J. OPERATOR THEORY
24(1990), 187–206

ON THE SUMMABILITY OF THE SPECTRAL SHIFT FUNCTION FOR PAIR OF CONTRACTIONS AND DISSIPATIVE OPERATORS

V. M. ADAMJAN and H. NEIDHARDT

1. INTRODUCTION

The spectral shift function can be regarded as one of the fundamental notions of the perturbation theory. This function, originally arisen in the physical literature, attracted attention of mathematicians after the pioneering work by I. M. Lifshitz [20]. The mathematical theory of this function was performed by M. G. Krein [12–14], who has introduced the name of this function in the literature. Let $\{H, H_0\}$ be a pair of self-adjoint operators on a separable Hilbert space $\mathfrak{H}$ which are different by a nuclear operator. In [12] M. G. Krein has proved the existence of a summable real function $\xi(\cdot)$ defined on $\mathbf{R}^1$ such that for a certain class of functions $\psi(\cdot)$ the relation

$$\mathrm{tr}\{\psi(H) - \psi(H_0)\} = \int_{-\infty}^{+\infty} \xi(\lambda)\psi'(\lambda)\mathrm{d}\lambda \tag{1.1}$$

holds. The function $\xi(\cdot)$ was called the spectral shift function of the pair $\{H, H_0\}$; and the relation itself, the trace formula. In [13] this result was extended to a pair of unitary operators $\{U, U_0\}$ which differ by a nuclear operator. It was shown that again for a suitable class of functions $\varphi(\cdot)$ there is a summable real function $\mu(\cdot)$ defined on $[-\pi, \pi)$ such that the trace formula

$$\mathrm{tr}\{\varphi(U) - \varphi(U_0)\} = \int_{-\pi}^{\pi} \mu(\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta \tag{1.2}$$

is valid. In contrast with the previous case the function $\mu(\cdot)$, which is again called the spectral shift function of $\{U, U_0\}$, is determined by (1.2) up to a real constant.

Using the Cayley transforms $U = \frac{iI + H}{iI - H}$ and $U_0 = \frac{iI + H_0}{iI - H_0}$ in [13], M. G. Krein established that (1.2) results in the relation (1.1) assuming that the resolvent difference of the self-adjoint operators H and H_0 is nuclear. But the generalization is accompanied by weakening of the summability of the spectral shift function because in this case $\xi(\cdot)$ is summable only with the weight $(1 + \lambda^2)^{-1}$ on $\mathbf{R}^1$, i.e. $\xi(\cdot) \in L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}d\lambda)$ instead of $\xi(\cdot) \in L^1(\mathbf{R}^1, d\lambda)$. Further investigations of the spectral shift function were carried out in [14] by M. G. Krein. Up to now there is a lot of papers on this subject in which detailed information on this function can be found [7], [17], [15], [24]. However, it must be remarked that in most of the papers either a pair of self-adjoint operators or a pair of unitary operators is considered. To include other classes of operators in the investigation of the spectral shift function is a rather new branch attracting more and more attention of mathematicians in the last years.

One of the first going in this direction was H. Langer [19]. A further attempt was made by L. A. Sahnovič [28]. R. V. Akopjan [2], [3], [4] tried to extend the concept of spectral shift function to self-adjoint operators on Krein spaces. In [2] he found a pair of non-negative bounded self-adjoint operators on a Krein space differing by a nuclear operator so that a summable spectral shift function did not exist. In [10], [11] P. Jonas was able to extend the trace formula to a broad class of operators generalizing the spectral shift function to a distribution.

In 1979, V. M. Adamjan and B. S. Pavlov [1] made a start to extend the trace formula to the class of maximal dissipative operators. In the following this development was continued by A. V. Rybkin [25–27]. Recently, a paper by M. G. Krein [16] has appeared which was devoted to the same problem. Attention to this problem was also paid by H. Neidhardt in [21–23]. In [21], [22] the pair $\{H = H_0 + V, H_0\}$ was considered where V is a nuclear dissipative operator. It was found that a trace formula

$$\mathrm{tr}\{(H - z)^{-1} - (H_0 - z)^{-1}\} = -\int_{-\infty}^{+\infty} \xi(\lambda)(\lambda - z)^{-2}d\lambda, \tag{1.3}$$

$\mathrm{Im}(z) > 0$, makes sense if the nuclearity of V is strengthened, for instance, if $\mathrm{Im}(V) \in \mathfrak{L}_{1/2}(\mathfrak{H})$. Here and in the following $\mathfrak{L}_p(\mathfrak{H})$, $0 < p < +\infty$, denotes the class of compact operators on $\mathfrak{H}$ whose elements are characterized by $\sum_{k=1}^{+\infty} s_k(A)^p < +\infty$ where $\{s_k(A)\}_{k=1}^{\infty}$ denotes the set of singular numbers of the compact operator A. Despite the stronger perturbation it was found that in contrast with [12] the real spectral shift function $\xi(\cdot)$ belongs only to $L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}d\lambda)$ and hence $\xi(\cdot)$ is determined up to a constant by (1.3). Furthermore, the representation

$$\xi(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \to +0} \operatorname{Im} \log \det(I + V(H_0 - \lambda - i\varepsilon)^{-1}) + \text{const.} \tag{1.4}$$

was proved for a.e. $\lambda \in \mathbf{R}^1$ (with respect to the Lebesgue measure $|\cdot|$ on $\mathbf{R}^1$) in [21], [22]. However, as it has been pointed out in [22], the right-hand side of (1.4) makes sense assuming only that V is a nuclear dissipative operator. Thus, the possibility arises to define a spectral shift function of $\{H, H_0\}$ forgetting the trace formula (1.3). But doing so, it has been found in [22, Example 3.10] that a spectral shift function thus defined is in general not locally summable (compare with [2]). However, the arbitrariness of the spectral shift function is limited by the condition that $\xi(\cdot)$ belongs to the class of weakly summable functions, i.e. $\xi(\cdot) \in L^1_w(\mathbf{R}^1, d\lambda)$. We recall that a measurable function $f(\cdot)$ on $\mathbf{R}^1$ belongs to $L^1_w(\mathbf{R}^1, d\lambda)$ if its distribution function $m_f(s) = |\{\lambda \in \mathbf{R}^1 : |f(\lambda)| > s > 0\}|$ obeys the condition $\sup_{s>0}(s\, m_f(s)) < +\infty$. Obviously, the inclusion $L^1(\mathbf{R}^1, d\lambda) \subseteq L^1_w(\mathbf{R}^1, d\lambda)$ holds. It is well-known that functions of $L^1_w(\mathbf{R}^1, d\lambda)$ allow singularities of the type $(\lambda - \lambda_0)^{-1}$.

