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A CLASSIFICATION OF OPERATOR ALGEBRAS

MICHAEL MARSALLI

1. INTRODUCTION

Throughout this paper $\mathscr{X}$ will denote a complex Banach space, $\mathscr{X}^*$ will denote the Banach space of norm continuous linear functionals on $\mathscr{X}$, and $\mathscr{L}(\mathscr{X})$ will denote the algebra of bounded linear operators on $\mathscr{X}$. For $\mathscr{S} \subset \mathscr{L}(\mathscr{X})$, $\mathscr{S}'$ will denote the set $\{T \in \mathscr{L}(\mathscr{X}) : ST = TS \text{ for all } S \in \mathscr{S}\}$, $\mathscr{S}''$ will denote $(\mathscr{S}')'$, and Lat $\mathscr{S}$ will denote the set $\{\mathscr{M} \subset \mathscr{X} : \mathscr{M}$ is a (closed) subspace of $\mathscr{X}$ and $S\mathscr{M} \subset \mathscr{M}$ for all $S \in \mathscr{S}\}$. For $T \in \mathscr{L}(\mathscr{X})$ with $\|T\| \leqslant 1$, it is easy to see that $\{x \in \mathscr{X} : \|T^n x\| \to 0\}$ is in Lat$\{T\}'$. We prove a version of this result for subalgebras of $\mathscr{L}(\mathscr{X})$. This leads to a classification of algebras analogous to the classification of contractions on a Hilbert space in [9, p. 72]. We give some applications to the theory of dual algebras. In certain cases there is a close connection between the type of a contraction and the type of the dual algebra it generates. We characterize the appropriate classes of self-adjoint algebras on a Hilbert space.

2. THE CLASSES $A_{\alpha\beta}$

$\mathscr{A}$ will denote a subalgebra of $\mathscr{L}(\mathscr{X})$, and $\mathscr{A}_1$ will denote the set $\{A \in \mathscr{A} : \|A\| \leqslant 1\}$. For $x \in \mathscr{X}$, define $\rho_x : \mathscr{L}(\mathscr{X}) \to \mathscr{X}$ by $\rho_x(T) = Tx$, $T \in \mathscr{L}(\mathscr{X})$. Let $\mathscr{M}_{\mathrm{p}}(\mathscr{A})$ denote the set $\{x \in \mathscr{X} : \rho_x(\mathscr{A}_1)$ is norm precompact$\}$. For properties of precompact sets in topological vector spaces see [7].

THEOREM 1. $\mathscr{M}_{\mathrm{p}}(\mathscr{A}) \in \mathrm{Lat}\mathscr{A} \cap \mathrm{Lat}\mathscr{A}'$.

Proof. Let $x \in \mathscr{M}_{\mathrm{p}}(\mathscr{A})$ and let $\lambda \in \mathbf{C}$. Then $\rho_{\lambda x}(\mathscr{A}_1) = \lambda\rho_x(\mathscr{A}_1)$, and scalar multiplication preserves precompactness. So $\lambda x \in \mathscr{M}_{\mathrm{p}}(\mathscr{A})$. Let $x, y \in \mathscr{M}_{\mathrm{p}}(\mathscr{A})$. Then $\rho_{x+y}(\mathscr{A}_1) \subset \rho_x(\mathscr{A}_1) + \rho_y(\mathscr{A}_1)$. Since the sum of two precompact sets is precompact, and precompactness is inherited by subsets, we have $x + y \in \mathscr{M}_{\mathrm{p}}(\mathscr{A})$.

Now we show $\mathscr{M}_{\mathrm{p}}(\mathscr{A})$ is closed. Let $\{x_n\}$ be a sequence in $\mathscr{M}_{\mathrm{p}}(\mathscr{A})$ which converges to x in norm. Given $\varepsilon > 0$, choose N such that $\|x_N - x\| < \varepsilon/2$. Since

$x_N \in \mathcal{M}_p(\mathcal{A})$, there exist $y_1, \ldots, y_m$ in $\mathcal{X}$ such that for each $A \in \mathcal{A}_1$ there is a y_i, $1 \leqslant i \leqslant m$, with $\|Ax_N - y_i\| < \varepsilon/2$. Then

$$\|Ax - y_i\| \leqslant \|Ax - Ax_N\| + \|Ax_N - y_i\| < \varepsilon.$$

So $\rho_x(\mathcal{A}_1)$ is totally bounded, and $x \in \mathcal{M}_p(\mathcal{A})$.

We show $\mathcal{M}_p(\mathcal{A}) \in \operatorname{Lat}\mathcal{A}$. Let $x \in \mathcal{M}_p(\mathcal{A})$ and let $A \in \mathcal{A}$. Then $\rho_{Ax}(\mathcal{A}_1) \subset \|A\|\rho_x(\mathcal{A}_1)$. Thus $Ax \in \mathcal{M}_p(\mathcal{A})$.

