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THE BEREZIN TRANSFORM AND HA-PLITZ OPERATORS

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By *quantization* one usually intends a procedure which to functions ("symbols") on a suitable manifold (Poisson, symplectic etc.) assigns linear operators in Hilbert space. Quantization is of interest not only in physics (quantum theory) but also in connection with pseudodifferential operators. One then speaks of *operator calculi.*

An approach to quantization of Kähler manifolds was suggested in the 70's by the late F. A. Berezin, who apparently was a very independent mind. Berezin is otherwise known as the father of "super mathematics" (see [8]). He died in an accident in 1980. His main papers on quantization are [4–6]. They contain a wealth of ideas which seem still to be largely unexplored. See also the survey [7], unfortunately written in peculiar, next to unintelligible English; it is amazing that such an unfinished product could have been accepted by an otherwise respectable (?) journal. A complete list of Berezin's publications can be found in the book [8]. A critique of Berezin's view can be found in e.g. [36] (cf. Section 6 of the present paper). The determination of all admissible values of "Planck's constant" in the case of symmetric domains begun in [6] was completed in [17]. An extension of these results to infinite dimension is given in [29]. See also [23], [24], [12]. Berezin quantization is furthermore mentioned in [38], [39], where in particular a Jordan theoretic formuation is given.

Even if Berezin's ideas on quantization should not be of real interest for physics, they undoubtedly have found a permanent place in "Ha-plitz" (Hankel + s+ Toeplitz) theory, an acronym invented by N. K. Nikol'skiĭ. In particular, the so-called *Berezin transform* is of vital interest there, and appears explicitly or implicitly in many recent papers (see e.g. [2], [3], [21], [11], [12], [13–15], [41–43], [30]). This has instigated us to write the present paper, which to a large extent is expository. We thus review those aspects of Berezin quantization which are of particular interest from the Ha-plitz point of view. However, this is by no means a fully objective presentation of the matter. I give rather a picture which is consciously distorted to fit my own personal prejudices. Throughout some directions for further work are pointed out. The possible physical aspects of the theory will be payed no attention to.

We begin in Section 1 by recalling briefly the salient facts about Hermitean line bundles over a complex manifold.

Then we go on in Sections 2–3 by summarizing, in a form convenient for our purposes, what we believe to be the essential contents of [4–6]. A novelty is perhaps the systematic use of the language of line bundles. Berezin himself works with scalar functions, not sections of line bundles, which makes some parts of his work look somewhat artificial. In the context of Hankel forms line bundles were used in [25].

In Section 4 we discuss some of the most important examples of manifolds quantizable in the present sense. In particular, it turns out that symmetric domains can be quantized.

In Section 5, following [5], [7], we express the Berezin transform in some of these cases in terms of the Laplacian. This allows us to extend to the case of the ball a result on (big) Hankel operators with non-analytic symbols previously proved in [2] for the disc.

In Section 6 we mention some alternative avenues to quantization, including A. Unterberger's vast program of quantization of symmetric spaces (see e.g. [31–37]).

Section 7 is devoted to some concluding observations.

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1. THE LANGUAGE OF LINE BUNDLES

Let Ω be a complex manifold. As is well-known a (holomorphic) Hermitean line bundle $\varkappa$ over Ω can be defined by the following data:

1) An open covering $\{U_\alpha\}_{\alpha\in I}$ of Ω.

2) A family of transition functions $\{g_{\alpha\alpha'}\}_{\alpha,\alpha'\in I}$, each function $g_{\alpha\alpha'}$ being holomorphic and nonvanishing in $U_\alpha \cap U_{\alpha'}$ (if $\neq \emptyset$) satisfying the cocycle condition:

$$g_{\alpha\alpha'}g_{\alpha'\alpha''} = g_{\alpha\alpha''} \quad \text{in } U_\alpha \cap U_{\alpha'} \cap U_{\alpha''} \quad (\text{if } \neq \emptyset).$$

3) A family of metric coefficients $\{e_\alpha\}_{\alpha\in I}$, each e_α being a positive smooth (C^∞, real analytic) function on U_α, satisfying

$$e_\alpha = |g_{\alpha\alpha'}|^{-2}e_{\alpha'} \quad \text{in } U_\alpha \cap U_{\alpha'} \quad (\text{if } \neq \emptyset).$$

Here and in the sequel I stands for a suitable index set, fixed throughout the discus-(sion.)

We denote by $\bar{\Omega}$ the same set with the opposite complex structure. Then the family of conjugates $\bar{g}_{\alpha\alpha'}$ defines an Hermitean line bundle $\bar{\varkappa}$ over $\bar{\Omega}$ (the covering $\{U_\alpha\}_{\alpha\in I}$ and the family $\{e_\alpha\}_{\alpha\in I}$ of metric coefficients are the same).

The fiber $\varkappa_z$ of $\varkappa$ over a point $z \in \Omega$ can be defined to consist of all families $x = \{x_\alpha\}_{\alpha\in I_z}$ of complex numbers such that

$$g_{\alpha\alpha'}(z)x_{\alpha'} = x_\alpha \quad \text{for } \alpha \in I_z.$$

Here I_z denotes the set of indices α such that $z \in U_\alpha$.) It follows that

$$|x_\alpha|^2 e_\alpha(z) = |x_{\alpha'}|^2 e_{\alpha'}(z) \quad \text{for } \alpha \in I_z.$$

Thus we can define the norm $\|x\|_z$ of $x \in \varkappa_z$, as the positive square root of the last expression. The corresponding inner product is denoted $(x, y)_z$.

Remark. The bundle, and the metric do not change if we replace the $g_{\alpha\alpha'}$ by $\tilde{g}_{\alpha\alpha'} = t_\alpha g_{\alpha\alpha'} t_{\alpha'}^{-1}$ and e_α by $\tilde{e}_\alpha = |t_\alpha|^{-2} e_\alpha$, where $\{t_\alpha\}$ is a family of nonvanishing holomorphic functions. Then $x = \{x_\alpha\}_{\alpha\in I_z}$ has to be replaced by $\tilde{x} = \{t_\alpha(z)x_\alpha\}_{\alpha\in I}$. In addition, one can of course also replace the given covering $\{U_\alpha\}_{\alpha\in I}$ by a finer one.

A holomorphic section f of $\varkappa$ is given by a family of functions $\{f_\alpha\}_{\alpha\in I}$, each function f_α being holomorphic in U_α, such that

$$f_\alpha = g_{\alpha\alpha'} f_{\alpha'} \quad \text{in } U_\alpha \cap U_{\alpha'} \ (\text{if} \neq \emptyset).$$

We denote further by $\bar{f}$ the corresponding section of $\bar{\varkappa}$ given by the family of conjugates $\{\bar{f}_\alpha\}_{\alpha\in I}$.

