J. OPERATOR THEORY
24(1990), 227–238

ON THE STRUCTURE OF THE C^*-ALGEBRA OF THE FOLIATION FORMED BY THE K-ORBITS OF MAXIMAL DIMENSION OF THE REAL DIAMOND GROUP

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INTRODUCTION

Recent works of D.N. Diep [5], J. Rosenberg [10], G. G. Kasparov [8], V. M. Son and H. H. Viet [12] ... have shown that K-functors are well adapted to the description (or characterization) of group C^*-algebras. Kirillov's method of orbits allows to find out the classes of Lie groups MD and $\widetilde{\mathrm{MD}}$ in terms of D. N. Diep, for which the group C^*-algebras can be described by means of suitable K-functors (see [12]).

In 1982, studying foliated manifolds, A. Connes [4] introduced the concept of the C^*-algebra associated to a measured foliation. Once again, in the case of Reeb foliations (see A. M. Torpe [14]), the method of K-functors has been proved as very effective in describing the structure of Connes' C^*-algebras.

In this paper, we attempt to combine the methods of Kirillov and Connes by considering the foliations formed by the generic K-orbits of solvable Lie groups. A special interesting case is the case of the real form of the diamond group $\mathbf{R}.H_3$.

In [16], we have studied the picture of K-orbits of $\mathbf{R}.H_3$ and showed that the family $\mathscr{F}$ of the K-orbits of maximal dimension forms a measured foliation $(V, \mathscr{F})$ in terms of Connes. We call $(V, \mathscr{F})$ *the real diamond foliation.*

The purpose of this paper is to describe the structure of the C^*-algebra $C^*(V, \mathscr{F})$. Note that we can not express $C^*(V, \mathscr{F})$ by a single extension of the form

$$0 \longrightarrow C_0(X) \otimes \mathscr{K} \longrightarrow C^*(V, \mathscr{F}) \longrightarrow C_0(Y) \otimes \mathscr{K} \longrightarrow 0.$$

We will express $C^*(V, \mathscr{F})$ by two repeated extensions of the form

$$0 \longrightarrow C_0(X_1) \otimes \mathscr{K} \longrightarrow C^*(V, \mathscr{F}) \longrightarrow B_1 \longrightarrow 0$$

$$0 \longrightarrow C_0(X_2) \otimes \mathscr{K} \longrightarrow B_1 \longrightarrow C_0(Y_2) \otimes \mathscr{K} \longrightarrow 0.$$

Then we will compute the invariant system of $C^*(V, \mathcal{F})$ with respect to these extensions. If the given C^*-algebra is isomorphic to the reduced crossed product of the form $C_0(V) \rtimes H$, where H is a Lie group, we can use the Thom-Connes isomorphism to compute the connecting maps δ_0, δ_1. Hence, we first will give a new effective description of the real diamond foliation $(V, \mathcal{F})$ by a smooth action of $\mathbf{R}^3$ on the foliated manifold V.

In another paper, we will be concerned with the similar problem for the other groups of the class MD.

The author would like to express his deepest gratitude to Dr. Do Ngoc Diep and Dr. Ho Huu Viet for suggesting this study and for their great help and encouragement.

1. THE REAL DIAMOND FOLIATION

Let $\mathfrak{G}$ be the Lie algebra with the basis $\{T, X, Y, Z\}$ such that

$$[T, X] = -X, \quad [T, Y] = Y, \quad [X, Y] = Z,$$

$$[T, Z] = [X, Z] = [Y, Z] = 0.$$

The corresponding simply connected Lie group, denoted by $\mathbf{R}.H_3$, is called the real form of the diamond group or, breafly, the real diamond group.

It is known that $\mathbf{R}.H_3$ belongs to the class MD (see [12]). We now recall the geometric description of the K-orbits in the dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ (see [16]). Let $\{T^*, X^*, Y^*, Z^*\}$ be the basic in $\mathfrak{G}^*$ dual to $\{T, X, Y, Z\}$. The two-dimensional orbit Ω_F including $F = (\delta, \alpha, \beta, \gamma)$ in $\mathfrak{G}^* \cong \mathbb{R}^4$ is one of the following forms:

If $\gamma = 0$, $\beta = 0$, $\alpha \neq 0$ then $\Omega_F = \{(t, x, 0, 0),\ t \in \mathbf{R},\ x\alpha > 0\}$ (the coordinate half-planes $\mathbf{R}T^* + \mathbf{R}_\pm X^*$);