Finally we show $\mathcal{M}_p(\mathcal{A}) \in \operatorname{Lat}\mathcal{A}'$. Let $x \in \mathcal{X}$ and let $S \in \mathcal{A}'$. Then $\rho_{Sx}(\mathcal{A}_1) = S(\rho_x(\mathcal{A}_1))$. Since precompactness is preserved by continuous linear operators, $Sx \in \mathcal{M}_p(\mathcal{A})$.

Of course $\mathcal{M}_p(\mathcal{A})$ might be (0) or $\mathcal{X}$. This motivates the following definition. For $T \in \mathcal{L}(\mathcal{X})$, T^* will denote the Banach space adjoint of T. $\mathcal{A}^*$ will denote the set $\{A^* : A \in \mathcal{A}\}$. So $\mathcal{A}^* \subset \mathcal{L}(\mathcal{X}^*)$.

DEFINITION 2. Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(\mathcal{X})$.
(a) $\mathcal{A} \in A_{0\cdot}(\mathcal{X})$ if $\mathcal{M}_p(\mathcal{A}) = \mathcal{X}$.
(b) $\mathcal{A} \in A_{1\cdot}(\mathcal{X})$ if $\mathcal{M}_p(\mathcal{A}) = (0)$.
(c) $\mathcal{A} \in A_{\cdot\alpha}(\mathcal{X})$ if $\mathcal{A}^* \in A_{\alpha\cdot}(\mathcal{X}^*)$ for $\alpha = 0,1$.
(d) $\mathcal{A} \in A_{\alpha\beta}(\mathcal{X})$ if $\mathcal{A} \in A_{\alpha\cdot}(\mathcal{X}) \cap A_{\cdot\beta}(\mathcal{X})$ for $\alpha,\beta = 0,1$.

We write $A_{\alpha\beta}$ for $A_{\alpha\beta}(\mathcal{X})$ when there is no possibility of confusion. When $\mathcal{X}$ is a Hilbert space and $T \in \mathcal{L}(\mathcal{X})$, T^* will henceforth denote the Hilbert space adjoint of T. The above definition is then equivalent to the corresponding definition using the Hilbert space adjoint.

REMARKS. One can show that the transitive algebra problem on a reflexive Banach space reduces to algebras in the four classes $A_{\alpha\beta}$, $\alpha,\beta = 0,1$. The transitive algebra problem on a nonseparable Banach space reduces to algebras in the class A_{11}.

We now assemble some useful properties of the classes $A_{\alpha\beta}$. Let $[\mathcal{A}x]$ denote the norm closure of the set $\{Ax : A \in \mathcal{A}\}$. For $\mathcal{C}$ a collection of subspaces of $\mathcal{X}$, $\operatorname{Alg}\mathcal{C}$ denotes the set $\{T \in \mathcal{L}(\mathcal{X}) : T\mathcal{M} \subset \mathcal{M}$ for all $\mathcal{M} \in \mathcal{C}\}$. $\mathcal{K}(\mathcal{X})$ will denote the ideal of compact operators on $\mathcal{X}$. The *weak topology* (or *weak operator topology*) on $\mathcal{L}(\mathcal{X})$ is the topology induced by the semi-norms $\varphi_{x,f}(T) = |f(Tx)|$, where $T \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. The *strong topology* (or *strong operator topology*) on $\mathcal{L}(\mathcal{X})$ is the topology induced by the semi-norms $\psi_x(T) = \|Tx\|$ where $T \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$. For $T \in \mathcal{L}(\mathcal{X})$, $\mathcal{W}_T$ will denote the weak closure of the polynomials in T. For $\mathcal{E}$ a subset of a Banach space, $\mathcal{E}^-$ will denote the norm closure of $\mathcal{E}$.

PROPOSITION 3. *Let $\mathcal{A}$, $\mathcal{B}$ be subalgebras of $\mathcal{L}(\mathcal{X})$ with $\mathcal{B} \subset \mathcal{A}$. Let $T \in \mathcal{L}(\mathcal{X})$.*
(a) $\mathcal{M}_p(\mathcal{A}) \subset \mathcal{M}_p(\mathcal{B})$.
(b) *If $\mathcal{A} \in A_{0\cdot}$, then $\mathcal{B} \in A_{0\cdot}$.*

(c) *If* $\mathscr{B} \in A_{1\cdot}$, *then* $\mathscr{A} \in A_{1\cdot}$.

(d) $\mathscr{M}_p(\mathscr{A}^-) = \mathscr{M}_p(\mathscr{A})$.

(e) *If* $[\mathscr{A}x]$ *is finite dimensional, then* $x \in \mathscr{M}_p(\mathscr{A})$.

(f) *If* $\mathscr{C}$ *is a collection of finite dimensional subspaces whose closed linear span is* $\mathscr{X}$, *then* $\operatorname{Alg}\mathscr{C} \in A_{0\cdot}$.