As it was pointed out in [23], these results allow an extension to a pair of contractions $\{T, T_0\}$ defined on $\mathfrak{H}$. Denoting by $\mathfrak{G}_\Pi$ the set of all functions on $\Pi = \{z \in \mathbf{C}^1 : |z| = 1\}$ derivation of which possesses absolutely convergent Fourier series and introducing the decomposition

$$\varphi(e^{it}) = \sum_{l=-\infty}^{+\infty} a_l e^{ilt} = \varphi_+(e^{it}) + \varphi_-(e^{-it}), \tag{1.5}$$

$$\varphi_+(e^{it}) = \sum_{l=0}^{+\infty} a_l e^{ilt}, \tag{1.6}$$

$$\varphi_-(e^{it}) = \sum_{m=1}^{+\infty} a_{-m} e^{imt}, \tag{1.7}$$

$t \in [-\pi, \pi)$, $\varphi(\cdot) \in \mathfrak{G}_\Pi$, the existence of a summable spectral shift function $\mu(\cdot) \in L^1(-\pi, \pi)$, such that the trace formula

$$\operatorname{tr}\{\varphi_+(T) + \varphi_-(T^*) - \varphi_+(T_0) - \varphi_-(T_0^*)\} = \int_{-\pi}^{\pi} \mu(\theta) \frac{d}{d\theta} \varphi(e^{i\theta}) d\theta, \tag{1.8}$$

holds, can be proved under the condition

$$T - T_0 \in \mathfrak{L}_1(\mathfrak{H}) \tag{1.9}$$

and the nuclearity of the defect operators D_T, D_{T^*}, D_{T_0}, $D_{T_0^*}$ [29].

An attempt to avoid the additional conditions $D_T^2, D_{T^*}^2, D_{T_0}^2, D_{T_0^*}^2 \in \mathfrak{L}_{1/2}(\mathfrak{H})$ was made in the same paper. Assuming only (1.9), a solution of this problem was obtained by using the concept of an integrated spectral shift function and a certain modification of the trace formula (1.8).

In the following, we are interested in the original form (1.8) of the trace formula. To this end we introduce the ideal $\mathfrak{L}_1^0(\mathfrak{H})$ of the algebra of all bounded operators $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{H}$ consisting of all compact operators A on $\mathfrak{H}$ singular number $\{s_j(A)\}_{j=1}^{\infty}$ of which satisfy the condition

$$(1.10) \qquad \sum_{j=1}^{\infty} s_j(A) \log^+ \frac{1}{s_j(A)} < +\infty.$$

Obviously, we have the inclusions

$$(1.11) \qquad \mathfrak{L}_r(\mathfrak{H}) \subset \mathfrak{L}_1^0(\mathfrak{H}) \subset \mathfrak{L}_1(\mathfrak{H}), \quad 0 < r < 1.$$

Assuming in addition to (1.9) $D_T^2, D_{T_0^*}^2 \in \mathfrak{L}_1^0(\mathfrak{H})$ we establish that there is a summable spectral shift function $\mu(\cdot)$ such that the trace formula (1.8) is valid. Via the Cayley transform we carry over the result to maximal dissipative operators. At the end, we give an application of the results to the dissipative Schrödinger operator on $L^2(\mathbf{R}^n, dx)$, $n = 1, 2, 3$, showing that the usual falling off condition $|q(x)| \leqslant \text{const.}(1 + |x|)^{-(n+\varepsilon)}$, $n = 1, 2, 3$, of the potential $q(\cdot)$ guaranteeing the nuclearity of the resolvent difference of the dissipative Schrödinger operator and the Laplace operator is strong enough to guarantee also the existence of a summable spectral shift function of the class $L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}d\lambda)$. We remark that because of [23], a natural modification of the Birman-Krein formula [6] is valid in these cases provided T_0 is a unitary operator or the unperturbed operator is a self-adjoint one.

2. TRACE FORMULA

Let $\{T, T_0\}$ be a pair of contractions on the separable Hilbert space $\mathfrak{H}$ obeying (1.9). In [23, Theorem 2.5] it was shown that the limit

$$(2.1) \qquad \mu(\theta) = -\lim_{r \to 1+0} \frac{1}{\pi} \operatorname{Im} \log \det(I + (T - T_0)(T_0 - re^{i\theta})^{-1})$$

exists for a.e. $\theta \in [-\pi, \pi)$, where a branch of the logarithm was fixed by the condition $\lim_{|z| \to +\infty} \log \det(I + (T - T_0)(T_0 - z)^{-1}) = 0$. In the following, we give a new proof of this fact under a slightly stronger condition working out in such a way further information about the limit (2.1) which will be useful in the sequel.

PROPOSITION 2.1. *If the pair of contractions* $\{T, T_0\}$ *obeys*

$$I - T^*T_0 \in \mathfrak{L}_1(\mathfrak{H}) \tag{2.2}$$

and

$$I - TT_0^* \in \mathfrak{L}_1(\mathfrak{H}), \tag{2.3}$$

then the limit (2.1) *exists for* a.e. $\theta \in [-\pi, \pi)$ *and* $\mu(\cdot)$ *belongs to* $L^1_w(-\pi, \pi)$.

Proof. First of all we note that the conditions (2.2) and (2.3) allow one to divide the existence problem into three independent problems with a special structure. Since

$$\begin{gathered} I - T^*T_0 + I - T_0^*T = \\ = I - T^*T + I - T_0^*T_0 + (T^* - T_0^*)(T - T_0) \in \mathfrak{L}_1(\mathfrak{H}) \end{gathered} \tag{2.4}$$

we obtain $I - T^*T \in \mathfrak{L}_1(\mathfrak{H})$ and $I - T_0^*T_0 \in \mathfrak{L}_1(\mathfrak{H})$. Considering the polar decompositions $T = V|T|$, $|T| = (T^*T)^{1/2}$, and $T_0 - V_0|T_0|$, $|T_0| = (T_0^*T_0)^{1/2}$, we find

$$I - |T| \in \mathfrak{L}_1(\mathfrak{H}), \tag{2.5}$$

$$I - |T_0| \in \mathfrak{L}_1(\mathfrak{H}) \tag{2.6}$$

and

$$I - V^*V_0 \in \mathfrak{L}_1(\mathfrak{H}). \tag{2.7}$$

Using (2.3), (2.5) and (2.6) we get

$$I - VV_0^* \in \mathfrak{L}_1(\mathfrak{H}). \tag{2.8}$$

But (2.7) and (2.8) imply the nuclearity of the defect operators $D_V, D_{V^*}, D_{V_0}, D_{V_0^*}$ which are projections in this special case. Obviously, we have

$$V - V_0 \in \mathfrak{L}_1(\mathfrak{H}). \tag{2.9}$$

We consider the pairs $\{T, V\}$, $\{V, V_0\}$ and $\{T_0, V_0\}$.