Let μ be a given positive measure on Ω satisfying certain assumptions[1]. We denote by $\mathscr{H}^2(\Omega, \mu, \varkappa)$ the Hilbert space of holomorphic sections with the norm

$$\|f\|_\mu^2 = \int_\Omega \|f(z)\|_z^2 \mathrm{d}\mu(z).$$

The corresponding inner product will be denoted $(f, g)_\mu$. Further we let $\overline{\mathscr{H}^2(\Omega, \mu, \varkappa)}$ be the space consisting of the conjugates of all elements of $\mathscr{H}^2(\Omega, \mu, \varkappa)$; these are thus *anti-holomorphic* sections of $\varkappa$ (or holomorphic sections of $\bar{\varkappa}$).

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[1] For most purposes it suffices to assume that μ has a smooth positive density.

REMARK. The space $\mathscr{H}^2(\Omega, \mu, \varkappa)$ and its metric do not change if we replace $d\mu$ by the measure $d\tilde{\mu} = \chi d\mu$ and, at the same time, each e_α by $\tilde{e}_\alpha = \chi^{-1}e_\alpha$. Here χ is some fixed positive smooth function on Ω. For this we have only to record the formula

$$(|x|_z^{\sim})^2 = \chi^{-1}(z)|x|_z^2 \quad \text{for} \quad x \in \varkappa_z,$$

expressing the change of metric on the bundle level. We refer to such a change of the set up as a *gauge transformation* (cf. [2], [21]). Perhaps, in retrospect, *conformal* would have been more appropriate.

Concluding the section, let us also recall why line bundles, and in particular vector spaces of sections, are of interest in (algebraic) geometry. Namely, they give all maps of the given manifold Ω into a projective space $\mathbf{P}^N$ (N = one minus the dimension of the vector space).

EXAMPLE. A conic in $\mathbf{P}^2$ is given (in inhomogeneous coordinates) by an equation of the type

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$$

In particular, the space of all conics is a $\mathbf{P}^5$. Fixing a point (x, y) consider the set of all conic through it. It is a hyperplane in $\mathbf{P}^5$. Thus we obtain a map of the plane into the dual projective space $\mathbf{P}^{5*}$ (Veronese map; cf. e.g. [18], p. 178–179). More generally, the (holomorphic) sections of any holomorphic line bundle over a projective space $\mathbf{P}^N$ of any dimension may be represented by homogeneous polynomials of suitable degree.

2. THE BEREZIN TRANSFORM IN HERMITEAN LINE BUNDLES

Maintain the scenario of Section 1 (Ω, $\varkappa$, μ etc.). Consider a linear operator from the Hilbert space $\mathscr{H}^2(\Omega, \mu, \varkappa)$ into itself. Then there exists for each $z \in \Omega$ a family $\{\mathscr{A}_{\beta w}\}_{\alpha \in I_w}$ of sections of the space $\overline{\mathscr{H}^2(\Omega, \mu, \varkappa)}$ such that

$$(1) \qquad (Tf)_\beta(w) = (f, \overline{\mathscr{A}_{\beta w}})_\mu \quad \text{if } w \in U_\beta.$$

It is clear that

$$(2) \qquad g_{\beta\beta'}\mathscr{A}_{\beta' w} = \mathscr{A}_{\beta w} \ \text{ in } U_\beta \cap U_{\beta'}.$$

Each $\mathscr{A}_{\beta w}$ again is given by a family $\mathscr{A}_{\beta\alpha w}(\cdot)$ of holomorphic functions. Thus we have *in toto* a doubly indexed family $\{\mathscr{A}_{\beta\alpha}(w, z)\}$ of functions on the product $\Omega \times \overline{\Omega}$, each function $A_{\beta\alpha}(w, z)$ being holomorphic in the first argument and anti-holomorphic in the second argument. This family plays the role of the *kernel* of the operator T.

It is clear that we now have the transformation rule

$$g_{\beta\beta'}\bar{g}_{\alpha\alpha'}\mathscr{A}_{\beta'\alpha'} = \mathscr{A}_{\beta\alpha} \quad \text{in } (U_\beta \cap U_{\beta'}) \times (U_\alpha \cap U_{\alpha'}). \tag{3}$$

In terms of $\mathscr{A}_{\beta\alpha}$ we can write the preceding "producing" formula in a perhaps more familiar looking form as

$$(Tf)_\beta(w) = \int_\Omega \sum_{\alpha\in I} \varphi_\alpha(z)\mathscr{A}_{\beta\alpha}(w, z)f_\alpha(z)e_\alpha(z)\mathrm{d}\mu(z). \tag{4}$$

Here $\{\varphi_\alpha\}$ is a partition of unity such that $\operatorname{supp}\varphi_\alpha \subset U_\alpha$.

Let us apply these considerations to the case when T is the identity operator on the space $\mathscr{H}^2(\Omega, \mu, \varkappa)$. Then we get a family $\{\mathscr{K}_{\beta w}\}$ of anti-holomorphic sections such that

$$f_\beta(w) = (f, \overline{\mathscr{K}_{\beta w}})_\mu \quad \text{if } w \in U_\beta \tag{1'}$$

satisfying

$$g_{\beta\beta'}\bar{g}_{\alpha\alpha'}\mathscr{K}_{\beta'\alpha'} = \mathscr{K}_{\beta\alpha} \quad \text{in } U_\beta \cap U_{\beta'}. \tag{2'}$$

There is a corresponding doubly indexed family $\{\mathscr{K}_{\beta\alpha}(w, z)\}$ satisfying

$$g_{\beta\beta'}\bar{g}_{\alpha\alpha'}\mathscr{K}_{\beta'\alpha'} = \mathscr{K}_{\beta\alpha} \quad \text{in } (U_\beta \cap U_{\beta'}) \times (U_\alpha \cap U_{\alpha'}), \tag{3'}$$

which obviously plays the role of the *reproducing kernel*. Formula (4) has the following counterpart:

$$f_\beta(w) = \int_\Omega \sum_{\alpha\in I} \varphi_\alpha(z)\mathscr{K}_{\beta\alpha}(w, z)f_\alpha(z)e_\alpha(z)\mathrm{d}\mu(z) \tag{4'}$$

We notice that

$$\mathscr{K}_{\beta\alpha}(w, z) = (\mathscr{K}_{\alpha z}, \mathscr{K}_{\beta w})_\mu .$$

In particular, it follows that

$$\mathscr{K}_{\alpha\beta}(z, w) = \overline{\mathscr{K}_{\beta\alpha}(w, z)}.$$

It is natural to set

$$\frac{\mathscr{A}_{\beta w}}{\mathscr{K}_{\beta w}} \overset{\text{def}}{=} A_w = A(w, \cdot).$$

Since the fibers are one dimensional vector spaces, this division makes sense. In other words, we also have

$$A(w, z) = \frac{\mathscr{A}_{\beta\alpha}(w, z)}{\mathscr{K}_{\beta\alpha}(w, z)}.$$

Notice that $A(w, z)$ is a globally defined function on $\Omega \times \bar{\Omega}$ which is analytic in the first argument and anti-analytic in the second argument. We will refer to it as the *Berezin "covariant" symbol* of the operator T. As it is uniquely determined by its restriction to the "diagonal" in $\Omega \times \bar{\Omega}$, we may likewise refer to $A(z) = A(z, z)$ as the Berezin "covariant" symbol. It contains thus all essential information about T. Below we will see ample illustrations of this.