If $\gamma = 0$, $\alpha = 0$, $\beta \neq 0$ then $\Omega_F = \{(t, 0, y, 0),\ t \in \mathbf{R},\ y\beta > 0\}$ (the coordinate half-planes $\mathbf{R}T^* + \mathbf{R}_\pm Y^*$);

If $\gamma = 0$, $\alpha\beta \neq 0$ then $\Omega_F = \{(t, x, y, 0),\ t \in \mathbf{R},\ xy = \alpha\beta,\ x\alpha > 0,\ y\beta > 0\}$ (the vertical cylinder with the diretrix-one of the branches of the hyperbola $xy = \alpha\beta$, $t = 0$).

If $\gamma \neq 0$ then $\Omega_F = \{(t, x, y, z),\ z = \gamma,\ \gamma(t - \delta) = xy - \alpha\beta\}$ (the hyperbolic paraboloid with the minimax-point $(\delta - \alpha\beta/\gamma, 0, 0, \gamma)$).

It is proved in [16] that the family $\mathcal{F}$ of the K-orbits of maximal dimension forms the real diamond foliation on the open submanifold

$$V = \{(t, x, y, z) \in \mathfrak{G}^*,\ x^2 + y^2 + z^2 \neq 0\} \approx \mathbf{R} \times (\mathbb{R}^3)^*.$$

PROPOSITION 1. *The real diamond foliation* $(V, \mathscr{F})$ *can be given by an action of the commutative Lie group* $\mathbf{R}^2$ *on the manifold* V.

Proof. One needs only to verify that the following action λ of $\mathbf{R}^2$ on V gives the real diamond foliation

$$\lambda: \mathbf{R}^2 \times V \to V$$

$$(r, s) \times (t, x, y, z) \mapsto (\bar{t}, \bar{x}, \bar{y}, \bar{z})$$

$$\bar{t} = t + r \cdot \frac{x^2 + y^2}{x^2 + y^2 + z^2} + r^2 \frac{xyz}{(x^2 + y^2 + z)^2}$$

$$\bar{x} = e^{-s}\left(x + r \frac{yz}{x^2 + y^2 + z^2}\right)$$

$$\bar{y} = e^{s}\left(y + r \frac{xz}{x^2 + y^2 + z^2}\right)$$

$$\bar{z} = z.$$

2. C*(V, $\mathscr{F}$) AS TWO REPEATED EXTENSIONS

2.1. Let V_1, W_1, V_2, W_2 be the following submanifolds of V

$$V_1 = \{(t, x, y, z) \in V, \ \ z \neq 0\} \cong \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^*,$$

$$W_1 = V \setminus V_1 = \mathbf{R} \times (\mathbf{R}^2)^*,$$

$$V_2 \stackrel{\text{def}}{=} \{(t, x, y, z) \in W_1, \ \ xy \neq 0\} \cong \mathbf{R} \times \mathbf{R}^* \times \mathbf{R}^*,$$

$$W_2 = W_1 \setminus V_2 \cong \mathbf{R} \times (\mathbf{R}^* \times \{0\} \cup \{0\} \times \mathbf{R}^*).$$

It is easy to see that the action λ in Proposition 1 preserves the subsets V_1, W_1, V_2, W_2. Let i_1, i_2, μ_1, μ_2 be the inclusions and the restrictions

$$i_1: C_0(V_1) \to C_0(V), \quad i_2: C_0(V_2) \to C_0(W_1)$$

$$\mu_1: C_0(V) \to C_0(W_1), \quad \mu_2: C_0(W_1) \to C_0(W_2)$$

where each function of $C_0(V_1)$ (resp. $C_0(V_2)$) is extended to the one of $C_0(V)$ (resp $C_0(W_1)$) by taking the value of zero outside V_1 (resp. V_2).

It is a known fact that i_1, i_2, μ_1, μ_2 are λ-equivariant and the following sequences are equivariantly exact

$$(2.1.1) \qquad 0 \longrightarrow C_0(V_1) \xrightarrow{i_1} C_0(V) \xrightarrow{\mu_1} C_0(W_1) \longrightarrow 0$$

$$(2.1.2) \qquad 0 \longrightarrow C_0(V_2) \xrightarrow{i_2} C_0(W_1) \xrightarrow{\mu_2} C_0(W_2) \longrightarrow 0.$$

2.2. Now we denote by $(V_1, \mathscr{F}_1), (W_1, \mathscr{F}_1), (V_2, \mathscr{F}_2), (W_2, \mathscr{F}_2)$ the foliations-restrictions of $(V, \mathscr{F})$ on V_1, W_1, V_2, W_2 respectively.

