}$ is a complete K'-spectral set for some K'. The proof remains essentially the same. The estimates involving integrals and traces are replaced by the corresponding dual pairings for the matrix case. Likewise, Theorem 2 still works in the complete case since the result of Douglas and Paulsen we use is also valid for complete K-spectral sets.

We are now in a position to generalize J. Stampfli's theorem.

THEOREM 3. *If X is a spectral set for T, $R(X)$ is a hypo-Dirichlet algebra, and G is any open set containing the spectrum of T, then T is similar to an operator for which $X \cap \bar{G}$ is a spectral set.*

Proof. Recall that $R(X)$ is a hypo-Dirichlet means that $\{\mathrm{Re} f|\partial X : f \in R(X)\}$ has finite codimension in the space of continuous real valued functions on ∂X.

It follows from [2] that X is in fact a complete K-spectral set for some K, hence by the version of Theorem 2 for complete spectral sets we have that $X \cap \bar{G}$ is a complete K'-spectral set for some K'. By a result of V. Paulsen [5] this is equivalent to T being similar to an operator for which $X \cap \bar{G}$ is a spectral set.

If X is finitely connected set, then $R(X)$ is a hypo-Dirichlet algebra [2], [3, p. 116] so this theorem contains J. Stampfli's result [8].

Of course, the problem to be solved is whether the intersection of two K-spectral sets is a K'-spectral set for some K'. Although it might not be possible to extend the techniques developed here to settle this question, I conjecture that it is possible to improve the estimates in Theorem 1 to get an affirmative answer when one of the spectral sets has rectifiable boundary.

REFERENCES

1. CONWAY, J., *Subnormal operators*, Pitman Advanced Pub. Program, Boston, 1981.

2. DOUGLAS, R. G.; PAULSEN, V. I., Completely bounded maps and hypo-Dirichlet algebras, *Acta. Sci. Math. (Szeged)*, 50(1986), 143–157.

3. GAMELIN, T., *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

4. VON NEUMANN, J., Eine Specktraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.*, 4(1951), 258–281.

5. PAULSEN, V. I., Every completely polynomially bounded operator is similar to a contraction, *J. Funct. Anal.*, 55(1984), 1–17.

6. PAULSEN, V. I., *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics, **146**, Great Britain, 1986.
7. STAMPFLI, J. G., Surgery on spectral sets, *J. Operator Theory*, **16**(1986), 235–243.
8. STAMPFLI, J. G., Surgery on spectral sets. II: the multiply connected case.

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