First let us however put into play an auxiliary quantity, already encountered in [21], [25]. Namely, let us set

$$\omega(z) \stackrel{\text{def}}{=} \frac{1}{e_\alpha(z)\mathscr{K}_{\alpha\alpha}(z, z)} \quad \text{for} \quad \alpha \in I_z.$$

Again, it is easy to verify that this gives a globally defined function on Ω. The measure

$$d\iota(z) = \omega^{-1}(z) d\mu(z)$$

is called the *Berezin invariant measure*.

REMARK. Similarly, $d\nu(z) = w(z)d\mu(z)$ is called the "associated" measure in [21], [25]. Really, for our present purposes it would have been more convenient to work with $\omega^{-1}(z)$, not $\omega(z)$. However, we did not want to depart here to much from already established notation. Sometimes $\omega(z)$ has the interpretation of "distance from z to the boundary".

Consider the composition $T = T_1 \circ T_2$ of two operators. We let $\mathscr{A}_{k\beta w}$, $\mathscr{A}_{k\beta\alpha}$, A_k correspond to T_k $(k = 1, 2)$, our goal being to express the Berezin symbol A of T in terms of the Berezin symbols A_1 and A_2 of T_1 and T_2.

Let $\{\varphi_\alpha\}$ be the previous partition of unity. Then, using (1) and (4), we can write

$$\begin{aligned}(Tf)_\beta(z) &= (T_1(T_2 f))_\beta(w) = (T_2 f, \mathscr{A}_{1\beta w})_\mu = \\ &= \int_\Omega \sum_{\gamma \in I} \varphi_\gamma(\zeta)(T_2 f)_\gamma(\zeta)\mathscr{A}_{1\beta\gamma}(w, \zeta)e_\gamma(\zeta)d\mu(\zeta) = \\ &= \int_\Omega \sum_{\gamma \in I} \varphi_\gamma(\zeta)(f, \overline{\mathscr{A}_{2\gamma\zeta}})_\mu \mathscr{A}_{1\beta\gamma}(w, \zeta)e_\gamma(\zeta)d\mu(\zeta) = \\ &= \left(f, \overline{\int_\Omega \sum_{\gamma \in I} \varphi_\gamma(\zeta)\mathscr{A}_{2\gamma\zeta}\mathscr{A}_{1\beta\gamma}(w, \zeta)e_\gamma(\zeta)d\mu(\zeta)}\right)_\mu.\end{aligned}$$

Comparison with (1) shows that $\mathscr{A}_{\beta w}$ equals the last (vector valued) integral. Thus division with $\mathscr{K}_{\beta w}$ yields

$$\frac{\mathscr{A}_{\beta w}}{\mathscr{K}_{\beta w}} = \int_{\Omega} \sum_{\gamma \in I} \varphi_\gamma(\zeta) \frac{\mathscr{A}_{2\gamma\zeta} \mathscr{K}_{\gamma\zeta} \mathscr{A}_{1\beta\gamma}(w, \zeta) \mathscr{K}_{\beta\gamma}(w, \zeta)}{\mathscr{K}_{\gamma\zeta} \mathscr{K}_{\beta w} \mathscr{K}_{\beta\gamma}(w, \zeta) \mathscr{K}_{\gamma\gamma}(\zeta, \zeta)} e_\gamma(\zeta) \mathscr{K}_{\gamma\gamma}(\zeta,\zeta) \mathrm{d}\mu(\zeta)$$

or

$$A(w, z) = \int_{\Omega} \sum_{\gamma \in I} \varphi_\gamma(\zeta) \frac{\mathscr{K}_{\gamma\alpha}(\zeta, z) \mathscr{K}_{\beta\gamma}(w, \zeta)}{\mathscr{K}_{\beta\alpha}(w, z) \mathscr{K}_{\gamma\gamma}(\zeta, \zeta)} A_1(w, \zeta) A_2(\zeta, z) \mathrm{d}\imath(\zeta).$$

We notice that

$$J(w, z, \zeta) \overset{\text{def}}{=} \frac{\mathscr{K}_{\gamma\alpha}(\zeta, z) \mathscr{K}_{\beta\gamma}(w, \zeta)}{\mathscr{K}_{\beta\alpha}(w, z) \mathscr{K}_{\gamma\gamma}(\zeta, \zeta)}$$

is independent of α, β, γ (see (3')). As $\sum_{\gamma \in I} \varphi_\gamma = 1$, we therefore conclude that

$$A(w, z) = \int_{\Omega} J(w, z, \zeta) A_1(w, \zeta) A_2(\zeta, z) \mathrm{d}\imath(\zeta)$$

– compare with the formula for matrix multiplication! – or, with $z = w$,

$$A(z) = \int_{\Omega} J(z, \zeta) A_1(z, \zeta) A_2(\zeta, z) \mathrm{d}\imath(\zeta),$$

where

$$J(z, \zeta) \overset{\text{def}}{=} J(z, z, \zeta) = \frac{\mathscr{K}_{\gamma\alpha}(\zeta, z) \mathscr{K}_{\gamma\alpha}(z, \zeta)}{\mathscr{K}_{\alpha\alpha}(z, z) \mathscr{K}_{\gamma\gamma}(\zeta, \zeta)}.$$

Thus we may write symbolically

$$\boxed{A = \operatorname{Ber}(A_1 \cdot \check{A}_2)},$$

where Ber stands for the *Berezin transform*, i.e. quite generally

$$\operatorname{Ber} F(z) = \int_{\Omega} J(z, \zeta) F(z, \zeta) \mathrm{d}\imath(\zeta)$$

and

$$\check{F}(z, \zeta) = F(\zeta, z).$$

It will be convenient to use also the alternative notation

$$A_1 * A_2 = \operatorname{Ber}(A_1 \cdot \check{A}_2).$$

Then the above formula can be written simply as

$$A = A_1 * A_2 .$$

We conclude thus that the covariant symbols thus form an *associative algebra*, which we in the following denote by $\mathscr{B}$ (Berezin algebra).