THEOREM 1. *$C^*(V, \mathscr{F})$ admits the following canonical repeated extensions*

$$(\gamma_1) \qquad 0 \longrightarrow J_1 \xrightarrow{\hat{i}_1} C^*(V, \mathscr{F}) \xrightarrow{\hat{\mu}_1} B_1 \longrightarrow 0,$$

$$(\gamma_2) \qquad 0 \longrightarrow J_2 \xrightarrow{\hat{i}_2} B_1 \xrightarrow{\hat{\mu}_2} B_2 \longrightarrow 0,$$

where

$$J_1 = C^*(V_1, \mathscr{F}_1) \cong C_0(V_1) \rtimes_\lambda \mathbf{R}^2 \cong C_0(\mathbf{R} \times \mathbf{R}^*) \otimes \mathscr{K},$$

$$J_2 = C^*(V_2, \mathscr{F}_2) \cong C_0(V_2) \rtimes_\lambda \mathbf{R}^2 \cong C_0(\mathbf{R}^* \cup \mathbf{R}^*) \otimes \mathscr{K},$$

$$B_2 = C^*(W_2, \mathscr{F}_2) \cong C_0(W_2) \rtimes_\lambda \mathbf{R}^2 \cong \mathbf{C}^4 \otimes \mathscr{K},$$

$$B_1 = C^*(W_1, \mathscr{F}_1) \cong C_0(W_1) \rtimes_\lambda \mathbf{R}^2,$$

$$C^*(V, \mathscr{F}) \cong C_0(V) \rtimes_\lambda \mathbf{R}^2,$$

and the homomorphismes $\hat{i}_1, \hat{i}_2, \hat{\mu}_1, \hat{\mu}_2$ are defined by

$$(\hat{i}_k f)(r, s) = i_k f(r, s) \quad k = 1, 2,$$

$$(\hat{\mu}_k f)(r, s) = \mu_k f(r, s) \quad k = 1, 2.$$

Proof. We note that the graph of $(V, \mathscr{F})$ is identical with $V \times \mathbf{R}^2$, so by [4, Section 5], $J_1 = C^*(V_1, \mathscr{F}_1) \cong C_0(V_1) \rtimes_\lambda \mathbf{R}^2$. Similarly, we have

$$B_1 = C_0(W_1) \rtimes_\lambda \mathbf{R}^2,$$

$$J_2 = C_0(V_2) \rtimes_\lambda \mathbf{R}^2,$$

$$B_2 = C_0(W_2) \rtimes_\lambda \mathbf{R}^2,$$

$$C^*(V, \mathscr{F}) = C_0(V) \rtimes_\lambda \mathbf{R}^2.$$

From the equivariantly exact sequences in 2.1 and by [3, Lemma I.1] we obtain the repeated extensions (γ_1) and (γ_2).

Furthermore, the foliation $(V_1, \mathscr{F}_1)$ can be derived from the submersion

$$p_1 : V_1 \cong \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^* \longrightarrow \mathbf{R} \times \mathbf{R}^*$$

$$p_1(t, x, y, z) = \left(t - \frac{xy}{z}, z\right).$$

Hence, by a result of [4, p. 562], we get $J_1 \cong C_0(\mathbf{R} \times \mathbf{R}^*) \otimes \mathscr{K}$. The same argument shows that

$$J_2 \cong C_0(\mathbf{R}^* \times \mathbf{R}^*) \otimes \mathscr{K}, \quad B_2 \cong \mathbf{C}^4 \otimes \mathscr{K}.$$

3. COMPUTING THE SYSTEM OF INVARIANTS OF $C^*(V, \mathscr{F})$

DEFINITION. The set of elements $\{\gamma_1, \gamma_2\}$ corresponding to the repeated extensions $(\gamma_1), (\gamma_2)$ in the Kasparov groups $\operatorname{Ext}(B_i, J_i)$, $i = 1,2$ is called the *system of invariants* of $C^*(V, \mathscr{F})$ and denoted by $\operatorname{Index} C^*(V, \mathscr{F})$.