1. Let $\{e_j\}_{j=1}^N$ be the orthonormal systems of eigenvectors of the operator $I - |T|$ corresponding to the nonzero eigenvalues $\{\lambda_j\}_{j=1}^N$, $N \leqslant +\infty$, such that

$$I - |T| = \sum_{j=1}^{N} \lambda_j(\cdot, e_j)e_j. \tag{2.10}$$

In order to extend the results pointed out in the introduction it is only sufficient to consider the case $N = +\infty$. With respect to the pair $\{T, V\}$ we set

(2.11)
$$\begin{cases} Q_0 = V, & n = 0, \\ Q_n = V - \sum_{j=1}^{n} \lambda_j(\cdot, e_j)Ve_j, & n = 1,2,\dots . \end{cases}$$

Since $I - |T| \in \mathfrak{L}_1(\mathfrak{H})$, i.e.

$$\sum_{j=1}^{\infty} \lambda_j = \mathrm{tr}\{I - |T|\} = \|I - |T|\|_1 < +\infty,$$

the sequence of contractions $\{Q_n\}_{n=0}^{\infty}$ converges to T in the trace norm. In particular, we obtain

(2.12)
$$\det(I + (T - V)(V - z)^{-1}) = \lim_{n\to+\infty} \det(I + (Q_n - V)(V - z)^{-1}),$$

$|z| > 1$. Taking into account the multiplicative property of the perturbation determinant we get

(2.13)
$$\det(I + (Q_n - V)(V - z)^{-1}) = \prod_{l=1}^{n} \det(I + (Q_l - Q_{l-1})(Q_{l-1} - z)^{-1}),$$

$|z| > 1$. By $(e_l, e_j) = 0$, $j = 1, 2, \dots, l - 1$, and (2.11) we find $Q_{l-1}e_l = Ve_l$ which yields

(2.14)
$$\det(I + (Q_l - Q_{l-1})(Q_{l-1} - z)^{-1}) = 1 - \lambda_l((Q_{l-1} - z)^{-1}Q_{l-1}e_l, e_l) =$$
$$= \left(1 - \frac{1}{2}\lambda_l\right)\left[1 + \frac{\lambda_l}{2 - \lambda_l}\left(\left(I + \frac{1}{z}Q_{l-1}\right)\left(I - \frac{1}{z}Q_{l-1}\right)^{-1}e_l, e_l\right)\right],$$

$|z| > 1$. Let

$$\mu_l = \frac{\lambda_l}{2 - \lambda_l}, \quad l = 1, 2, \dots .$$

Obviously, we have $0 < \mu_l \leqslant 1$, $0 < \mu_l \leqslant \lambda_l \leqslant 1$, $l = 1, 2, \dots,$ and

(2.15)
$$\sum_{l=1}^{\infty} \mu_l \leqslant \sum_{l=1}^{\infty} \lambda_l < +\infty.$$

We set

$$X_l(z) = ((I + zQ_{l-1})(I - zQ_{l-1})^{-1}e_l, e_l), \quad |z| < 1.$$

Since Q_{l-1} is a contraction and e_l is a normalized vector $X_l(z)$ obeys the properties

$$|X_l(z)| \leqslant \frac{1 + |z|}{1 - |z|}, \quad |z| < 1, \tag{2.16a}$$

$$\mathrm{Re}(X_l(z)) \geqslant 0, \quad |z| < 1, \tag{2.16b}$$

$$X_l(0) = 1. \tag{2.16c}$$

Moreover, by (2.12)–(2.14) we obtain the representation

$$\det(I + (T - V)(V - z)^{-1}) = CF(1/z), \quad |z| > 1, \tag{2.17}$$

introducing the quantities

$$C = \prod_{l=1}^{\infty}\left(1 - \frac{1}{2}\lambda_l\right) = \det\left(\frac{1}{2}(I + |T|)\right)$$

and

$$F(z) = \prod_{l=1}^{\infty}(1 + \mu_l X_l(z)), \quad |z| < 1. \tag{2.18}$$

Notice that by (2.15) and (2.16a) the last infinite product converges absolutely and uniformly in every disk $|z| < r < 1$. On account of $\mathrm{Re}(1 + \mu_l X_l(z)) \geqslant 1$ and $1 + \mu_l X_l(0) = 1 + \mu_l > 1$ the function $1 + \mu_l X_l(z)$ is an outer one [9] and, hence, the representation

$$\log(1 + \mu_l X_l(z)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log|1 + \mu_l X(e^{it})| dt \tag{2.19}$$

takes place, where the branch of the logarithm is fixed by the condition $\mathrm{Im}\log(1 + \mu_l X_l(0)) = 0$. From (2.18) and (2.19) and the inequality

$$\log|1 + \mu_l X_l(e^{it})| \geqslant \log \mathrm{Re}(1 + \mu_l X_l(e^{it})) \geqslant 0$$

we derive the representation

$$\log(F(z)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} h(t) dt, \tag{2.20}$$

where $h(\cdot) \in L^1(-\pi, \pi)$ is given by

$$h(t) = \sum_{l=1}^{\infty} \log|1 + \mu_l X_l(e^{it})| \geqslant 0 \tag{2.21}$$

for a.e. $t \in [-\pi, \pi)$. Since $h(\cdot) \in L^1(-\pi, \pi)$ the limit

$$\lim_{r\to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2r\sin(\theta - t)}{1 + r^2 - 2r\cos(\theta - t)}\, h(-t)\mathrm{d}t$$

exists for a.e. $\theta \in [-\pi, \pi)$ [18, III C 1° and 2°]. Hence, we obtain the existence of the limit $\mu_{\{T,V\}}(\theta)$,

$$\begin{aligned}\mu_{\{T,V\}}(\theta) &= -\lim_{r\to 1-0} \frac{1}{\pi} \operatorname{Im}\log\det(I + (T - V)(V - r^{-1}e^{i\theta})^{-1}) = \\ &= \lim_{r\to 1-0} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \frac{2r\sin(\theta - t)}{1 + r^2 - 2r\cos(\theta - t)}\, h(-t)\mathrm{d}t\end{aligned} \tag{2.22}$$

for a.e. $\theta \in [-\pi, \pi)$. Moreover, taking into account the Kolmogorov Theorem [18, V C 1°], we find that the distribution function $m_{\mu_{\{T,V\}}}(\lambda) = |\{\theta \in [-\pi, \pi) : |\mu_{\{T,V\}}(\theta)| \geqslant \lambda\}|$ can be estimated by

$$m_{\mu_{\{T,V\}}}(\lambda) \leqslant \frac{K}{\lambda} \|\log|F|\|_{L^1}. \tag{2.13}$$

But (2.23) implies $\mu_{\{T,V\}}(\cdot) \in L^1_w(-\pi, \pi)$.

2. Similarly, prescribing zero-index to the quantities connected with the pair $\{T_0, V_0\}$ we establish the existence of $\mu_{\{T_0,V_0\}}(\theta)$ for a.e. $\theta \in [-\pi, \pi)$, the inclusion $\mu_{\{T_0,V_0\}}(\cdot) \in L^1_w(-\pi, \pi)$ and representations of type (2.20) and (2.21).

3. Concerning the pair $\{V, V_0\}$ we note that on account of $D_V, D_{V^*}, D_{V_0}, D_{V_0^*} \in \mathfrak{L}_1(\mathfrak{H})$ and (2.9) the limit $\mu_{\{V,V_0\}}(\theta)$ exists for a.e. $\theta \in [-\pi, \pi)$ and $\mu_{\{V,V_0\}}(\cdot) \in L^1(-\pi, \pi) \subseteq L^1_w(-\pi, \pi)$, as was pointed out in the introduction.