(g) *If* $\mathscr{A}_i \in A_{0\cdot}(\mathscr{X}_i)$, $1 \leqslant i < \infty$, *then* $\bigoplus_{i=1}^{\infty} \mathscr{A}_i \in A_{0\cdot}\left(\bigoplus_{i=1}^{\infty} \mathscr{X}_i\right)$.

(h) *If* $K \in \mathscr{A}' \cap \mathscr{K}(\mathscr{X})$, *then* $(K\mathscr{X})^- \subset \mathscr{M}_p(\mathscr{A})$.

(i) *If* $\mathscr{A} \in A_{1\cdot}$, *then* $\mathscr{A}' \cap \mathscr{K}(\mathscr{X}) = (0)$.

(j) *If* $\{Ax : A \in \mathscr{A}\} = \mathscr{X}$ *and* $\mathscr{X}$ *is infinite dimensional, then* $x \notin \mathscr{M}_p(\mathscr{A})$.

(k) *The closed linear span of the set* $\{x \in \mathscr{X} : (T - \lambda)^n x = 0$ *for some* $n \in \mathbf{N}$, $\lambda \in \mathbf{C}\}$ *is contained in* $\mathscr{M}_p(\mathscr{W}_T)$.

(l) *If* $S \in \mathscr{L}(\mathscr{X})$ *is invertible, then* $\mathscr{M}_p(S^{-1}\mathscr{A}S) = S^{-1}\mathscr{M}_p(\mathscr{A})$. *So the classes* $A_{\alpha\beta}$ *are invariant under similarities.*

Proof. (a) – (c) are obvious. To show (d), note that $(\mathscr{A}_1)^- = (\mathscr{A}^-)_1$. Let $x \in \mathscr{M}_p(\mathscr{A})$. Then $\rho_x((\mathscr{A}^-)_1) = \rho_x((\mathscr{A}_1)^-) \subset (\rho_x(\mathscr{A}_1))^-$, since ρ_x is norm continuous. But the closure of a precompact set is precompact, so $\mathscr{M}_p(\mathscr{A}^-) \subset \mathscr{M}_p(\mathscr{A})$. The reverse inclusion follows from (a). (e) is clear. We show (f). Let $\mathscr{M} \in \mathscr{C}$, $x \in \mathscr{M}$. Then $x \in \mathscr{M}_p(\operatorname{Alg}\mathscr{C})$ by (e). Since $\mathscr{M}_p(\mathscr{A})$ is a subspace, the closed linear span of the elements of $\mathscr{C}$ is contained in $\mathscr{M}_p(\operatorname{Alg}\mathscr{C})$. Thus (f) holds. The proof of (g) is similar to the proof of (f). For (h), observe that $\rho_{Kx}(\mathscr{A}_1) = K(\rho_x(\mathscr{A}_1))$. Since $\rho_x(\mathscr{A}_1)$ is bounded, $K(\rho_x(\mathscr{A}_1))$ is precompact. Thus $K\mathscr{X} \subset \mathscr{M}_p(\mathscr{A})$. Since $\mathscr{M}_p(\mathscr{A})$ is closed, (h) follows. (i) is an easy consequence of (h). To establish (j), it suffices to show that $x \in \mathscr{M}_p(\mathscr{A}^-)$ by (d). Clearly $\{Ax : A \in \mathscr{A}^-\} = \mathscr{X}$. Thus ρ_x maps $\mathscr{A}^-$ onto $\mathscr{X}$. By the Open Mapping Theorem, $\rho_x((\mathscr{A}^-)_1)$ contains an open ball. Thus if $\rho_x((\mathscr{A}^-)_1)$ were precompact, $\mathscr{X}$ would be finite dimensional. We now prove (k). Since $\mathscr{M}_p(\mathscr{W}_T)$ is a subspace, it is enough to show that $\{x \in \mathscr{X} : (T - \lambda)^n x = 0\} \subset \mathscr{M}_p(\mathscr{W}_T)$ for each $n \in \mathbf{N}$, $\lambda \in \mathbf{C}$. So fix λ and n. Since $\mathscr{W}_T = \mathscr{W}_{T-\lambda}$ and $\mathscr{W}_{T-\lambda}$ is the strong closure of the polynomials in $T - \lambda$, $[\mathscr{W}_T x]$ is spanned by $x, (T - \lambda)x, \ldots, (T - \lambda)^{n-1}x$ whenever $(T - \lambda)^n x = 0$. The result now follows from (e). Finally, we show (l). Let $\mathscr{R} = S^{-1}\mathscr{A}S$. Since S is a similarity, there are positive real numbers a and b such that $a\mathscr{R}_1 \subset S^{-1}\mathscr{A}_1 S \subset b\mathscr{R}_1$. Thus $Sx \in \mathscr{M}_p(\mathscr{A})$ if and only if $x \in \mathscr{M}_p(\mathscr{R})$, and the result follows.