There is a second way how the Berezin transform enters. Let us observe that the "reproducing" family $\{\bar{\mathscr{K}}_{\alpha z}\}_{\alpha \in I_z}$ for each $z \in \Omega$ spans a one dimensional subspace $\mathscr{H}_z$ of $\mathscr{H}^2(\Omega, \mu, \varkappa)$. Let P_z denote orthogonal projection in $\mathscr{H}^2(\Omega, \mu, \varkappa)$ onto $\mathscr{H}_z$. That is, explicitly:

$$P_z f = \frac{(f, \bar{\mathscr{K}}_{\alpha z})_\mu}{(\mathscr{K}_{\alpha z}, \mathscr{K}_{\alpha z})_\mu} \bar{\mathscr{K}}_{\alpha z} \quad \text{for } f \in \mathscr{H}^2(\Omega, \mu, \varkappa). \tag{5}$$

If now a linear operator T in that space can be written in the form

$$T = \int_\Omega \mathring{A}(\zeta) P_\zeta \,d\iota(\zeta) \tag{6}$$

for some function $\mathring{A}$ then we say that $\mathring{A}$ is the *Berezin "contravariant" symbol* of T. Thus, a given operator may have two "symbols"[2]. What is then the relation A and $\mathring{A}$? It turns out that, indeed, the answer can be stated in terms of the Berezin transform Ber. Using (5) we can write (6) as

$$Tf = \int_\Omega \mathring{A}(\zeta) \sum_{\gamma \in I} \varphi_\gamma(\zeta) \frac{(f, \bar{\mathscr{K}}_{\gamma\zeta})_\mu}{(\mathscr{K}_{\gamma\zeta}, \mathscr{K}_{\gamma\zeta})_\mu} \bar{\mathscr{K}}_{\gamma\zeta} \,d\iota(\zeta)$$

or

$$(Tf)_\alpha(z) = \int_\Omega \mathring{A}(\zeta) \sum_{\gamma \in I} \varphi_\gamma(\zeta) \frac{(f, \bar{\mathscr{K}}_{\gamma\zeta})_\mu}{(\mathscr{K}_{\gamma\zeta}, \mathscr{K}_{\gamma\zeta})_\mu} \mathscr{K}_{\alpha\gamma}(z, \zeta) \,d\iota(\zeta) =$$

$$= \left(f, \overline{\int_\Omega \mathring{A}(\zeta) \sum_{\gamma \in I} \varphi_\gamma(\zeta) \frac{\mathscr{K}_{\alpha\gamma}(z, \zeta)}{\mathscr{K}_{\gamma\gamma}(\zeta, \zeta)} \mathscr{K}_{\gamma\zeta}}\right)_\mu \quad \text{for } \alpha \in I_z . \tag{7}$$

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[2]) Why Berezin picks the words "covariant" and "contravariant" is a mystery to us.

It follows that

$$A(z)=\frac{(Tf)_\alpha(z)}{\mathcal{K}_{\alpha z}(z)}=\int_\Omega \mathring{A}(\zeta)\,\frac{\mathcal{K}_{\alpha\gamma}(z,\zeta)\mathcal{K}_{\gamma\alpha}(\zeta,z)}{\mathcal{K}_{\alpha\alpha}(z,z)\mathcal{K}_{\gamma\gamma}(\zeta,\zeta)}\,\mathrm{d}\iota(z)=$$

$$=\int_\Omega J(z,\zeta)\mathring{A}(\zeta)\mathrm{d}\iota(z),$$

that is, as predicted

$$\boxed{A=\operatorname{Ber}(\mathring{A})}\,.$$

Remark (Berezin-Toeplitz operators). Formula (6) can be given yet another interpretation. Let P denote orthogonal projection onto $\mathcal{H}^2(\Omega,\mu,\varkappa)$ in the space $L^2(\Omega,\mu,\varkappa)$, the space of square integrable (not necessarily having any regularity properties whatsoever) sections of the bundle $\varkappa$. Explicitly, in our previous notation:

$$(8)\quad (Pf)_\beta(w)=(f,\overline{\mathcal{K}_{\beta w}})_\mu=\int_\Omega\sum_{\alpha\in I}\varphi_\alpha(z)\mathcal{K}_{\beta\alpha}(w,z)f_\alpha(z)e_\alpha(z)\mathrm{d}\mu(z)\qquad\text{for } f\in L^2(\Omega,\mu,\varkappa).$$

Using the measure μ instead of ι, we can rewrite (7) as

$$(Tf)_\alpha(z)=\int\mathring{A}(\zeta)\sum_{\gamma\in I}\varphi_\gamma(\zeta)\mathcal{K}_{\gamma\alpha}(z,\zeta)f_\alpha(\zeta)e_\gamma(\zeta)\mathrm{d}\mu(\zeta)\qquad\text{for } f\in\mathcal{H}^2(\Omega,\mu,\varkappa).$$

Comparing with (8) it follows that

$$T=P\circ M_{\mathring{A}}\,,$$

where $M_{\mathring{A}}$ stands for the operation of multiplication by the function $\mathring{A}$. In other words, the operator in (6) may be viewed as a kind of Toeplitz operator, a truncated multiplication operator.

Finally, let us that the operator T is given by a formula like (1) or (1′) but with a family $\{\tilde{\mathcal{A}}_{\beta w}\}$ or $\{\tilde{\mathcal{A}}_{\beta\alpha}(w,z)\}$ where we have given up the requirement of anti-holomorphicity in the variable z. How to restore this? We may define a "symbol" $\tilde{A}$ in an analogous way as we defined the Berezin (covariant) symbol A. Again the answer is in terms of the Berezin transform. Using our previous partition of unity $\{\varphi_\alpha\}_{\alpha\in I}$, we can write

$$(Tf)_\beta(w) = \int_\Omega \sum_{\alpha\in I} \varphi_\alpha(z)\tilde{\mathscr{A}}_{\beta\alpha}(w, z)f_\alpha(z)e_\alpha(z)\mathrm{d}\mu(z) =$$

$$= \int_\Omega \sum_{\gamma\in I} \varphi_\gamma(\zeta)\tilde{\mathscr{A}}_{\beta\gamma}(w, \zeta)(f, \mathscr{K}_{\gamma\zeta})_\mu e_\gamma(\zeta)\mathrm{d}\mu(\zeta) =$$

$$= \left(f, \overline{\int_\Omega \sum_{\gamma\in I} \varphi_\gamma(\zeta)\tilde{\mathscr{A}}_{\beta\gamma}(w, \zeta)\mathscr{K}_{\gamma\zeta}e_\gamma(\zeta)\mathrm{d}\mu(\zeta)}\right)_\mu .$$

It follows that

$$A(w, z) = \int_\Omega \sum_{\gamma\in I} \varphi_\gamma(\zeta)\frac{\mathscr{K}_{\gamma\alpha}(\zeta, z)\mathscr{K}_{\beta\gamma}(w, \zeta)}{\mathscr{K}_{\beta\alpha}(w, z)\mathscr{K}_{\gamma\gamma}(\zeta, \zeta)}\tilde{A}(w, \zeta)e_\gamma(\zeta)\mathscr{K}_{\gamma\gamma}(\zeta, \zeta)\mathrm{d}\mu(\zeta) =$$

$$= \int_\Omega J(w, z, \zeta)\tilde{A}(w, \zeta)\mathrm{d}\iota(\zeta); \quad A(z) = \int_\Omega J(z, \zeta)\tilde{A}(z, \zeta)\mathrm{d}\iota(\zeta),$$

that is,

$$\boxed{A = \mathrm{Ber}(\tilde{A})} .$$

These three framed formulae constitute the main findings of this section.