We complete the proof using the multiplication property of the perturbation determinant, i.e.

$$\begin{aligned}\det(I + (T - T_0)(T_0 - z)^{-1}) &= \det(I + (T - V)(V - z)^{-1})\cdot \\ &\cdot\det(I + (V - V_0)(V_0 - z)^{-1})(\det(I + (T_0 - V_0)(V_0 - z)^{-1}))^{-1},\end{aligned} \tag{2.24}$$

$|z| > 1$. On account of the previous steps formula (2.24) immediately yields the existence of the limit (2.1) for a.e. $\theta \in [-\pi, \pi)$. Moreover, the equality

$$\mu(\theta) = \mu_{\{T,V\}}(\theta) + \mu_{\{V,V_0\}}(\theta) - \mu_{\{T_0,V_0\}}(\theta) \tag{2.25}$$

holds for a.e. $\theta \in [-\pi, \pi)$. Obviously, from (2.25) we obtain $\mu(\cdot) \in L^1_w(-\pi, \pi)$. ▨

In accordance with [23] we call every real function on $[-\pi, \pi)$, which differs from $\mu(\cdot)$ by a constant, a spectral shift function of the pair $\{T, T_0\}$. In particular, the function $\mu(\cdot)$ itself is a spectral shift function of $\{T, T_0\}$.

It is easy to see from the proof of Proposition 2.1 that the nonsummability of the spectral shift function $\mu(\cdot)$ arises from the nonsummability of the spectral shift functions $\mu_{\{T,V\}}(\cdot)$ and $\mu_{\{T_0,V_0\}}(\cdot)$.

THEOREM 2.2. *Let V be a partial isometry and let $T = V|T|$ be a contraction such that $T^*T = |T|^2$ and $I - |T| \in \mathfrak{L}_1(\mathfrak{H})$. The pair $\{T, V\}$ possesses a summable spectral shift function $\mu_{\{T,V\}}(\cdot)$,* i.e. $\mu_{\{T,V\}} \in L^1(-\pi, \pi)$, *such that the trace formula* (1.8) *(where T_0 is replaced by V) holds for every $\varphi(\cdot) \in \mathfrak{G}_\Pi$ if and only if the condition*

$$h(t)\log^+ h(t) \in L^1(-\pi, \pi) \tag{2.26}$$

is satisfied.

Proof. We note that on account of (2.22) $\mu_{\{T,V\}}(\cdot)$ is the harmonic conjugate function of $\frac{1}{\pi} h(-t)$. Moreover, since $h(-t) > 0$ Zygmund and Riesz theorems [18, V C 3° and 4°] imply that $\mu_{\{T,V\}}(\cdot) \in L^1(-\pi, \pi)$ and (2.26) are equivalent.

In order to prove the trace formula (1.8) we introduce the function $H(\cdot)$,

$$H(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} h(t)dt, \quad |z| < 1. \tag{2.27}$$

Since $h(t)\log^+ h(t) \in L^1(-\pi, \pi)$ we get $H(\cdot) \in H^1$ [18, V C 3°]. Moreover, by (2.17) and (2.20) we have

$$H\left(\frac{1}{z}\right) = \log\det(I + (T - V)(V - z)^{-1}) - \log\det\left(\frac{1}{2}(I + |T|)\right), \tag{2.28}$$

$|z| > 1$. Now $H(z)$ can be represented by its imaginary part,

$$H(z) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \operatorname{Im} H(e^{it})dt + \operatorname{Re} H(0), \tag{2.29}$$

$|z| < 1$. Calculating $\operatorname{Im} H(e^{it})$ we find

$$\operatorname{Im} H(e^{i\theta}) = \lim_{r\to 1-0} \operatorname{Im} H(re^{i\theta}) = -\pi\mu_{\{T,V\}}(-\theta) \tag{2.30}$$

for a.e. $\theta \in [-\pi, \pi)$. Thus, the representation

$$H\left(\frac{1}{z}\right) = \frac{i}{2}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,\mu_{\{T,V\}}(t)\mathrm{d}t - \operatorname{Re} H(0), \tag{2.31}$$

$|z| > 1$, holds. Deriving (2.28) by z and using formula (1.14) of [9, IV § 1] we get

$$\operatorname{tr}\{(T-z)^{-1} - (V-z)^{-1}\} = -i\int_{-\pi}^{\pi} \frac{e^{it}}{(e^{it}-z)^2}\,\mu_{\{T,V\}}(t)\mathrm{d}t, \tag{2.32}$$

$|z| > 1$. But from (2.32) we derive the trace formula (1.8) in a standard manner. ▨

In general, the following theorem takes place.

THEOREM 2.3. *The pair of contractions $\{T, T_0\}$ obeying* (2.2) *and* (2.3) *possesses a summable spectral shift function $\mu(\cdot)$ such that the trace formula* (1.8) *is valid for every $\varphi(\cdot) \in \mathfrak{G}_\Pi$ if the condition*

$$|h(t) - h_0(t)|\log^+|h(t) - h_0(t)| \in L^1(-\pi, \pi) \tag{2.33}$$

is satisfied.

Proof. Taking into account (2.22) and a similar relation for $\mu_{\{T_0,V_0\}}(\cdot)$ we get

$$\begin{aligned} \nu(\theta) &\overset{\text{def}}{=} \mu_{\{T,V\}}(\theta) - \mu_{\{T_0,V_0\}}(\theta) = \\ &= \lim_{r\to 1-0} \frac{1}{2\pi^2}\int_{-\pi}^{\pi} \frac{2r\sin(\theta - t)}{1 + r^2 - 2r\cos(\theta - t)}\,[h(-t) - h_0(-t)]\mathrm{d}t \end{aligned} \tag{2.34}$$

for a.e. $\theta \in [-\pi, \pi)$. By the Zygmund theorem we conclude that $\nu(\cdot) \in L^1(-\pi, \pi)$. Introducing the analytic function

$$H(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,[h(t) - h_0(t)]\mathrm{d}t, \quad |z| < 1. \tag{2.35}$$

we again obtain $H(\cdot) \in H^1$ and repeating the considerations of the previous theorem we find

$$\operatorname{tr}\{(T-z)^{-1}-(T_0-z)^{-1}\} = -\mathrm{i}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-z)^2}\,\nu(t)\mathrm{d}t + \tag{2.36}$$

$$+\operatorname{tr}\{(V-z)^{-1}-(V_0-z)^{-1}\},$$

$|z|>1$, $\nu(\theta) = -\frac{1}{\pi}\operatorname{Im}(H(\mathrm{e}^{-\mathrm{i}\theta}))$, $\theta\in[-\pi,\pi)$. Since $\{V, V_0\}$ obeys the trace formula

$$\operatorname{tr}\{(V-z)^{-1}-(V_0-z)^{-1}\} = -\mathrm{i}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}t}}{(\mathrm{e}^{\mathrm{i}t}-z)^2}\,\mu_{\{V,V_0\}}(t)\mathrm{d}t, \tag{2.37}$$

$|z|>1$, we obtain (1.8) using (2.36) and (2.37) and introducing the spectral shift function $\mu(t) = \mu_{\{V,V_0\}}(t) + \nu(t)$, $t\in[-\pi,\pi)$. ▨

3. SIMPLE SUFFICIENT CONDITIONS

The assumptions of Theorem 2.2 or Theorem 2.3 are far from being simple. Naturally, the problem arises to find conditions involving only T and T_0 in a simple manner and guaranteeing the summability of the spectral shift function. The task of the present chapter will be to give a satisfactory solution of this problem.