If F is finite rank, then $\mathscr{W}_F \in A_{0\cdot}$ by Proposition 4(e). If $K \in \mathscr{K}(\mathscr{X})$ and $(K\mathscr{X})^- = \mathscr{X}$, then $\mathscr{W}_K \in A_{0\cdot}$ by Proposition 4(h). However, not every compact operator generates an $A_{0\cdot}$ algebra. There exists a compact operator K on an infinite dimensional space such that $\{Ax : A \in \mathscr{W}_K\} = \mathscr{X}$ for some $x \in \mathscr{X}$ [6, p. 336]. Thus by Proposition 4(j), $\mathscr{W}_K \notin A_{0\cdot}$. Nevertheless the class $A_{0\cdot}$ is quite rich. For example, if T is a backward weighted unilateral shift on a Hilbert space, then

$\mathscr{W}_T \in A_0$. by Proposition 4(k). Obviously all algebraic operators generate A_{00} algebras. We now exhibit some topological properties of the algebras in A_0.. For $T \in \mathscr{L}(\mathscr{X})$ and $n \in \mathbf{N}$, $T^{(n)}$ will denote the direct sum of n copies of T acting on $\mathscr{X}^{(n)}$, the direct sum of n copies of $\mathscr{X}$. $\mathscr{A}^{(n)}$ will denote the set $\{A^{(n)} : A \in \mathscr{A}\}$.

THEOREM 4. *$\mathscr{A} \in A_0$. if and only if $\mathscr{A}_1$ is strongly precompact.*

Proof. For sufficiency, observe that $\rho_x : (\mathscr{L}(\mathscr{X}), \text{strong}) \to (\mathscr{X}, \text{norm})$ is a continuous linear operator for each $x \in \mathscr{X}$. Since such operators preserve precompactness, $\mathscr{A} \in A_0$..

We show necessity. Let $x = (x_1, \ldots, x_n) \in \mathscr{X}^{(n)}$. Let $\mathscr{U} = \{T \in \mathscr{L}(\mathscr{X}) : \|Tx_i\| < 1, \ 1 \leqslant i \leqslant n\}$ be a basic strong neighborhood of 0. Since $\mathscr{A} \in A_0.(\mathscr{X})$, $\mathscr{A}^{(n)} \in A_0.(\mathscr{X}^{(n)})$ by Proposition 4, parts (b) and (g). Thus $x \in \mathscr{M}_p(\mathscr{A}^{(n)})$. So there exist $A_1, \ldots, A_m \in \mathscr{A}_1$ such that for each $A \in \mathscr{A}$ there is an A_j, $1 \leqslant j \leqslant m$, with $\|A^{(n)}x - A_j^{(n)}x\| < 1$. Consequently, $\mathscr{A}_1 \subset \bigcup_j (A_j + \mathscr{U})$. Since $\mathscr{U}$ was arbitrary, $\mathscr{A}_1$ is strongly precompact.

THEOREM 5. *If $\mathscr{S} \subset \mathscr{L}(\mathscr{X})$ is strongly precompact, then the strong and weak topologies coincide on $\mathscr{S}$.*

Proof. The strong topology is stronger than the weak topology on $\mathscr{L}(\mathscr{X})$, so it suffices to show that weak convergence in $\mathscr{S}$ implies strong convergence in $\mathscr{S}$. Let $\{S_i\}$ be a net in $\mathscr{S}$ which converges weakly to $S \in \mathscr{S}$. Let $x \in \mathscr{X}$. Then $\rho_x : (\mathscr{L}(\mathscr{X}), \text{strong}) \to (\mathscr{X}, \text{norm})$ and $\rho_x : (\mathscr{L}(\mathscr{X}), \text{weak}) \to (\mathscr{X}, \text{weak})$ are continuous linear operators. Thus $\rho_x(\mathscr{S})$ is norm precompact. Now $\{S_i x\} \subset \rho_x(\mathscr{S})$ and $S_i x \to Sx$ weakly. So the only norm limit point of $\{S_i x\}$ is Sx. It follows that $S_i x \to Sx$ in norm.

For definitions and properties of the ultraweak and ultrastrong topologies see [4].

COROLLARY 6. *If $\mathscr{A} \in A_0.(\mathscr{X})$, then the weak and strong topologies coincide on bounded subsets of $\mathscr{A}$. If $\mathscr{X}$ is a Hilbert space and $\mathscr{A} \in A_0.(\mathscr{X})$, then the weak, strong, ultraweak and ultrastrong topologies coincide on bounded subsets of $\mathscr{A}$.*

Proof. By Theorem 4, $\mathscr{A}_1$ is strongly precompact. Thus any bounded set is strongly precompact, so the first assertion follows from Theorem 5. Since the weak (respectively, strong) topology coincides with the ultraweak (respectively, ultrastrong) topology on bounded subsets of $\mathscr{L}(\mathscr{X})$, the second assertion is clear.