3. QUANTIZATION OF KÄHLER MANIFOLDS

In the previous section we have explained how to assign to a Hermitean line bundle $\varkappa$ on a complex manifold Ω and a positive measure μ a structure of associative algebra for functions on Ω, the algebra $\mathscr{B}$ (see the first framed formula in that section). From Berezin's point of view quantization is about *deformation* of this structure. The question is thus to select a family line bundles $\varkappa = \varkappa_h$ depending on a positive parameter h ("Planck's constant") and to pass to the limit ($h \to 0$). Optimally, one then expects to recover in the limit a given Poisson or symplectic structure on Ω ("correspondence principle"). Thus, quantization is also about deformation of line bundles.

From now on, we assume that Ω comes equipped with a *Kähler* structure. Out of the many available of the many available definitions of a Kähler metric we select the following one: Ω is an Hermitean manifold such that its metric (an Hermitean form on the tangent bundle of Ω) can locally be written $ds^2 = \sum g_{ij} dz^i d\bar{z}^j$ with $g_{ij} = \dfrac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}$ is a real valued smooth function ("Kähler potential"). If we introduce the alternating differential form $\sigma = i \sum g_{ij} dz^i \wedge d\bar{z}^j$, then this requirement can be condensed into the equivalent condition $d\sigma = 0$; σ is then known as the "Kähler form". Thus a Kähler manifold carries automatically also a *symplectic* structure.

The Kähler potential is of course not unique: we can add to Φ any pluriharmonic function, that is, a function of the type $\varphi + \bar{\varphi}$ where φ is holomorphic. More precisely, we thus have the following data on our manifold Ω:

1) An open covering $\{U_\alpha\}_{\alpha \in I}$ of Ω.

2) A family $\{z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n)\}_{\alpha \in I}$ of corresponding local coordinates ($n = \dim \Omega$).

3) A family $\{\Phi_\alpha\}_{\alpha \in I}$ of real valued smooth functions satisfying the following compatibility condition:

$$\Phi_\alpha - \Phi_{\alpha'} = \varphi_{\alpha\alpha'} + \overline{\varphi_{\alpha\alpha'}} \quad \text{in } U_\alpha \cap U_{\alpha'}. \tag{1}$$

where $\varphi_{\alpha\alpha'}$ is holomorphic.

How to get a line bundle from these data?

We observe that the $\varphi_{\alpha\alpha'}$ are not unique: we can add to them arbitrary purely imaginary numbers. Adding up two relations of the type (1) we obtain

$$\Phi_\alpha - \Phi_{\alpha''} = \varphi_{\alpha\alpha'} + \varphi_{\alpha'\alpha''} + \overline{\varphi_{\alpha\alpha'} + \varphi_{\alpha'\alpha''}}.$$

From this we conclude that the $\varphi_{\alpha\alpha'}$ do not give quite an ("additive") cocycle:

$$\varphi_{\alpha\alpha'} + \varphi_{\alpha'\alpha''} = \varphi_{\alpha'\alpha''} + ik_{\alpha\alpha'\alpha''} \tag{2}$$

with $k_{\alpha\alpha'\alpha''} \in \mathbf{R}$. However, we deduce the cocycle relation

$$k_{\alpha'\alpha''\alpha'''} - k_{\alpha\alpha''\alpha'''} + k_{\alpha\alpha'\alpha'''} - k_{\alpha\alpha'\alpha''} = 0.$$

Thus we obtain an element in the cohomology group $H^2(\Omega, \mathbf{R})$.

REMARK. It is easy to see that this cohomology class coincides with the one defined by the Kähler form σ. One has just to adapt the usual proof of de Rham's theorem (see e.g. [18]).

We now make the basic assumption ("quantization condition") that there is a number $h > 0$ (Planck's constant) such that the above cohomology class is h times an element in $H^2(\Omega, \mathrm{Z})$. This is a strong requirement.

EXAMPLE. For instance, if Ω is compact then it follows by the above remark that Ω can be imbedded in a projective space $\mathbf{P}^N$ for N sufficiently big (Koidara's theorem; see again [18]). If Ω is a plane domain ($n = 1$) bounded by finitely many smooth curves then $H^2(\Omega, \mathbf{R}) = 0$ so any h and any metric will do. A natural choice is the "Poincaré metric", which is an Hermitean metric of constant negative curvature whose existence follows from the uniformization theorem. (Cf. infra Section 6).

Let the above "integer valued" cohomology class be defined by a family of integers $\tilde{k}_{\alpha\alpha'\alpha''}$. Then we can write with suitable numbers $l_{\alpha\alpha'}$

$$k_{\alpha\alpha'\alpha''} = h\tilde{k}_{\alpha\alpha'\alpha''} + l_{\alpha'\alpha''} - l_{\alpha\alpha''} + l_{\alpha\alpha'} . \tag{3}$$

Upon replacing the given functions $\varphi_{\alpha\alpha'}$ by $\varphi_{\alpha\alpha'} + il_{\alpha\alpha'}$ we can without loss of generality assume that (3) holds with $l_{\alpha\alpha'} = 0$, i.e. that

$$k_{\alpha\alpha'\alpha''} = h\tilde{k}_{\alpha\alpha'\alpha''} . \tag{3'}$$

Thus we obtain our Hermitean line bundle $\varkappa = \varkappa_h$ if we put

$$g_{\alpha\alpha'} = e^{-\frac{1}{\hbar}\varphi_{\alpha\alpha'}}, \quad e_\alpha = e^{-\frac{1}{\hbar}\Phi_\alpha} \quad \left(\hbar = \frac{h}{2\pi}\right).$$

(Cf. Section 1.) Furthermore, we have a natural choice for the measure μ, namely the Kähler-Liouville measure: $\mu = \text{const} \cdot \underbrace{\sigma \wedge \ldots \wedge \sigma}_{n \text{ times}}$. Thus the Hilbert space $\mathscr{H}(\Omega, \mu, \varkappa)$ formally makes sense.

REMARK. If we replace the $\varphi_{\alpha\alpha'}$ by $\varphi_{\alpha\alpha'} + u_\alpha - u_{\alpha'}$ where the u_α are holomorphic, then $g_{\alpha\alpha'}$ gets replaced by $e^{-\frac{1}{\hbar}u_\alpha} g_{\alpha\alpha'} e^{\frac{1}{\hbar}u_{\alpha'}}$ and e_α gets replaced by $e_\alpha e^{-\frac{1}{\hbar}(u_\alpha + \bar{u}_\alpha)}$. Therefore we get essentially the same bundle (an element of $x = \{x_\alpha\}_{\alpha \in I}$ the fiber of $\varkappa$ over $z \in \Omega$ corresponds under this change to $\{x_\alpha e^{-\frac{1}{\hbar}u_\alpha}\}_{\alpha \in I}$).