THEOREM 3.1. *Let V be a partially isometric operator and let $T = V|T|$ be a contraction such that $T^*T = |T|^2$. If $I - |T| \in \mathfrak{L}_1^0(\mathfrak{H})$, then the pair $\{T, V\}$ possesses a summable spectral shift function such that the trace formula* (1.8) *(where T_0 is replaced by V) holds for every $\varphi(\cdot)\in\mathfrak{G}_\Pi$.*

Proof. By (2.20)–(2.22) it is sufficient to prove that $\log F(z)\in H^1$. To this end we use the representation

$$\log F(z) = \sum_{l=1}^{\infty}\log(1+\mu_l X_l(z)), \quad |z|<1. \tag{3.1}$$

We consider the harmonic functions

$$\psi_l(r\mathrm{e}^{\mathrm{i}\theta}) = \log|1+\mu_l X_l(r\mathrm{e}^{\mathrm{i}\theta})| \tag{3.2}$$

and

$$\tilde{\psi}_l(r\mathrm{e}^{\mathrm{i}\theta}) = \operatorname{Im}\log(1+\mu_l X_l(r\mathrm{e}^{\mathrm{i}\theta})), \tag{3.3}$$

$0 < r < 1$. By (2.16) we have $|1 + \mu_l X_l(re^{i\theta})| \geqslant 1$ and, hence, we find

$$|\tilde{\psi}_l(re^{i\theta})| \leqslant \frac{\pi}{2}. \tag{3.4}$$

Taking into account (2.19) we get

$$\|\psi_l\|_{L^1} = \int_{-\pi}^{\pi} |\psi_l(e^{i\theta})| d\theta = \int_{-\pi}^{\pi} \psi_l(e^{i\theta}) d\theta = 2\pi \log(1 + \mu_l). \tag{3.5}$$

But (3.4) and (3.5) imply $\psi_l + i\tilde{\psi}_l \in H^1$. Moreover, the estimate

$$\begin{aligned} \|\psi_l + i\tilde{\psi}_l\|_{L^1} &= \int_{-\pi}^{\pi} |\psi_l(e^{i\theta}) + i\tilde{\psi}_l(e^{i\theta})| d\theta \leqslant \\ &\leqslant \|\psi_l\|_{L^1} + \int_{-\pi}^{\pi} |\tilde{\psi}_l(e^{i\theta})| d\theta \end{aligned} \tag{3.6}$$

holds. Let $m_l(\cdot)$ be the distribution function of $\tilde{\psi}_l(\theta)$,

$$m_l(\lambda) = |\{\theta \in [-\pi, \pi) : |\tilde{\psi}_l(e^{i\theta})| > \lambda\}|.$$

Since $\psi_l(\theta) \geqslant 0$, the improved Kolmogorov theorem [18, V C 1°] yields the estimate

$$m_l(\lambda) \leqslant \frac{4\pi \|\psi_l\|_{L^1}}{\|\psi_l\|_{L^1} + 2\pi\lambda}. \tag{3.7}$$

Using (3.7) we get

$$\begin{aligned} \int_{-\pi}^{\pi} |\tilde{\psi}_l(e^{i\theta})| d\theta &= -\int_0^{\pi/2} \lambda \, dm_l(\lambda) = \int_0^{\pi/2} m_l(\lambda) d\lambda \leqslant \\ &\leqslant 2\|\psi_l\|_{L^1} \log \frac{\|\psi_l\|_{L^1} + \pi^2}{\|\psi_l\|_{L^1}}. \end{aligned} \tag{3.8}$$

But (3.5), (3.6) and (3.8) yield

$$\|\psi_l + i\tilde{\psi}_l\|_{L^1} \leqslant 2\pi \log(1 + \mu_l)\left[1 + 2\log\left(1 + \frac{\pi}{2\log(1 + \mu_l)}\right)\right].$$

Now the function $g(x) = \pi x[1 + 2\log(1 + \pi/x)]$, $x > 0$, is nondecreasing. Using this fact and the estimate $\log(1 + \mu_l) \leqslant \mu_l \leqslant \lambda_l$ we obtain

$$\|\psi_l + i\tilde{\psi}_l\|_{L^1} \leqslant 2\pi\lambda_l\left[1 + 2\log\left(1 + \frac{\pi}{2\lambda_l}\right)\right] \leqslant 6\pi\lambda_l + 4\pi\lambda_l\log\frac{1}{\lambda_l}, \tag{3.9}$$

where by $\{\lambda_l\}_{l=1}^{\infty}$ we denote the eigenvalues of $I - |T|$. Now the condition $I - |T| \in \mathfrak{L}_1^0(\mathfrak{H})$ implies

$$\sum_{l=1}^{\infty} \lambda_l \log\frac{1}{\lambda_l} < +\infty.$$

Hence the series (3.1) converges in H^1 and, consequently, $\log F(z) \in H^1$. ▨

REMARK 3.2. Example 3.10 of [22] shows that in order to obtain a summable spectral shift function the assumptions of Theorem 3.1 cannot be essentially improved.

THEOREM 3.3. *If* $I - |T| \in \mathfrak{L}_1^0(\mathfrak{H})$, $I - |T_0^*| \in \mathfrak{L}_1^0(\mathfrak{H})$ *and* $T - T_0 \in \mathfrak{L}_1(\mathfrak{H})$, *then* $\{T, T_0\}$ *possesses a summable spectral shift function* $\mu(\cdot)$ *such that the trace formula* (1.8) *holds for every* $\varphi(\cdot) \in \mathfrak{G}_\Pi$.

Proof. First of all we establish (2.2) and (2.3). We have $I - T^*T_0 = (I + |T|)(I - |T|) - T^*(T_0 - T) \in \mathfrak{L}_1(\mathfrak{H})$ and $I - TT_0^* = (I + |T_0^*|)(I - |T_0^*|) - (T - T_0^*)T_0^* \in \mathfrak{L}_1(\mathfrak{H})$. Furthermore, using the polar decompostion $T_0 = V_0|T_0|$ we find $I - |T_0^*| = I - V_0^*V_0 + V_0^*(I - |T_0|)V_0$. Since $I - V_0^*V_0 \in \mathfrak{L}_1(\mathfrak{H})$ this operator is a finite dimensional projection. Thus, $I - |T_0| \in \mathfrak{L}_1^0(\mathfrak{H})$. From Theorem 3.1 and Theorem 2.2 we conclude $h(t)\log^+ h(t) \in L^1(-\pi, \pi)$ and $h_0(t)\log^+ h_0(t) \in L^1(-\pi, \pi)$. Since $h(t), h_0(t) \geqslant 0$ we find

$$\begin{aligned} &|h(t) - h_0(t)|\log^+|h(t) - h_0(t)| \leqslant \\ &\leqslant h(t)\log^+ h(t) + h_0(t)\log^+ h(t), \end{aligned} \tag{3.10}$$

$t \in [-\pi, \pi)$. Hence we have shown $|h(t) - h_0(t)|\log^+|h(t) - h_0(t)| \in L^1(-\pi, \pi)$. Applying Theorem 2.3 we complete the proof. ▨

REMARK 3.4. Since the inclusion (1.11) holds all conclusions of Theorem 3.1 and Theorem 3.2 remain true if the conditions $I - T^*T_0 \in \mathfrak{L}_p(\mathfrak{H})$ and $I - TT_0^* \in \mathfrak{L}_q(\mathfrak{H})$, $0 < p, q < 1$, are satisfied.