3. DUAL ALGEBRAS

We apply some of the above ideas to the theory of dual algebras. For information on dual algebras see [1]. $\mathscr{H}$ will denote a separable, complex Hilbert space $\mathscr{C}_1(\mathscr{H})$ will denote the ideal of trace class operators on $\mathscr{H}$. $\mathscr{L}(\mathscr{H})$ is the dual of

$\mathscr{C}_1(\mathscr{H})$ via the pairing $\langle T, L\rangle = \mathrm{tr}(TL)$, $T \in \mathscr{L}(\mathscr{H})$, $L \in \mathscr{C}_1(\mathscr{H})$, where $\mathrm{tr}(TL)$ is the trace of TL. The weak* topology induced by this pairing coincides with the ultraweak topology on $\mathscr{L}(\mathscr{H})$. A *dual algebra* is a weak* closed, unital subalgebra of $\mathscr{L}(\mathscr{H})$. $\mathscr{A}$ will henceforth denote a dual algebra. $\mathscr{A}$ is the dual of $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, where ${}^{\perp}\mathscr{A} = \{L \in \mathscr{C}_1(\mathscr{H}) : \langle A, L\rangle = 0 \text{ for all } A \in \mathscr{A}\}$, via the pairing $\langle A, [L]_{\mathscr{A}}\rangle = \mathrm{tr}(AL)$. We use $[L]_{\mathscr{A}}$, or $[L]$ when there is no possibility of confusion, to denote the coset of L in $Q_{\mathscr{A}}$. $Q_{\mathscr{A}}$ is called the *predual* of $\mathscr{A}$. For $x, y \in \mathscr{H}$, $x \otimes y$ is the element of $\mathscr{C}_1(\mathscr{H})$ defined by $(x \otimes y)(u) = (u, y)x$, $u \in \mathscr{H}$. A computation shows that $\langle T, x \otimes y\rangle = (Tx, y)$, $T \in \mathscr{L}(\mathscr{H})$. We sharpen Theorem 4 for dual algebras.

PROPOSITION 7. *A dual algebra $\mathscr{A}$ is in $A_{0\cdot}(\mathscr{H})$ if and only if $\mathscr{A}_1$ is strongly compact.*

Proof. Sufficiency follows from Theorem 4. To prove necessity, use Corollary 6 and the fact that $\mathscr{A}_1$ is ultraweakly compact by Alaoglu's Theorem.

The next result provides a link between $\mathscr{M}_p(\mathscr{A})$ and $Q_{\mathscr{A}}$.

PROPOSITION 8. *Let $\mathscr{A}$ be a dual algebra, and let $x \in \mathscr{H}$. The following are equivalent.*

(a) $x \in \mathscr{M}_p(\mathscr{A})$.

(b) $\rho_x : \mathscr{A} \to \mathscr{H}$ *is a compact operator.*

(c) *If $\{y_n\}$ is a sequence in $\mathscr{H}$ with $y_n \to 0$ weakly, then* $\|[x \otimes y_n]\| \to 0$.

Proof. The equivalence of (a) and (b) is clear. We show the equivalence of (b) and (c). Let $J: \mathscr{H}^* \to \mathscr{H}$ be the conjugate linear isomorphism such that $f(u) = (u, Jf)$, $u \in \mathscr{H}$, $f \in \mathscr{H}^*$. For $x \in \mathscr{H}$, let $\sigma_x : \mathscr{H}^* \to Q_{\mathscr{A}}$ be defined by $\sigma_x(f) = [x \otimes (Jf)]$, $f \in \mathscr{H}^*$. A calculation shows that ρ_x is the Banach space adjoint of σ_x. It follows that ρ_x is compact if and only if σ_x is compact. Since $\mathscr{H}^*$ is reflexive, σ_x is compact if and only if $\|\sigma_x(f_n)\| \to 0$ whenever $\{f_n\}$ is a sequence in $\mathscr{H}^*$ with $f_n \to 0$ weakly. Since $y_n \to 0$ weakly in $\mathscr{H}$ if and only if $J^{-1}y_n \to 0$ weakly in $\mathscr{H}^*$, (b) and (c) are equivalent.

For $T \in \mathscr{L}(\mathscr{H})$ a contraction, i.e. $\|T\| \leqslant 1$, let $\mathscr{M}_0(T) = \{x \in \mathscr{H} : \|T^n x\| \to 0$ as $n \to \infty\}$. We have the following definition from [9, p. 72].

(a) $T \in C_{0\cdot}$ if $\mathscr{M}_0(T) = \mathscr{H}$.

(b) $T \in C_{1\cdot}$ if $\mathscr{M}_0(T) = (0)$.