However $\mathscr{H}(\Omega, \mu, \varkappa)$ may "degenerate" in some way or other. (A priori it may consist of the zero section only). To avoid such pathology in Berezin [5] (the "flat" case when the metric is given by a single globally defined Kähler potential Φ) a number of assumptions labeled A–D are imposed. Here we shall concentrate on (the analogue) of hypothesis A, the remaining three are to some extent of a more technical nature. More precisely, it is (in our case) question of the following:

$$\boxed{\mathscr{K}^h_\alpha(z, z) = \text{const} \cdot e_\alpha^{-1}(z) = \text{const} \cdot e^{\frac{1}{\hbar}\Phi_\alpha(z)}},$$

where $\{\mathscr{K}_\alpha^h\}_{\alpha \in I}$ is the reproducing family associated with the space $\mathscr{H}(\Omega, \mu, \varkappa)$ (see Section 1). In other words, using the function ω (see again Section 1) and choosing the constant in the definition of the Kähler-Liouville measure μ (a different choice for each h) we are requiring that

$$\boxed{\omega(z) \equiv 1}\,.$$

REMARK. It is of interest to note that a similar assumption appears also in the theory of *Hankel forms*. It is known as Hypothesis (V) ("V" for Swedish "villkor" (= "condition")). There also a "weak" form of it, Hypothesis (weak-V), connected with "weak factorization", occurs (see [21], in particular Appendix 1). Thus, we may ask what weak factorization does mean for quantization.

Having made such assumptions Berezin [5] can then rigorously establish the "correspondence principle". Let us briefly recall what this is about.

We recall (see Section 2) that corresponding to any holomorphic line bundle $\varkappa$ there is a commutative algebra of symbols $\mathscr{B}$, the corresponding multiplication being denoted by $*$. In our present case (viz. the for the bunldle $\varkappa_h$) we write, in order to indicate the h-dependence, $\mathscr{B}_h$ and $*_h$ respectively. In suitable assumptions it can then be proved that

$$(4) \qquad A_1 *_h A_2 \to A_1 \cdot A_2 \quad (\text{for } h \to 0),$$

$$(5) \qquad \frac{1}{h}(A_1 *_h A_2 - A_2 *_h A_1) \to \frac{1}{i}[A_1, A_2]_P \quad (\text{for } h \to 0),$$

where in the first formula the dot $\cdot$ serves to indicate ordinary multiplication of functions and in the last formula $[\,,\,]_P$ stands for the Poisson bracket. That is, in the limit we recover indeed the symplectic (Kählerian) structure.

A better interpretation of these relations goes as follows. Consider families $A^* = \{A(h, \cdot)\}$ labeled by the parameter h, running through all "admissible" values (cf. infra Section 4), and consisting of functions (symbols) depending on h, with $A(h, \cdot) \in \mathscr{B}_h$ for each fixed admissible h, and admitting an asymptotic expansion of the type

$$A(h, z) \sim A(0, z) + \bar{h}B(z) + \bar{h}^2C(z) + \dots\ .$$

The set of all such families A^* forms a associative algebra $\mathscr{B}^*$ if we set

$$(A_1^* * A_2^*)(h, \cdot) = A_1(h, \cdot) *_h A_2(h, \cdot).$$

Then a more exact version of (4) reads

$$(4') \qquad (A_1^* * A_2^*)(h, z) = A_1(0, z) \cdot A_2(0, z) + o(1)$$

and, similarly, a better version of (5), reads

$$(5') \qquad \frac{1}{h}(A_1^* * A_2^* - A_2^* * A_1^*)(h, z) = \frac{1}{i}[A_1, A_2](0, z) + o(1).$$

See [5], p. 1136 for more details. Berezin does not discuss higher order terms, but this might be of some interest. (This is a question of vital importance in the theory of pseudodifferential operators and to which Unterberger, in his approach to quantization, pays a great importance (cf. Section 6).)

4. EXAMPLES OF QUANTIZED MANIFOLDS

It is high time that we give some more concrete examples to the theory developed in the previous two sections.

EXAMPLE 1. $\Omega = \mathbf{C}^n$ with the canonical flat Hermitean structure, i.e. $\mathbf{C}^n$ comes with the Hermitean form $z^1\bar{z}^1 + \ldots + z^n\bar{z}^n$. As (local) coordinates we can take any linear coordinates obtained from the "identical" ones upon application of an arbitrary unitary transformation. In other words, we can let the index set I be the unitary group $\mathrm{U}(n)$, all the U_α's being equal to $\mathbf{C}^n$ as sets. Notice that in all of these coordinates we can use one and the same Kähler potential, namely

$$\Phi = \log(z^1\bar{z}^1 + \ldots + z^n\bar{z}^n).$$

The spaces $\mathscr{H}^2(\Omega, \mu, \varkappa_h)$ are then all canonically isomorphic and reduce to the famous Fock space.[3] (The latter consists of all entire functions square integrable with respect to a Gaussian measure.) This is a unique situation where everybody agrees upon what quantization is (quantization of a flat symplectic manifold). Let us remark that covariant and contravariant symbols in this case are intimately linked to the *Wick* and *anti-Wick calculus* respectively (cf. [9], [10]). This is, historically speaking, the origin of the entire theory.

EXAMPLE 1'. It is also of interest to consider [5], [7] quotients of a vector space, $\mathbf{C}^n$ modulo a discrete subgroup. Typical examples are the quantized cylinder and the quantized torus. This connects our subject with the theory of *theta functions* (cf. [20], [26]).

EXAMPLE 2. $\Omega = \mathbf{D}$ = the unit disk in the complex plane $\mathbf{C}$. In this case there is a canonical choice for the Kähler metric and the Kähler potential, namely the Poincaré metric

$$(1) \qquad ds^2 = \frac{dz d\bar{z}}{(1 - z\bar{z})^2}$$

[3] Perhaps "Segal space" might have been more appropriate.