4. DISSIPATIVE CASE

Let H be a maximal dissipative operator on $\mathfrak{H}$, i.e. $\mathrm{Im}(Hf, f) \leqslant 0, f \in \mathrm{dom}(H)$. By T we denote its Cayley transform, i.e. $T = \dfrac{iI + H}{iI - H}$. It is well-known that T is a contraction on $\mathfrak{H}$. Furthermore, we introduce the set of functions $\mathfrak{F}_{\mathbf{R}}$. We say $\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$ if $\psi\left(\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}t}-1}{\mathrm{e}^{\mathrm{i}t}+1}\right) \in \mathfrak{G}_{\Pi}$, $t \in [-\pi, \pi)$, and $\lim_{\lambda\to\pm\infty} \psi(\lambda) = 0$. Let $\varphi(\mathrm{e}^{\mathrm{i}t}) = \psi\left(\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}t}-1}{\mathrm{e}^{\mathrm{i}t}+1}\right)$, $t \in [-\pi, \pi)$, and $\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$. Taking into account the decomposition (1.5) – (1.7) and setting $\psi_{\pm}(\lambda) = \varphi_{\pm}\left(\frac{\mathrm{i}+\lambda}{\mathrm{i}-\lambda}\right)$ we obtain a decomposition in $\mathfrak{F}_{\mathbf{R}}$,

$$\psi(\lambda) = \psi_+(\lambda) + \psi_-(-\lambda), \tag{4.1}$$

$\lambda \in \mathbf{R}^1$. We set $\psi_+(H) = \varphi_+(T)$ and $\psi_-(-H^*) = \varphi_-(T^*)$.

THEOREM 4.1. *Let $\{H, H_0\}$ be a pair of maximal dissipative operators on $\mathfrak{H}$. If the conditions*

$$(H^* + \mathrm{i})^{-1} - (H - \mathrm{i})^{-1} + 2\mathrm{i}(H^* + \mathrm{i})^{-1}(H - \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H}), \tag{4.2}$$

$$(H_0^* + \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} + 2\mathrm{i}(H_0 - \mathrm{i})^{-1}(H_0^* + \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H}), \tag{4.3}$$

and

$$(H - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} \in \mathfrak{L}_1(\mathfrak{H}) \tag{4.4}$$

are satisfied, then there is a real measurable function $\xi(\cdot) \in L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}\mathrm{d}\lambda)$, which is called the spectral shift function of the pair $\{H, H_0\}$, such that the trace formula

$$\begin{aligned} &\operatorname{tr}\{\psi_+(H) + \psi_-(-H^*) - \psi_+(H_0) - \psi_-(-H_0^*)\} = \\ &= \int_{-\infty}^{+\infty} \xi(\lambda) \frac{\mathrm{d}}{\mathrm{d}\lambda} \psi(\lambda)\mathrm{d}\lambda \end{aligned} \tag{4.5}$$

is valid for every $\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$.

Proof. Denoting by T and T_0 the Cayley transform of H and H_0, respectively, it is easy to see that (4.2) and (4.3) are equivalent to $I - |T| \in \mathfrak{L}_1^0(\mathfrak{H})$, $I - |T_0^*| \in \mathfrak{L}_1^0(\mathfrak{H})$ and $T - T_0 \in \mathfrak{L}_1(\mathfrak{H})$. Thus, using Theorem 3.3 we get

$$\begin{aligned} &\operatorname{tr}\{\psi_+(H) + \psi_-(-H^*) - \psi_+(H_0) - \psi_-(-H_0^*)\} = \\ &= \operatorname{tr}\{\varphi_+(T) + \varphi_-(-T^*) - \varphi_+(T_0) - \varphi_-(T^*)\} = \\ &= \int_{-\pi}^{\pi} \mu(\theta) \frac{\mathrm{d}}{\mathrm{d}\theta} \varphi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta, \end{aligned} \tag{4.6}$$

$\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$, where $\mu(\cdot)$ is the spectral shift function of the pair $\{T, T_0\}$, $\mu(\cdot) \in L^1(-\pi, \pi)$. Setting $\theta = -2\operatorname{arctg}\lambda$ and $\xi(\lambda) = -\mu(-2\operatorname{arctg}\lambda)$, $\lambda \in \mathbf{R}^1$, we find $\xi(\cdot) \in L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}d\lambda)$ and (4.5). ▨

COROLLARY 4.2. *If the conditions*

$$(4.7) \qquad (H^* + \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} + 2\mathrm{i}(H^* + \mathrm{i})^{-1}(H_0 - \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H}),$$

$$(4.8) \qquad (H - \mathrm{i})^{-1} - (H_0^* + \mathrm{i})^{-1} - 2\mathrm{i}(H - \mathrm{i})^{-1}(H_0^* + \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H}),$$

are satisfied, then $\{H, H_0\}$ *possesses a spectral shift function* $\xi(\cdot)$ *of class* $L^1(\mathbf{R}^1, (1 + \lambda^2)d\lambda)$ *such that the trace formula* (4.5) *holds for every* $\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$.

Proof. In a standard manner we show that the conditions (4.7) and (4.8) imply (4.2) and (4.3) and $(H - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H})$. Since $\mathfrak{L}_1^0(\mathfrak{H}) \subset \mathfrak{L}_1(\mathfrak{H})$ we complete the proof applying Theorem 4.1. ▨

COROLLARY 4.3. *Let* V *be a nuclear dissipative operator*, i.e. $V \in \mathfrak{L}_1(\mathfrak{H})$, *and let* H_0 *be a self-adjoint operator. Denoting by* V_1 *the imaginary part of* V, i.e. $V_1 = \frac{1}{2\mathrm{i}}(V - V^*) \leqslant 0$, *the pair* $\{H = H_0 + V, H_0\}$ *possesses a summable spectral shift function* $\xi(\cdot) \in L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1} d\lambda)$ *such that the trace formula* (4.5) *holds if*

$$(4.9) \qquad V_1 \log(-V_1) \in \mathfrak{L}_1(\mathfrak{H}).$$

Proof. Since $V \in \mathfrak{L}_1(\mathfrak{H})$ the condition (4.4) is fulfilled. Taking into account that H_0 is self-adjoint the condition (4.3) is obvious. In order to apply Theorem 4.1 it remains to verify the condition (4.2). To this end we use the representation

$$(4.10) \qquad \begin{aligned} &(H^* + \mathrm{i})^{-1} - (H - \mathrm{i})^{-1} + 2\mathrm{i}(H^* + \mathrm{i})^{-1}(H - \mathrm{i})^{-1} = \\ &= 2\mathrm{i}(H^* + \mathrm{i})^{-1}V_1(H - \mathrm{i})^{-1}. \end{aligned}$$

By (4.9) we obtain $V_1 \in \mathfrak{L}_1^0(\mathfrak{H})$. Since $\mathfrak{L}_1^0(\mathfrak{H})$ is an ideal in $\mathfrak{B}(\mathfrak{H})$ we have shown (4.2). ▨

REMARK 4.4. Corollary 4.3 essentially improves the results of [22].