REMARK. $\operatorname{Index} C^*(V, \mathscr{F})$ determines the so-called "stable type" of $C^*(V, \mathscr{F})$ in the set of all repeated extensions

$$0 \longrightarrow J_1 \longrightarrow E \longrightarrow B_1 \longrightarrow 0$$

$$0 \longrightarrow J_2 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow 0.$$

The main result of the paper is the following

THEOREM 2. $\operatorname{Index} C^*(V, \mathscr{F}) = \{\gamma_1, \gamma_2\}$, *where:*

$$\gamma_1 = (1, 1) \text{ in the group } \operatorname{Ext}(B_1, J_1) = \mathbf{Z}^2,$$

$$\gamma_2 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ in the group } \operatorname{Ext}(B_2, J_2) = \operatorname{Hom}(\mathbf{Z}^4, \mathbf{Z}^4).$$

To prove this theorem, we shall need some lemmas. With the notation of Section 2, we have

LEMMA 1. *Set*

$$I_1 = C_0(\mathbf{R}^2 \times \mathbf{R}^*) \quad \text{and} \quad A_1 = C_0(\mathbf{R}^2 \setminus (0, 0)).$$

The following diagram is commutative

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_j(I_1) & \longrightarrow & K_jC_0(\mathbf{R}^3 \setminus (0,0,0)) & \longrightarrow & K_j(A_1) & \longrightarrow & K_{j+1}(I_1) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & \\
\cdots \longrightarrow & K_{j+1}C_0(V_1) & \longrightarrow & K_{j+1}C_0(V) & \longrightarrow & K_{j+1}C_0(W_1) & \longrightarrow & K_jC_0(V_1) & \longrightarrow \cdots
\end{array}$$

where β_1 *is the isomorphism defined in* [13, Theorem 9.7] *or in* [3, Corollary VI. 3], $j \in \mathbf{Z}/2\mathbf{Z}$.

Proof. Let

$$k_1 : I_1 = C_0(\mathbf{R}^2 \times \mathbf{R}^*) \longrightarrow C_0(\mathbf{R}^3 \setminus (0,0,0))$$

$$v_1 : C_0(\mathbf{R}^3 \setminus (0,0,0)) \longrightarrow A_1 = C_0(\mathbf{R}^2 \setminus (0,0))$$

be the inclusion and restriction defined similarly as in 2.1.

One gets the exact sequence

$$0 \longrightarrow I_1 \xrightarrow{k_1} C_0(\mathbf{R}^3 \setminus (0,0,0)) \xrightarrow{v_1} A_1 \longrightarrow 0.$$

Note that

$$C_0(V_1) \cong C_0(\mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^*) \cong C_0(\mathbf{R}) \otimes I_1,$$

$$C_0(W_1) \cong C_0(\mathbf{R} \times (\mathbf{R}^2 \setminus (0,0))) \cong C_0(\mathbf{R}) \otimes A_1,$$

$$C_0(V) \cong C_0(\mathbf{R} \times (\mathbf{R}^3 \setminus (0,0,0))) \cong C_0(\mathbf{R}) \otimes C_0(\mathbf{R}^3 \setminus (0,0,0)).$$

The extension (2.1.1) thus can be identified to the following one

$$0 \longrightarrow C_0(\mathbf{R}) \otimes I_1 \xrightarrow{\mathrm{id}\otimes k_1} C_0(\mathbf{R}) \otimes C_0(\mathbf{R}^3 \setminus (0,0,0)) \xrightarrow{\mathrm{id}\otimes v_1} C_0(\mathbf{R}) \otimes A_1 \longrightarrow 0.$$

Now, by using [13, Theorem 9.7 and Corollary 9.8] we obtain the assertion of Lemma 1.

LEMMA 2. *Set*

$$I_2 = C_0\left(\mathbf{S}^1 \setminus \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}\right),$$

where

$$\mathbf{S}^1 = \{e^{i\varphi},\ \varphi \in [0, 2\pi]\}.$$

Let

$$k_2 : I_2 \to C(\mathbf{S}^1), \quad v_2 : C(\mathbf{S}^1) \to C\left(\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}\right) \cong \mathbf{C}^4$$

be the inclusion and restriction given as in 2.1. *There is the following commutative diagram*

$$\begin{array}{ccccccccc}
\dots \longrightarrow & K_j(I_2) & \xrightarrow{k_{2*}} & K_jC(S^1) & \xrightarrow{r_{2*}} & K_j(C^1) & \longrightarrow & K_{j+1}(I_2) & \longrightarrow \dots \\
& \downarrow \beta_2 & & \downarrow \beta_2 & & \downarrow \beta_2 & & \downarrow \beta_2 & \\
\dots \longrightarrow & K_jC_0(V_2) & \longrightarrow & K_jC_0(W_1) & \longrightarrow & K_jC_0(W_2) & \longrightarrow & K_{j+1}C_0(V_2) & \longrightarrow \dots
\end{array}$$

where β_2 *is the Bott isomorphism,* $j \in \mathbf{Z}/2\mathbf{Z}$.