(c) $T \in C_{\cdot\alpha}$ if $T^* \in C_{\alpha\cdot}$, $\alpha = 0,1$.

(d) $T \in C_{\alpha\beta}$ if $T \in C_{\alpha\cdot} \cap C_{\cdot\beta}$, $\alpha, \beta = 0,1$.

For $T \in \mathscr{L}(\mathscr{H})$, $\mathscr{A}_T$ will denote the ultraweak closure of the polynomials in T. In certain cases a $C_{\alpha\cdot}$ contraction generates an $A_{\alpha\cdot}$ algebra. However, every singly generated dual algebra is generated by a $C_{0\cdot}$ contraction. (If T is a generator, choose $r > 0$ such that $\|rT\| < 1$. Then rT is a $C_{0\cdot}$ generator.) For $T \in \mathscr{L}(\mathscr{H})$, T is an *absolutely continuous contraction*, which we abbreviate as $T \in (\mathrm{ACC})$, if $\|T\| \leqslant 1$ and the unitary part of T is absolutely continuous or acts on (0). If $T \in (\mathrm{ACC})$, then the Sz.-Nagy – Foiaş functional calculus is a contractive, weak* continuous algebra homomorphism from H^∞, the algebra of bounded analytic functions on the unit disk, to $\mathcal{A}_T$ [1, Chapter IV]. The class $\mathrm{A}(\mathcal{H})$ consists of those $T \in (\mathrm{ACC})$ for which this functional calculus is an isometry. We collect some results relating the type of a contraction to the type of the dual algebra it generates. First we require a proposition which is known, but we state it for convenience.

PROPOSITION 9. *If $T \in (\mathrm{ACC})$, then $T^n \to 0$ ultraweakly as $n \to \infty$.*

Proof. Let $\Phi: H^\infty \to \mathcal{A}_T$ be the Sz.-Nagy–Foiaş functional calculus. Then Φ is continuous when H^∞ and $\mathcal{A}_T$ are given their weak* topologies. Since $\Phi(\lambda^n) = T^n$ and $\lambda^n \to 0$ weak* by the Riemann-Lebesgue Lemma, the result follows.

THEOREM 10. *Let $T \in (\mathrm{ACC})$.*

(a) *If $T \in C_{0\cdot} \cap \mathrm{A}(\mathcal{H})$, then $\mathcal{A}_T \in A_{0\cdot}$.*

(b) *Let $\mathcal{A} \in A_{0\cdot}$. If $T \in \mathcal{A}$, then $T \in C_{0\cdot}$.*

(c) *If $T \in C_{1\cdot}$, then $\mathcal{A}_T \in A_{1\cdot}$.*

Proof. We show (a). Let $x \in \mathcal{H}$. By [2, Lemma 4.5] for every sequence $\{y_n\}$ in $\mathcal{H}$ with $y_n \to 0$ weakly, we have $\|[x \otimes y_n]\| \to 0$. Apply Proposition 8.

For (b), note that $T^n \to 0$ ultraweakly by Proposition 9. So $T^n \to 0$ strongly by Corollary 6.

To show (c), let $X \in \mathcal{M}_p(\mathcal{A}_T)$. As a consequence of Proposition 9, $T^n x \to 0$ weakly. As in the proof of Theorem 5, this implies $\|T^n x\| \to 0$. Thus $x = 0$.

The above theorem yields the following examples. Let S be a forward unilateral shift, and let W be a forward bilateral shift. Then $S \in C_{10} \cap \mathrm{A}(\mathcal{H})$, $S^* \in C_{01} \cap \mathrm{A}(\mathcal{H})$ and $W \in C_{11} \cap \mathrm{A}(\mathcal{H})$. Thus $\mathcal{A}_S \in A_{10}$, $\mathcal{A}_{S^*} \in A_{01}$ and $\mathcal{A}_W \in A_{11}$.

We note the following variant of Theorem 10(a). For $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{U}_T$ will denote the norm closure of the polynomials in T, and $\sigma(T)$ will denote $\{\lambda \in \mathbf{C} : T - \lambda$ is not invertible$\}$. Let $\mathbf{T} = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$.