corresponding to the potential

(2) $$\Phi(z) = -\log(1 - z\bar{z}).$$

The Kähler-Liouville measure obviously coincides with the Poincaré measure. We can take the index set I to consist of one single element. Again no worry about coverings! But this makes the definition of a special choice of (local) coordinate so a slightly more sophisticated approach is the following, similar to the one in the flat case (Example 1). The corresponding automorphism group is the projective group $G = \mathrm{PSU}(1, 1)$ of all conformal maps of $\mathbf{D}$ onto itself. Its elements are fractional linear transformations ξ of the form $\xi(z) = \dfrac{az + b}{\bar{b}z + \bar{a}}$ where $|a|^2 - |b|^2 = 1$. However, as this group is not simply connected, it is better to work with its universal cover $\tilde{G}$. The elements of $\tilde{G}$ are ordered pairs $\tilde{\xi} = (\xi, l)$, the first member ξ being a transformation $\xi = \xi(z)$ in G and the second one a determination $l = l(z)$ of the analytic function $\log \dfrac{1}{\sqrt{\xi'(z)}}$ $(= \log(\bar{b}z + \bar{a}))$. We can now take as I the set $\tilde{G}$. If $\tilde{\xi}_1$, $\tilde{\xi}_2$ are any two members, of $\tilde{G}$, then the corresponding transition function is precisely this determination of $\log \dfrac{1}{\sqrt{\xi'(z)}}$ for $\xi = \xi_2\xi_1^{-1}$. It is easy to identify the Hilbert space $\mathscr{H}(\Omega, \mu, \varkappa) = \mathscr{H}^2(\Omega, \mu, \varkappa_h)$ in this case. It is just the Drzhbashyan (or weighted Bergman) space: its elements can be identified as holomorphic functions square integrable with respect to the density $\left(\dfrac{1}{\bar{h}} - 1\right)(1 - \bar{z}z)^{\frac{1}{\bar{h}} - 2}$ (with respect to normalized Euclidean area measure on $\mathbf{D}$, $\bar{h} = h/2\pi$). Thus, one could a priori think that only numbers h with $\bar{h} < 1$ were "admissible". However, if one interprets the integral defining the norm in a generalized sense (analytic continuation of the parameter $\dfrac{1}{\bar{h}}$; this is the method of Marcel Riesz), one sees that any $h > 0$ will do. Thus in this case the set of all admissible values of Planck's constant is the full interval $(0, \infty)$.

EXAMPLE 3. $\Omega = \mathbf{S}^2 = \mathbf{P}^1 =$ Riemann sphere. This is the dual symmetric space (the compact version of the hyperbolic space in Example 2). We know already what the holomorphic line bundles on $\mathbf{P}^1$ are (see Example in Section 1), and a natural Kähler structure is obtained by inducing the Euclidean metric in the standard imbedding of $\mathbf{S}^2$ in $\mathbf{R}^3$. The transition to $\mathbf{P}^1$ is obtained by the standard stereographic transformation. The metric and the potential are obtained formally just by changing sign in the formulae (1) and (2):

(1′) $$ds^2 = \frac{dzd\bar{z}}{(1 + z\bar{z})^2}$$

and

$$\Phi(z) = \log(1 + z\bar{z}). \tag{2'}$$

Using the "inhomogeneous" parameter z we can then realize holomorphic sections as *polynomials* in z. More precisely, the requirement of "admissibility" on Planck's constant h is now that the number $1/h$ should be a positive integer, $1/h = 1, 2, 3, \ldots$, and the Hilbert space $\mathscr{H}(\Omega, \mu, \varkappa_h)$ can be identified with the space of polynomials of degree less or equal to this integer with the metric

$$\|f\|^2 = \left(\frac{1}{h} + 1\right)\int_{\mathbf{C}} |f(z)|^2(1 + z\bar{z})^{-\frac{1}{h}-2} \, dx dy/\pi.$$

It is thus in particular finite dimensional of dimension $\frac{1}{h} + 1$. (By Liouville's theorem such an integral can be convergent only if f is a polynomial). In Berezin's own treatment [5], [7] this author is obliged to work with the affine line, the projective line $\mathbf{P}^1$ with one point removed, that is, the plane $\mathbf{C}$. This makes the presentation look somewhat artificial.

It is clear that the considerations of the above Examples 2 and 3 generalize to arbitrary Hermitean symmetric spaces, that is, both to symmetric domains ("Cartan domains") and their compact counterparts. The problem of admissibility of Planck's constant in this case was taken up by Berezin in [6] and then solved completely by Gindikin [17], in fact in the more general context of homogeneous Siegel domains. It is – in the case of symmetric domains – essentially question of deciding which powers of the Bergman kernel are positive definite. In [38], [39] one finds a formulation of the result in terms of Jordan theoretic invariants. The same question is also encountered in the theory of group representations (see e.g. [28]) and then the range of admissible values is known as the *Wallach set*. See further [23], [24], [16].

5. EXPRESSING THE BEREZIN TRANSFORM IN TERMS OF THE LAPLACIAN

In the case of a symmetric domain (space) it is clear that the Berezin transform intertwines with the group actions. Therefore, by general principles, we expect it to be a function of "Laplace operators". At least in the rank one case, it was shown by Berezin [6] (see also [7]) at the hand of concrete examples that this was indeed the case. As we have one little point to add, let us briefly review these results. Thus we return to the examples of the previous section.

EXAMPLE 1. The reproducing kernel in Fock space equals $e^{\frac{1}{h}(z,w)}$ where $(\cdot,\cdot)$ equals the standard Hermitean inner product in $\mathbf{C}^n$, the corresponding metric being denoted by $|\cdot|$. Therefore the "Berezin kernel" J (see Section 2) equals the "Gauss-Weierstrass kernel" $e^{-\frac{1}{h}|z-w|^2}$. That is, we can symbolically write

$$\mathrm{Ber}_h = e^{-h\Delta}, \tag{1}$$

where Δ is the standard Laplace operator in $\mathbf{C}^n$.

EXAMPLE 2. A generalization of the formula (1) to the case of the disk $\mathbf{D}$ can be found in [5], [7]: Ber_h can in this case be expressed as an infinite product of the "invariant Laplacian", again written Δ,

$$\Delta = (1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

The proof depends on the formula

$$\mathrm{Ber}_h = \left(1 - \frac{\Delta}{\frac{1}{h} \frac{1}{h} - 1} \right)^{-1} \mathrm{Ber}_{h'},$$

where h' corresponds to the transition $\frac{1}{h} \mapsto \frac{1}{h} + 1$. It is convenient to write here $\alpha = \frac{1}{h} - 2$. (This is the parameter often used in a Hankel context. To $h > 0$ then corresponds $\alpha > -2$.) Iterating one finds, as a generalization of (1),

$$\mathrm{Ber}_h = \frac{\prod_{k=1}^{\infty} (\alpha + k) \prod_{k=1}^{\infty} (\alpha + 1 + k)}{\prod_{k=1}^{\infty} \prod_{\pm} \left(\alpha + \frac{1}{2} + k \pm A \right)}, \tag{2}$$

with $A = \sqrt{-\Delta - \frac{1}{4}}$. Invoking Euler's gamma function, especially the well-known approximation

$$\Gamma(x + a) \sim x^a \Gamma(x), \quad x \to \infty,$$

in turn an easy consequence of Stirling's formula, one can write this equivalently as

$$\mathrm{Ber}_h = \frac{\left| \Gamma \left(\alpha + \frac{3}{2} + iA \right) \right|^2}{\Gamma(\alpha + 1)(\Gamma(\alpha + 2)}. \tag{3}$$

Using further the convexity properties of the gamma function it follows that

$$0 < \mathrm{Ber}_h < c < 1, \tag{4}$$

in the sense of self-adjoint operators in Hilbert space. This was proved in [2], where (3) was established independently without knowledge of Berezin's work. (Berezin himself [5] proves that, in very general assumptions,

$$\|\mathrm{Ber}\| \leqslant 1,$$

that is, the operator Ber is a contraction).