Proof. The proof is similar to that of Lemma 1, by using the exact sequence (2.1.2).

Before computing the K-groups, we need the following notations. Let $\eta : \mathbf{R} \to \mathbf{S}^1$ be the map

$$\eta(z) = e^{2\pi i\left(z/\sqrt{1+z^2}\right)}, \quad z \in \mathbf{R}.$$

Denote by u_+ (resp. u_-) the restriction of η on $\mathbf{R}_+$ (resp. $\mathbf{R}_-$). Note that the class $[u_+]$ (resp. $[u_-]$) is the canonical generator of $K_1C_0(\mathbf{R}_+) \cong \mathbf{Z}$ (resp. $K_1C_0(\mathbf{R}_-) \cong \mathbf{Z}$). Let us consider the matrix valued function $p : \mathbf{R}^2 \setminus (0,0) \to M_2(\mathbf{C})$ defined by:

$$p(x, y) = \frac{1}{2}\begin{pmatrix} 1 - \cos\pi\sqrt{x^2+y^2} & \dfrac{x+iy}{\sqrt{x^2+y^2}}\sin\pi\sqrt{x^2+y^2} \\ \dfrac{x-iy}{\sqrt{x^2+y^2}}\sin\pi\sqrt{x^2+y^2} & 1 + \cos\pi\sqrt{x^2+y^2} \end{pmatrix}.$$

Then p is an idempotent of rank 1 for each $(x, y) \in \mathbf{R}^2 \setminus (0,0)$. Let $[b] \in K_0C_0(\mathbf{R}^2)$ be the Bott element, [1] be the generator of $K_0C(\mathbf{S}^1) \cong \mathbf{Z}$, $[u]$ be the generator of $K_1C(\mathbf{S}^1) \cong \mathbf{Z}$ where $u = \mathrm{id}_{\mathbf{S}^1} \in C(\mathbf{S}^1)$. The vectors $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ form a basis of $K_0\mathbf{C}^4 = \mathbf{Z}^4$.

Let $\zeta \in C(\mathbf{S}^1)$ be the map $\zeta(e^{i\varphi}) = e^{4i\varphi}$, $\varphi \in [0, 2\pi]$ and u_1, u_2, u_3, u_4 be the restrictions of ζ on $(0, \pi/2)$, $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$ respectively. Then the class $[u_1]$ is a generator of $K_1C_0(0, \pi/2) \cong \mathbf{Z}$ and similarly for $[u_i]$, $i = 2, 3, 4$. The elements $[u_i]$, $i = 1, 2, 3, 4$ are, therefore, generators of $K_1(I_2)$,

$$K_1(I_2) = K_1C_0(0, \pi/2) \oplus K_1C_0(\pi/2, \pi) \oplus K_1C_0(\pi, 3\pi/2) \oplus K_1C_0(3\pi/2, 2\pi) \cong \mathbf{Z}^4$$

where I_2 is defined in Lemma 2.

LEMMA 3.

(i) $K_0C^*(V, \mathscr{F}) \cong \mathbf{Z}^2$; $K_1C^*(V, \mathscr{F}) = 0$,

(ii) $K_0(J_1) \cong Z^2$ *is generated by* $\varphi_0\beta_1([b] \boxtimes [u_+])$ *and* $\varphi_0\beta_1([b] \boxtimes [u_-])$; $K_1(J_1) = 0$,

(iii) $K_0(B_1) \cong Z$ *is generated by* $\varphi_0\beta_1([1] \boxtimes [u_+])$, $K_1(B_1) \cong Z$ *is generated by* $\varphi_1\beta_1([p] - [\varepsilon_1])$,
where φ_j, $j \in Z/2Z$, *is the Thom-Connes isomorphism (*see [3]*),* β_1 *is the isomorphism in Lemma* 1, ε_1 *is the constant matrix* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\boxtimes$ *is the external tensor product (*see, *for example,* [3, VI, 2]*).*

Proof. Consider the extension

$$0 \to C_0(\mathbf{R}^3 \setminus (0, 0, 0)) \to C_0(\mathbf{R}^3) \to \mathbf{C} \to 0.$$

Using the K-theoretical exact sequence associated to this extension. we obtain $K_0C_0(\mathbf{R}^3 \setminus (0, 0, 0)) = 0$ and $K_1C_0(\mathbf{R}^3 \setminus (0, 0, 0)) \cong Z^2$.