PROPOSITION 11. *If $T \in C_{0\cdot}$ and $\mathbf{T} \subset \sigma(T)$, then $\mathcal{U}_T \in A_{0\cdot}$.*

Proof. Since $T \in C_{0\cdot}$, there is a forward unilateral shift S on a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$, $S^*\mathcal{H} \subset \mathcal{H}$ and $T = S^*|\mathcal{H}$ [5, p. 81]. Define $\Phi: \mathcal{U}_{S^*} \to \mathcal{U}_T$ by $\Phi(A) = A|\mathcal{H}$, $A \in \mathcal{U}_{S^*}$. Let p be a polynomial. Since $S^* \in \mathrm{A}(\mathcal{H})$, $\|p(S^*)\| = \sup\{|p(\lambda)| : \lambda \in \mathbf{T}\}$. Since $\mathbf{T} \subset \sigma(T)$, $\|p(T)\| \geqslant \sup\{|p(\lambda)| : \lambda \in \mathbf{T}\}$ by the Spectral Mapping Theorem. Thus $\|p(S^*)\| \leqslant \|p(T)\|$ for any polynomial p. Since the reverse inequality is clear, Φ is a surjective isometry. Obviously $\Phi: (\mathcal{U}_{S^*}, \text{strong}) \to (\mathcal{U}_T, \text{strong})$ is continuous. Since $\mathcal{U}_{S^*} \subset \mathcal{A}_{S^*} \in A_{0\cdot}$, $(\mathcal{U}_{S^*})_1$ is strongly precompact by Theorem 4. Thus $(\mathcal{U}_T)_1 = \Phi((\mathcal{U}_{S^*})_1)$ is strongly precompact. So $\mathcal{U}_T \in A_{0\cdot}$ by Theorem 4.

4. SELF-ADJOINT ALGEBRAS

Let $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$. $\mathscr{A}$ is *self-adjoint* if $\mathscr{A} = \mathscr{A}^*$. We characterize the self-adjoint algebras in $A_{\alpha\alpha}$, $\alpha = 0,1$. This reduces to the problem of characterizing the von Neumann algebras in $A_{\alpha\alpha}$, $\alpha = 0,1$. (For information on von Neumann algebras see [4].) We begin with this reduction. For $\mathscr{S} \subset \mathscr{L}(\mathscr{H})$, $\mathscr{S}^s$ will denote the strong closure of $\mathscr{S}$.

PROPOSITION 12. *Let $\mathscr{A}$ be a self-adjoint algebra.*

(a) $\mathscr{M}_p(\mathscr{A}) = \mathscr{M}_p(\mathscr{A}^s) = \mathscr{M}_p(\mathscr{A}'')$.

(b) $\mathscr{A} \in A_{\alpha\alpha}$ *if and only if* $\mathscr{A}^s \in A_{\alpha\alpha}$, $\alpha = 0,1$.

(c) *There exist orthogonal projections P_0, P_1 in $\mathscr{A}'$ such that $\mathscr{H} = P_0\mathscr{H} + P_1\mathscr{H}$ and $\mathscr{A}P_\alpha \in A_{\alpha\alpha}(P_\alpha\mathscr{H})$ for $\alpha = 0,1$.*

Proof. For (a) we clearly have $\mathscr{M}_p(\mathscr{A}'') \subset \mathscr{M}_p(\mathscr{A}^s) \subset \mathscr{M}_p(\mathscr{A})$. Since $\mathscr{A}^s$ is a self-adjoint, strongly closed algebra, there is an orthogonal projection $E \in \mathscr{A}^s$ such that $EA = AE = A$ for $A \in \mathscr{A}^s$ [8, Theorem 1]. Let $\mathscr{E} = \{\lambda(I - E) : \lambda \in \mathbf{C}\}$. Let $\mathscr{R} = \mathscr{E} + \mathscr{A}^s$. One easily verifies that $\mathscr{R}$ is a von Neumann algebra and $\mathscr{A}' = \mathscr{R}'$. Thus $\mathscr{A}'' = \mathscr{R}$. It follows that for each $x \in \mathscr{H}$, $\rho_x((\mathscr{A}'')_1) = \rho_x((\mathscr{A}^s)_1) + \rho_x(\mathscr{E}_1)$. Since $\mathscr{E}$ is finite dimensional $\rho_x(\mathscr{E}_1)$ is norm compact for every $x \in \mathscr{H}$. So if $x \in \mathscr{M}_p(\mathscr{A}^s)$, then $\rho_x((\mathscr{A}'')_1)$ is precompact. Thus $\mathscr{M}_p(\mathscr{A}^s) \subset \mathscr{M}_p(\mathscr{A}'')$. Now we show $\mathscr{M}_p(\mathscr{A}) \subset \mathscr{M}_p(\mathscr{A}^s)$. Suppose $x \in \mathscr{M}_p(\mathscr{A})$. By the Kaplansky Density Theorem, $(\mathscr{A}_1)^s = (\mathscr{A}^s)_1$. Thus $\rho_x((\mathscr{A}^s)_1) = \rho_x(\mathscr{A}_1)^s \subset \rho_x(\mathscr{A}_1)^-$. This shows $x \in \mathscr{M}_p(\mathscr{A}^s)$.

(b) follows easily from (a).

To show (c), let P_0 be the orthogonal projection onto $\mathscr{M}_p(\mathscr{A})$, and let $P_1 = I - P_0$. By Theorem 1 and the Double Commutant Theorem, $P_\alpha \in \mathscr{A}'$, $\alpha = 0,1$. The rest is straightforward.