Formula (4) has an application to big Toeplitz operators with non-analytic symbols [2], which we now briefly recall. Denoting the operation of multiplication by a function B by M_B and letting P denote the "Dzhrbashyan projection", we define (see [1], [2], [3]) the big Hankel operator with symbol B as the commutator $H_B = [P, M_B]$. As is shown in [2], there are essentially three types of symbols: 1° analytic symbols, 2° antianalytic symbols, 3° symbols which "vanish at the boundary". That is, more precisely, the space of big Hankel operators of Hilbert-Schmidt class splits up into a direct sum of three spaces corresponding to symbols of each of these three types. Let us confine attention to symbols of the last (third) type only. Then one can prove, with practically "no" assumptions, that

$$\|H_B\|^2_{\mathrm{H.S.}} = 2(\|B\|^2_\iota - \|\mathrm{Ber}(B)\|^2_\iota),$$

where $\|\cdot\|_{\mathrm{H.S.}}$ stands for the Hilbert-Schmidt norm of an operator and $\|\cdot\|_\iota$ for the "invariant" norm of a function (cf. Section 1). Thus if a formula such as (4) holds true one can always draw the conclusion that *the Hilbert-Schmidt norm of the big Hankel operator H_B is equivalent to the invariant norm of its symbol B.* In other words the last component in the orthogonal expansion can be identified with the Hilbert space $L^2(\Omega, \iota)$. If $\Omega = \mathbf{D}$ such a conclusion, thus, was made in [2].

EXAMPLE 2′. Formula (2) can be extended to the case of the unit ball in $\mathbf{C}^n$ [6], p. 377. As in the case of the disk (Example 2) the result can also be expressed, by the same token, in terms of gamma factors. Namely, we have the result

$$\mathrm{Ber}_h = \frac{\left|\Gamma\left(\alpha + \frac{n}{2} + 1 + i\Lambda\right)\right|^2}{\Gamma(\alpha + 1)\Gamma(\alpha + n + 1)}, \tag{3′}$$

where this time $\frac{1}{h} = \alpha + n + 1$, $\Lambda = \sqrt{-\Delta - \left(\frac{n}{2}\right)^2}$. In particular, the inequalities (4) are valid in this case. Therefore, we can draw the conclusion that *the theorem for big Hankel operators over a disk described in the last paragraph* *extends to the case of the ball.* Thus, it seems to be of utmost importance to find a concrete expression the Berezin transform in terms of "Laplace operators" also in the case symmetric domains of higher rank.

6. OTHER APPROACHES TO QUANTIZATION

First, there is something known as "geometric quantization" associated with the names Soriau, Kirillov, Kostant, Sternberg etc., see e.g. the book [40]. In default of sufficient knowledge of this topic we must refrain from a detailed comparison of this subject with what is set out in the present paper.

Let us, however, say a few words about A. Unterberger's quantization program (see [31 – 37] and other works quoted there; the basic underlying idea can however be found already in [7], p. 169). Roughly speaking, it is question of the following. Consider any (Riemannian) symmetric space Ω (that we have a complex manifold is at present not essential anymore) and let W be a unitary representation of the group $G = \mathrm{Aut}(\Omega)$ in some Hilbert space $\mathscr{H}$. Then we can associate with any function (symbol) $\overset{1/2}{A}$ defined on Ω a linear operator $W(\overset{1/2}{A})$ on $\mathscr{H}$ given by the formula

$$W(\overset{1/2}{A}) = \int_{\Omega} W(s_z)\overset{1/2}{A}(z)\mathrm{d}\imath(z).$$

Here s_z stands for the symmetry about the point $z \in \Omega$, an element of G. This may be conceived as a generalization of the well-known *Weyl calculus*, to which it reduces in the case $G = \mathbf{C}^n$ (see Example 1 in Section 5). In this case the relation between the three kinds of symbols A, $\overset{1/2}{A}$ and $\overset{\circ}{A}$ is expressed by the formulae

$$A = \mathrm{e}^{-\frac{1}{2}\hbar\Delta}\overset{1/2}{A}, \quad \overset{1/2}{A} = \mathrm{e}^{-\frac{1}{2}\hbar\Delta}\overset{\circ}{A}.$$

This explains the strange notation $\overset{1/2}{A}$ for this "intermediate" symbol. Comparing with (1) in Section 5 we thus see that the Weyl calculus in a way sits "half way" between the two Berezin calculi. That much about the "flat" case. In other cases this relationship is not so simple. See e.g. [35], the case $\Omega = \mathbf{D}$, [37], the case of a Lie ball.

Leaving Unterberger's theory, let us now mention two entirely different directions in which the theory in the case of disk or a ball can, potentially, be extended.

One possibility consists of taking *strictly pseudo-convex domains* Ω. In fact, this is already in passing mentioned by Berezin in [5], p. 1141. Hypothesis A is not fulfilled in general so his idea is to use directly powers $\mathscr{K}^{\frac{1}{\hbar}}$ of the Bergman kernel $\mathscr{K}$ of Ω to define the required Hilbert spaces. For a related approach in a Hankel context, see [22].

Second, sticking to the case of one complex dimesion ($n = 1$), we may instead consider, as a generalization of the unit disk, *multiply connected plane domains* (or, more generally, bounded "open" Riemann surfaces) Ω, the boundary consisting of finitely many smooth curves. Representing Ω as a quotient of the disk $\mathbf{D}$, by virtue of the uniformization theorem, and pulling back the Poincaré metric on $\mathbf{D}$ to Ω, we obtain an Hermitean metric $ds = |dz|/\omega(z)$ of constant negative curvature on Ω. We then obtain using the recipe of Section 3 Hilbert spaces of holomorphic functions corresponding to the conformally invariant integral norm

$$\|f\|^2 = \int_\Omega |f(z)|^2(\omega(z))^\alpha dxdy,$$

with as before $\alpha = \frac{1}{h} - 2$. (In the context of automorphic forms, this is known as the Peterson metric.) The study Hankel forms over such a space was begun in [19]. In particular, it is conceivable that one should have an analogue of the correspondence principle (see Section 3) in this case, although hypothesis A is not fulfilled (unless Ω is simply connected, genus zero). The case of genus one (an "annulus") should be susceptible to a very explicit treatment, as then all relevant data can be expressed in terms of *elliptic functions* (cf. [27]). We intend to return to this situation in a future publication.

7. CONTENTIONS

Concluding, let us summarize some of the points made in the preceding compilation as follows.

1° Hermitean line bundles are of interest in connection with Berezin quantization, as well as in a Ha-plitz context.

2° A quantization procedure can be based on the use of Kähler structures. This, optimally, leads to a "correspondence principle", the ultimate test for the whole theory.

3° The Berezin transform is an interesting object of study, irrespective of the present context. In particular, one has the problem of expressing it in terms of "Laplace operators".

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