On the other hand, for $j \in Z/2Z$ we have

$$K_j(I_1) = K_jC_0(\mathbf{R}^2 \times \mathbf{R}^*) \cong K_j(C_0(\mathbf{R}^2) \otimes C_0(\mathbf{R}_+)) \oplus K_j(C_0(\mathbf{R}^2) \otimes C_0(\mathbf{R}_-)).$$

Thus, $K_0(I_1) = 0$ and $K_1(I_1) \cong Z^2$ is generated by $[b] \boxtimes [u_+]$ and $[b] \boxtimes [u_-]$ (see [3, Corollary VI. 3]).

Using the polar coordinates we can identify $\mathbf{R}^2 \setminus (0, 0)$ with $S^1 \times \mathbf{R}_+$, hence $A_1 = C_0(\mathbf{R}^2 \setminus (0, 0)) \cong C(S^1) \otimes C_0(\mathbf{R}_+)$. By the Künneth formula (see [11, Theorem 4.14]), $[u] \boxtimes [u_+]$ and $[1] \boxtimes [u_+]$ are generators of $K_0(A_1) \cong Z$ and $K_1(A_1) \cong Z$, respectively. Moreover, it follows from [3, Lemma VI.2] that $[u] \boxtimes [u_+] = [p] - [\varepsilon_1]$. Now the proof is complete by means of the isomorphisms β_1 and φ_j, $j \in Z/2Z$. ∎

LEMMA 4. (i) $K_0(B_1) \cong Z$ *is generated by* $\varphi_0\beta_2([1])$, $K_1(B_1) \cong Z$ *is generated by* $\varphi_1\beta_2([u])$,

(ii) $K_0(J_2) = 0$, $K_1(J_2) = Z^4$ *is generated by* $\varphi_1\beta_2([u_i])$, $i = 1, 2, 3, 4$,

(iii) $K_0(B_2) = Z^4$ *is generated by* $\varphi_0\beta_2([e_i])$, $i = 1, 2, 3, 4$, $K_1(B_2) = 0$,
where β_2 *is the Bott isomorphism in Lemma* 2 *and* φ_j, $j \in Z/2Z$, *is the Thom-Connes isomorphism (*see [3]*).*

Proof. The proof is straightforward because β_2 and φ_j are isomorphisms.

Proof of Theorem 2.

1. *Computation of* γ_1. Recall that the extension (γ_1) in Theorem 1 gives rise to a six-term exact sequence

$$\begin{array}{ccccc}
K_0J_1 & \longrightarrow & K_0C^*(V, \mathscr{F}) & \longrightarrow & K_0B_1 \\
{\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\
K_1B_1 & \longleftarrow & K_1C^*(V, \mathscr{F}) & \longleftarrow & K_1J_1 = 0.
\end{array}$$

By [11, Theorem 4.14], the isomorphism $\mathrm{Ext}(B_1, J_1) \cong \mathrm{Hom}(\mathrm{K}_1B_1, \mathrm{K}_0J_1) \cong \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}^2)$ associates the invariant $\gamma_1 \in \mathrm{Ext}(B_1, J_1)$ to the connecting map $\delta_1 : \mathrm{K}_1B_1 \to \mathrm{K}_0J_1$.

Since the Thom-Connes isomorphism commutes with K-theoretical exact sequence (see [14, Lemma 3.4.3]), we have the following commutative diagram ($j \in \mathbf{Z}/2\mathbf{Z}$):

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{K}_j(J_1) & \longrightarrow & \mathrm{K}_jC^*(V, \mathscr{F}) & \longrightarrow & \mathrm{K}_j(B_1) & \xrightarrow{\delta_j} & \mathrm{K}_{j+1}(J_1) & \longrightarrow \ldots \\
& \uparrow{\scriptstyle\varphi_j} & & \uparrow{\scriptstyle\varphi_j} & & \uparrow{\scriptstyle\varphi_j} & & \uparrow{\scriptstyle\varphi_{j+1}} & \\
\ldots \longrightarrow & \mathrm{K}_jC_0(V_1) & \longrightarrow & \mathrm{K}_jC_0(V) & \longrightarrow & \mathrm{K}_jC_0(W_1) & \xrightarrow{\delta_j} & \mathrm{K}_{j+1}C_0(V_1) & \longrightarrow \ldots .
\end{array}$$