We now characterize the von Neumann algebras in A_{00}.

THEOREM 13. *Let $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ be a von Neumann algebra. The following are equivalent.*

(a) $\mathscr{A} \in A_{00}$.

(b) $\mathscr{A}_1$ *is strongly compact.*

(c) $\mathscr{P}_{\mathscr{A}} = \{P \in \mathscr{A} : P = P^2 = P^*\}$ *is strongly compact.*

(d) $\mathscr{U}_{\mathscr{A}} = \{U \in \mathscr{A} : U^* = U^{-1}\}$ *is strongly compact.*

(e) $\mathscr{H}$ *is an orthogonal direct sum of finite dimensional elements of* Lat $\mathscr{A}$.

(f) $\mathscr{A}$ *is a direct sum of finite dimensional von Neumann algebras.*

Proof. Since a von Neumann algebra is a dual algebra, (a) implies (b) by Proposition 7. For (b) implies (c), observe that $\mathscr{P}_{\mathscr{L}(\mathscr{H})}$ is strongly closed. Thus $\mathscr{P}_{\mathscr{A}}$ is strongly closed, and so strongly compact.

We show (c) implies (d). Note that $\mathscr{U}_{\mathscr{A}}$ is contained in $\mathscr{D}$, the strong closure of the absolutely convex hull of $\mathscr{P}_{\mathscr{A}}$, by the Spectral Theorem. Since $(\mathscr{L}(\mathscr{H}))_1$ is strongly complete, $\mathscr{D}$ is strongly compact. So we must show $\mathscr{U}_{\mathscr{A}}$ is strongly closed. Let $\{U_i\}$ be a net in $\mathscr{U}_{\mathscr{A}}$ which converges strongly to V. Then V is an isometry, and $U_i^* \to V^*$ weakly. Since the weak and strong topologies agree on $\mathscr{D}$ by Theorem 5, $U_i^* \to \to V^*$ strongly. Thus V^* is an isometry, and so V is unitary.

We establish (d) implies (e). The identity map on $\mathscr{U}_{\mathscr{A}}$ is a strongly continuous unitary representation of the compact group $\mathscr{U}_{\mathscr{A}}$. As is well known [3, Theorem 15.1.3], such a representation is a direct sum of finite dimensional representations. So $\mathscr{H}$ is an orthogonal direct sum of finite dimensional elements of Lat $\mathscr{A}$.

That (e) implies (f) is straigtforward. Finally by Proposition 3, parts (e) and (g), we obtain (f) implies (a).

COROLLARY 14. *Let $\mathscr{A}$ be a self-adjoint algebra. $\mathscr{M}_p(\mathscr{A})$ is the span of the finite dimensional elements of* Lat $\mathscr{A}$.

Proof. $\mathscr{M}_p(\mathscr{A}) = \mathscr{M}_p(\mathscr{A}'')$ by Proposition 12(a). It is easy to see that Lat $\mathscr{A} = $ Lat $\mathscr{A}''$. So we may assume that $\mathscr{A}$ is a von Neumann algebra. Let E be the orthogonal projection onto $\mathscr{M}_p(\mathscr{A})$. Then $E \in \mathscr{A} \cap \mathscr{A}'$ by Theorem 1. It follows that $\mathscr{A}E$ is a von Neumann algebra in $A_{00}(E\mathscr{H})$. So $\mathscr{M}_p(\mathscr{A}) = E\mathscr{H}$ is the span of the finite dimensional elements of Lat$(\mathscr{A}E)$ by Theorem 13(e). Since $E \in \mathscr{A}$, an element of Lat$(\mathscr{A}E)$ is an element of Lat $\mathscr{A}$. Furthermore, a finite dimensional element of Lat $\mathscr{A}$ is in Lat$(\mathscr{A}E)$ by Proposition 3(e). Thus Lat $\mathscr{A}$ and Lat$(\mathscr{A}E)$ have the same finite dimensional elements, and the result follows.

Finally we characterize the self-adjoint algebras in A_{11}.

COROLLARY 15. *Let $\mathscr{A}$ be a self-adjoint algebra. The following are equivalent.*
(a) $\mathscr{A} \in A_{11}$.
(b) $\mathscr{A}' \cap \mathscr{K}(\mathscr{H}) = (0)$.
(c) Lat $\mathscr{A}$ *contains no non-zero finite dimensional elements.*

Proof. That (a) implies (b) is just Proposition 3(i). For (b) implies (c), assume Lat $\mathscr{A}$ has a non-zero finite dimensional element. Then the orthogonal projection onto this element is a non-zero finite rank projection in $\mathscr{A}'$. By Corollary 14, (c) implies (a).

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MICHAEL MARSALLI
Department of Mathematics,
Illinois State University,
Normal, Illinois 61761,
U.S.A.

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