5. APPLICATION

In the following we give a simple application of the abstract results to the Schrödinger operator. To this end we set $\mathfrak{H} = L^2(\mathbf{R}^n, dx)$, $n = 1, 2, 3$, and $H_0 = -\Delta$, where $-\Delta$ is the Laplace operator on $L^2(\mathbf{R}^n, dx)$, $n = 1, 2, 3$, and V is a multiplication operator induced by the bounded complex function $q(\cdot)$ via $(Vf)(x) =$

$= q(x)f(x)$, $x \in \mathbf{R}^n$, $f \in L^2(\mathbf{R}^n, \mathrm{d}x)$, $n = 1,2,3$. In order to obtain a maximal dissipative operator we assume $\mathrm{Im}(q(x)) \leqslant 0$ for a.e. $x \in \mathbf{R}^n$, $n = 1, 2, 3$. Setting $H = H_0 + V$, $\mathrm{dom}(H) = \mathrm{dom}(H_0)$, we obviously obtain a maximal dissipative operator.

THEOREM 5.1. *If there are constants* $C > 0$ *and* $\varepsilon > 0$ *such that*

$$|q(x)| \leqslant C(1 + |x|)^{-(n+\varepsilon)}, \tag{5.1}$$

is valid for a.e. $x \in \mathbf{R}^n$, $n = 1, 2, 3$, *then the pair* $\{H, H_0\}$ *possesses a spectral shift function* $\xi(\cdot)$ *belonging to* $L^1(\mathbf{R}^1, (1 + \lambda^2)^{-1}\mathrm{d}\lambda)$ *such that the formula* (4.5) *holds for every* $\psi(\cdot) \in \mathfrak{F}_{\mathbf{R}}$.

Proof. Since H_0 is a self-adjoint operator we find that the conditions (4.7) and (4.8) are equivalent to

$$(H - \mathrm{i})^{-1} - (H_0 - \mathrm{i})^{-1} \in \mathfrak{L}_1^0(\mathfrak{H}). \tag{5.2}$$

In accordance with the factorization $q = |q|^{1/2} \frac{q}{|q|} |q|^{1/2}$, where we agree to set $\frac{q}{|q|} = 1$ for $x \in \{x \in \mathbf{R}^n : q(x) = 0,\ n = 1, 2, 3\}$, we obtain a factorization $V = |V|^{1/2}W|V|^{1/2}$ where W is a unitary operator induced by $\frac{q}{|q|}$. Introducing the bounded operator $G = (H_0 - \mathrm{i})(H^* + \mathrm{i})^{-1}$ we get the representation

$$(H - \mathrm{i})^{-1}(H_0 - \mathrm{i})^{-1} = G^*(H_0 + \mathrm{i})^{-1}|V|^{1/2}W|V|^{1/2}(H_0 - \mathrm{i})^{-1}. \tag{5.3}$$

Taking into account that for $A_1 \in \mathfrak{L}_r(\mathfrak{H})$ and $A_2 \in \mathfrak{L}_r(\mathfrak{H})$ the product $A_1A_2 \in \mathfrak{L}_{r/2}(\mathfrak{H})$, $r > 0$, and that $\mathfrak{L}_{r/2}(\mathfrak{H}) \subset \mathfrak{L}_1^0(\mathfrak{H})$, $0 < r < 2$, we prove (5.2) showing $|V|^{1/2}(H_0 - \mathrm{i})^{-1} \in \mathfrak{L}_r(\mathfrak{H})$, $r < 2$. Denoting by $f(\cdot)$ and $g(\cdot)$ the functions $f(x) = |q(x)|^{1/2}$, $x \in \mathbf{R}^n$, $n = 1, 2, 3$, and $g(k) = \frac{1}{k^2 + 1}$, $k \in \mathbf{R}^n$, $n = 1, 2, 3$, we have to investigate the operator $|V|^{1/2}(H_0 - \mathrm{i})^{-1} = f(x)g(-\mathrm{i}\nabla)$, $\nabla = \frac{\mathrm{d}}{\mathrm{d}x}$. To this end, we introduce the sets

$$\mathcal{O}_m^{(n)} = \{x \in \mathbf{R}^n : \sqrt{n}\, m \leqslant |x| \leqslant \sqrt{n}(m + 1)\}, \tag{5.4}$$

$m = 0,1,2, \ldots, n = 1,2,3$. By $N_{(n)}(m)$ we denote the number of unit cubes Δ_α with the center at $\alpha \in \mathbf{Z}^n$ such that $\Delta_\alpha \cap \mathcal{O}_m^{(n)} \neq \emptyset$, $n = 1, 2, 3$. Because of $N_n(m) \leqslant |\mathcal{O}_{m-1}^{(n)} \cup \mathcal{O}_m^{(n)} \cup \mathcal{O}_{m+1}^{(n)}|$, $m = 1,2,3, \ldots$, there is a constant C_n such that the estimate $N_n(m) \leqslant C_n m^{n-1}$, $m = 1, 2, 3, \ldots$, $n = 1,2,3$, holds. Taking into account the estimate $|f(x)| \leqslant C(1 + |x|)^{-(n+\varepsilon)/2}$, $n = 1,2,3$, we get

$$\sum_{\alpha}\left(\int_{\Delta_\alpha}|f(x)|^2\mathrm{d}x\right)^{r/2} \leqslant C^{r/2}\sum_{\alpha}\left(\int_{\Delta_\alpha}(1+|x|)^{-(n+\varepsilon)}\mathrm{d}x\right)^{r/2} \leqslant$$

$$\leqslant C^{r/2}\sum_{m=0}^{+\infty}\sum_{\Delta_\alpha\in\mathcal{O}_m^{(n)}}\left(\int_{\Delta_\alpha}(1+|x|)^{-(n+\varepsilon)}\mathrm{d}x\right)^{r/2} \leqslant$$

(5.5)
$$\leqslant C^{r/2}\Bigg(\sum_{\Delta_\alpha\in\mathcal{O}_0^{(n)}}\left\{\int_{\Delta_\alpha}(1+|x|)^{-(n+\varepsilon)}\mathrm{d}x\right\}^{r/2} +$$

$$+\sum_{m=1}^{+\infty}N_n(m)\,(1+\sqrt{n}(m-1))^{-(n+\varepsilon)r/2}\Bigg) \leqslant$$

$$\leqslant \text{const.} + C^{r/2}C_n\sum_{m=1}^{+\infty}m^{n-1}(1+\sqrt{n}(m-1))^{-(n+\varepsilon)r/2}.$$