In view of Lemma 1, the following diagram is commutative

$$\begin{array}{ccccccccc}
\ldots \to & \mathrm{K}_jC_0(V_1) & \longrightarrow & \mathrm{K}_jC_0(V) & \longrightarrow & \mathrm{K}_jC_0(W_1) & \xrightarrow{\delta_j} & \mathrm{K}_{j+1}C_0(V_1) & \to \ldots \\
& \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta_1} & & \uparrow{\scriptstyle\beta_1} & \\
\ldots \to & \mathrm{K}_{j-1}I_1 & \to & \mathrm{K}_{j-1}C_0(\mathbf{R}^3\setminus(0,0,0)) & \to & \mathrm{K}_{j-1}A_1 & \xrightarrow{\delta_{j-1}} & \mathrm{K}_jI_1 & \longrightarrow \ldots ,
\end{array}$$

($j \in \mathbf{Z}/2\mathbf{Z}$).

Consequently, instead of computing $\delta_1 : \mathrm{K}_1B_1 \to \mathrm{K}_0J_1$, it is sufficient to compute $\delta_0 : \mathrm{K}_0A_1 \to \mathrm{K}_1I_1$. Thus, by the proof of Lemma 3, we have to define $\delta_0([p] - [\varepsilon_1]) = \delta_0[p]$ because $\delta_0([\varepsilon_1]) = 0$. By the usual definition (see [13, p. 170]), for $[p] \in \mathrm{K}_0A_1$, $\delta_0([p]) = [\mathrm{e}^{2\pi i\tilde{p}}] \in \mathrm{K}_1I_1$ where $\tilde{p}$ is a preimage of p in (a matrix algebra over) $C_0(\mathbf{R}^3\setminus(0,0,0))$ i.e. $v_1\tilde{p} = p$.

We can choose $\tilde{p}(x, y, z) = \dfrac{z}{\sqrt{1+z^2}}\, p(x, y)$, $(x, y, z) \in \mathbf{R}^3\setminus(0,0,0)$.

Let $\tilde{p}_+$ (resp. $\tilde{p}_-$) be the restriction of $\tilde{p}$ on $\mathbf{R}^2 \times \mathbf{R}_+$ (resp. $\mathbf{R}^2 \times \mathbf{R}_-$). Then we have

$$\delta_0([p]) = [\mathrm{e}^{2\pi i\tilde{p}}] = [\mathrm{e}^{2\pi i\tilde{p}_+}] + [\mathrm{e}^{2\pi i\tilde{p}_-}] \in$$

$$\in \mathrm{K}_1(C_0(\mathbf{R}^2) \otimes C_0(\mathbf{R}_+)) \oplus \mathrm{K}_1(C_0(\mathbf{R}^2) \otimes C_0(\mathbf{R}_-)) = \mathrm{K}_1I_1 .$$

By [13, Section 4], for each function $f: \mathbf{R}_\pm \to Q_n\widetilde{C_0(\mathbf{R}^2)}$ such that $\lim\limits_{t\to\pm 0} f(t) = \lim\limits_{t\to\pm\infty} f(t)$, where $Q_n\widetilde{C_0(\mathbf{R}^2)} = \{a \in M_n\widetilde{C_0(\mathbf{R}^2)},\ \mathrm{e}^{2\pi i a} = \mathrm{Id}\}$, the class $[f] \in \mathrm{K}_1(C_0(\mathbf{R}^2) \otimes C_0(\mathbf{R}_\pm))$ can be determined by

$$[f] = W_f \cdot [b] \boxtimes [u_\pm],$$

where $W_f = \dfrac{1}{2\pi i} \displaystyle\int_{\mathbf{R}_\pm} \mathrm{Tr}(f'(z)f^{-1}(z))\mathrm{d}z$ is the winding number of f.

By simple computation, we get $\delta_0[p] = [b]_{+}^{-1}[u_+] + [b]_{-}^{-1}[u_-]$. Thus $\gamma_1 = (1, 1) \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}, \mathbf{Z}^2) \cong \mathbf{Z}^2$.

2. *Computation of* γ_2. The extension (γ_2) gives rise to a six-term exact sequence

$$\begin{array}{ccccc} 0 = \mathrm{K}_0 J_2 & \longrightarrow & \mathrm{K}_0 B_1 & \longrightarrow & \mathrm{K}_0 B_2 \\ \uparrow{\scriptstyle \delta_1} & & & & \downarrow{\scriptstyle \delta_0} \\ K_1 B_2 & \longleftarrow & \mathrm{K}_1 B_1 & \longleftarrow & \mathrm{K}_1 J_2 . \end{array}$$

By [11, Theorem 4.14], $\gamma_2 = \delta_0 \in \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_0 B_2, \mathrm{K}_1 J_2) = \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^4, \mathbf{Z}^1)$. Similarly to part 1, taking account of Lemma 2, we have the following commutative diagram ($j \in \mathbf{Z}/2\mathbf{Z}$):

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{K}_j J_2 & \longrightarrow & \mathrm{K}_j B_1 & \longrightarrow & \mathrm{K}_j B_2 & \xrightarrow{\delta_0} & \mathrm{K}_{j+1} J_2 & \longrightarrow & \ldots \\ & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_j} & & \uparrow{\scriptstyle \varphi_{j+1}} & & \\ \ldots & \longrightarrow & \mathrm{K}_j C_0(V_2) & \longrightarrow & \mathrm{K}_j C_0(W_2) & \longrightarrow & \mathrm{K}_j C_0(W_2) & \longrightarrow & \mathrm{K}_{j+1} C_0(V_2) & \longrightarrow & \ldots \\ & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \uparrow{\scriptstyle \beta_2} & & \\ \ldots & \longrightarrow & \mathrm{K}_j I_2 & \longrightarrow & \mathrm{K}_j C_0(\mathrm{S}^1) & \longrightarrow & \mathrm{K}_j \mathrm{C}^4 & \xrightarrow{\delta_0} & \mathrm{K}_{j+1} I_2 & \longrightarrow & \ldots . \end{array}$$

Thus we can compute $\delta_0 : \mathrm{K}_0 \mathrm{C}^4 \to \mathrm{K}_1 I_2$ instead of $\delta_0 : \mathrm{K}_0 B_2 \to \mathrm{K}_1 J_2$. First, we will compute $\delta_0[e_1]$. An element $\tilde{e}_1 \in C(\mathrm{S}^1)$ with the property that $v_2(\tilde{e}_1) = e_1$ is given by $\tilde{e}_1(e^{i\varphi}) = l_1(\varphi)$, where $l_1 \in C[0, 2\pi]$ is a function satisfying

$$\begin{cases} l_1(0) = l_1(2\pi) = 1 \\ l_1(\varphi) = 0 \quad \text{if } \varphi \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2} \right]. \end{cases}$$

Let $\tilde{e}_{11}$ (resp. $\tilde{e}_{12}$, $\tilde{e}_{13}$, $\tilde{e}_{14}$) be the restriction of $\tilde{e}_1$ on $(0, \pi/2)$ (resp. $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$).

We get

$$\delta_0([e_1]) = [e^{2\pi i \tilde{e}_1}] = \sum_{i=1}^{4} [e^{2\pi i \tilde{e}_{1i}}] \in \mathrm{K}_1(I_2).$$

It is immediate that $\tilde{e}_{12}$, $\tilde{e}_{13}$ are zero, so

$$[e^{2\pi i \tilde{e}_{12}}] = [e^{2\pi i \tilde{e}_{13}}] = 0.$$

By [3, Lemma VI. 5], we get

$$[e^{2\pi i \tilde{e}_{11}}] = w_{11}[u_1], \quad [e^{2\pi i \tilde{e}_{14}}] = w_{14}[u_4]$$

where w_{11} and w_{14} are the winding numbers given by

$$w_{11} = \frac{1}{2\pi i} \int_0^{\pi/2} \frac{d}{d\varphi} (e^{2\pi i \tilde{e}_{11}}) e^{-2\pi i \tilde{e}_{11}} d\varphi = -1$$

$$w_{14} = \frac{1}{2\pi i} \int_{3\pi/2}^{2\pi} \frac{d}{d\varphi} (e^{2\pi i \tilde{e}_{14}}) e^{-2\pi i \tilde{e}_{14}} d\varphi = 1.$$

Thus, $\delta_0([e_1]) = -[u_1] + [u_4]$.

Similarly, $\delta_0[e_i] = [u_{i-1}] - [u_i]$, $i = 2, 3, 4$. Then $\gamma_2 = \delta_0 \in \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}^4, \mathbf{Z}^4)$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

The proof is complete.

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