Now the last series converges if $(n+\varepsilon)r/2-(n-1)>1$, i.e. $2n/(n+\varepsilon)<r$, $n=1,2,3$. Thus, $f(\cdot)\in\ell^r(L^2)$, $2n/(n+\varepsilon)<r$, $n=1,2,3$. Because of $|g(k)| \leqslant C(1+|k|)^{-2} \leqslant C(1+|k|)^{-(n+\varepsilon)/2}, k\in\mathbf{R}^n$, $n=1,2,3$, we obtain $g(\cdot)\in\ell^r(L^2)$, $n/2<r, n=1,2,3$. Since $\varepsilon>0$, r can be chosen such that the inequality $\max\{n/2, 2n/(n+\varepsilon)\}<r<2$, $n=1,2,3$, is satisfied. By Theorem 4.5 of [30] we get $|V|^{1/2}(H_0-\mathrm{i})^{-1}=f(x)g(-\mathrm{i}\nabla)\in\mathfrak{L}_r(\mathfrak{H})$, $r<2$. Hence, we have proved (5.2). Applying Corollary 4.2 we complete the proof. ▧

REFERENCES

1. Adamjan, V. M.; Pavlov, B. S., A trace formula for dissipative operators (Russian), *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.*, 7(1979), 5–9.
2. Akopjan, R. V., On a trace formula in perturbation theory for J-positive operators (Russian), *Dokl. Akad. Nauk Armenian SSR*, **57**(1973), 193–199.
3. Akopjan, R. V., On the perturbation theory of J-non-negative operators (Russian), Thesis, Univ. of Erevan.
4. Akopjan, R. V., On a trace formula for J-positive operators by nuclear perturbation (Russian), *Dokl. Akad. Nauk Armenian SSR*, **77**(1983), 195–200.
5. Baumgärtel, H.; Wollenberg, M., *Mathematical scattering theory*, Akademie-Verlag, Berlin, 1983.
6. Birman, M. Š.; Krein, M. G., On the theory of wave and scattering operators (Russian), *Dokl. Akad. Nauk SSSR*, **144**(1962), 475–480.

7. BIRMAN, M. Š.; SOLOMJAK, M. Z., Remarks on the spectral shift function (Russian), *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov. (LOMI)*, **27**(1972), 5–19.

8. GOHBERG, I. Z.; KREIN, M. G., *Introduction in the theory of linear nonself-adjoint operators* (Russian), Nauka, Moscow, 1965.

9. HOFFMAN, K., *Banach spaces of analytic functions*, Prentice-Hall Inc., Englewood Cliffs., New York, 1965.

10. JONAS, P., Die Spurformel der Störungstheorie für einige Klassen unitärer und selbstadjungierter Operatoren im Kreinraum, Report R-MATH-06/86, Karl Weierstrass Institut für Mathematik, AdW der DDR, Berlin, 1986.

11. JONAS, P., On the trace formula of perturbation theory, I, Preprint P-MATH-16/87, Karl Weierstrass Institut für Mathematik, AdW der DDR, Berlin 1987.

12. KREIN, M. G., On the trace formula in perturbation theory (Russian), *Mat. Sb.*, **33**(1953), 597–626.

13. KREIN, M. G., On the perturbation determinant and the trace formula for unitary and self-adjoint operators (Russian), *Dokl. Akad. Nauk SSSR*, **144**(1962), 268–271.

14. KREIN, M., G., *About some new investigations in perturbation theory for self-adjoint operators* (Russian), First Summer School of Mathematics, Kanaev, 1963.

15. KREIN, M. G.; JAVRJAN, V. A., On the spectral shift function arising from the perturbation of a positive operator (Russian), *J. Operator Theory*, **6**(1981), 155–191.

16. KREIN, M. G., On the perturbation determinant and the trace formula for some classes of pairs of operators (Russian), *J. Operator Theory*, **17**(1988), 129–187.

17. KOPLIENKO, L. S., Local existence conditions for the spectral shift function (Russian), *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)*, **73**(1977), 102–177.

18. KOOSIS, P., *Introduction to H^p spaces*, London Math. Soc. Lecture Note Series **40**, Cambridge Univ. Press, London, 1980.

19. LANGER, H., Eine Erweiterung der Spurformel der Strörungstheorie, *Math. Nachr.*, **30**(1965), 123–135.

20. LIFSHITZ, I. M., About one problem of perturbation theory connected with quantum statistic, (Russian), *Usp. Mat. Nauk.*, **7**(1952), 171–180.

21. NEIDHARDT, H., Eine mathematische Streutheorie für maximal dissipative Operatoren, Report R-MATH-03/86, Karl Weierstrass Institut für Mathematik, AdW der DDR, Berlin, 1986.

22. NEIDHARDT, H., Scattering matrix and spectral shift of the nuclear dissipative scattering theory, in *Operators in indefinite metric spaces, scattering theory and other topics*, Birkhäuer Verlag, Basel, 1987, pp. 237–250.

23. NEIDHARDT, H., Scattering matrix and spectral shift of the nuclear dissipative scattering theory. II, *J. Operator Theory*, **19**(1988), 43–62.

24. PELLER, V. V., Hankel operators in perturbation theory of unitary and self-adjoint operators (Russian), *Funkcional. Anal. i. Priložen.*, **19**(1985), 37–51.

25. RYBKIN, A. V., A trace formula for dissipative and self-adjoint operators and spectral identities for resonances (Russian), *Vestnik Leningrad. Univ. Math. Astronom.*, **19**(1984), 97–99.

26. RYBKIN, A. V., Spectral shift function for dissipative and self-adjoint operators and a trace formula for resonances (Russian), *Mat. Sb.*, **125**(1984), 420–430.

27. RYBKIN, A. V., A trace formula for contractive and unitary operators (Russian), *Funkcional. Anal. i. Priložen.*, **21**(1977), 85–87.

28. SAHNOVIČ, L. A., Dissipative operators with absolutely continuous spectrum (Russian), *Trudy Moscov, Mat. Obšč.*, **19**(1968), 211–270.

29. Sz.-Nagy, B.; Foiaş, C., *Harmonic analysis of linear operators in Hilbert space*, North-Holland, Amsterdam, 1970.
30. Simon, B., *Trace ideals and their applications*, Cambridge Univ. Press, Cambridge—London—New York — Melbourne, 1979.

V. M. ADAMJAN
Department of Theoretical Physics,
University of Odessa,
270057 Odessa,
U.S.S.R.

H. NEIDHARDT
Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research,
Head Post Office P.O. Box 79,
Moscow,
U.S.S.R.

Permanent address:

Karl Weierstrass Institut für Mathematik
der AdW der DDR,
Mohrenstrasse 39,
1089 Berlin,
D.D.R.

Received July 3, 1989.