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SPECTRAL PROPERTIES OF LINEAR VOLTERRA OPERATORS

BRUCE A. BARNES

INTRODUCTION

Let T be a bounded linear Volterra integral operator acting on some reasonable Banach space of functions. It seems probable that the following list of properties holds for T:

(1) The spectrum of T is connected;
(2) The Weyl spectrum of T is connected;
(3) If $\lambda \in \mathbf{C}$ and $\lambda - T$ is Fredholm, then $\mathrm{ind}(\lambda - T) \leqslant 0$.

In this paper some of these properties are verified for certain Volterra operators on L^p-spaces. Also, although this property does not hold for all Volterra operators, it is shown that for a large class of Volterra operators

(4) T has no nonzero eigenvalues.

The basic situations considered here are as follows. First, let $I = [0, 1]$ or $I = [0, \infty)$, and $\gamma(t)$ be a continuous nonnegative weight function on $I = [0, \infty)$, or on $[0, 1)$ when $I = [0, 1]$. Let $K(x, t)$ be a Volterra kernel on $I \times I$, that is $K(x, t)$ is a measurable function with $K(x, t) = 0$ whenever $t > x$. We consider Volterra operators of the form

$$T(f)(x) = \int_0^x K(x, t) f(t) \gamma(t) \, \mathrm{d}t \quad (f \in L^p(I, \gamma))$$

with the assumption that T is a bounded operator on L^p. Under certain hypotheses concerning K, it is shown that (1)—(4) hold. The second situation considered here is when the underlying Banach space is a sequence space $\ell^p(\gamma)$, where $\{\gamma(n)\}$ is some positive sequence of weights, and T is the Volterra operator determined by a lower triangular matrix kernel $K(n, m)$, $K(n, m) = 0$ if $m \geqslant n$. It is shown that in general (1)—(4) hold for T.

When the underlying space is Hilbert space, Volterra operators can be studied using the extensive theory of nest algebras: see K. Davidson's book [6] for a complete account. The methods used in this paper are largely single-operator methods, although some Banach algebra theory is used as a tool.

At this point we set some notation. If X is a Banach space, then $\mathscr{B}(X)$ is the algebra of all bounded linear operators on X, and $\mathscr{K}(X)$ is the space of all compact linear operators on X. Also, $\Phi(X)$ and $\Phi^0(X)$ denote the set of Fredholm operators and the set of Fredholm operators of index zero, respectively. For $T \in \mathscr{B}(X)$:

$\mathscr{R}(T)$ denotes the range of T;

$\sigma(T)$ is the usual spectrum of T in $\mathscr{B}(X)$;

$\omega(T) = \{\lambda \in \mathbf{C} : \lambda - T \notin \Phi(X)\}$;

$W(T) = \{\lambda \in \mathbf{C} : \lambda - T \notin \Phi^0(X)\}$;

$\Phi^-(T) = \{\lambda \in \mathbf{C} : \lambda - T \in \Phi(X)$ with $\operatorname{ind}(\lambda - T) < 0\}$.

$\sigma_{\mathscr{B}}(b)$ denotes the spectrum of an element b relative to a Banach algebra $\mathscr{B}$.

1. BASIC TOOLS

In this section we develop the basic results concerning spectral properties of operators which we later apply to Volterra operators. The first is certainly known, but is presented here for convenience.

PROPOSITION 1. *Assume $\{T_n\} \subseteq \mathscr{B}(X)$ with $T_n \to T$ in operator norm.*

(1) *If $\sigma(T_n)$ is connected for all n, then $\sigma(T)$ is connected;*

(2) *if $\omega(T_n)$ is connected for all n, then $\omega(T)$ is connected;*

(3) *if for each n, T_n has the property that whenever $\lambda \neq 0$ and $\lambda - T_n \in \Phi(X)$, then $\operatorname{ind}(\lambda - T_n) \leqslant k$, then T has this same property;*

(4) *if $\sigma(T_n) = \{0\}$ for all n, then $\sigma(T)$ is connected, $\omega(T)$ is connected, and whenever $\lambda \neq 0$ and $\lambda - T \in \Phi(X)$, then $\operatorname{ind}(\lambda - T) = 0$. Therefore in this case $\omega(T) = W(T)$.*

Proof. Let $\mathscr{B}$ be a Banach algebra with unit, and assume $\{b_n\} \subseteq \mathscr{B}$ and $b_n \to b$ in $\mathscr{B}$. Let Λ be a nonemtpty open and closed subset of $\sigma_{\mathscr{B}}(b)$, and let U be an open set with $\Lambda \subseteq U$. By [10, Theorem 2, p. 168] it follows that $\sigma_{\mathscr{B}}(b_n) \cap U$ is nonempty for all n sufficiently large. Therefore if $\sigma_{\mathscr{B}}(b)$ is disconnected, then $\sigma_{\mathscr{B}}(b_n)$ is disconnected for all n sufficiently large. Both (1) and (2) are consequences of this argument.

Now assume each T_n has the propety in (3). Suppose $\lambda \neq 0$ and $\lambda - T \in \Phi(X)$. Then for all n sufficiently large $\lambda - T_n \in \Phi(X)$, and so by hypothesis $\operatorname{ind}(\lambda - T_n) \leqslant k$. Thus $\operatorname{ind}(\lambda - T) \leqslant k$ by the continuity of the index [12, Theorem 3.2, p. 115].

(4) follows from (1), (2), and the continuity of the index.

The next theorem is the main tool applied to prove spectral properties of Volterra operators.

THEOREM 2. *Let X be a Banach space. Assume Y is a dense subspace of X, Y is a Banach space, and the embedding of Y into X is continuous. Assume $T \in \mathscr{B}(Y)$ has an extension $\overline{T} \in \mathscr{B}(X)$ with $\sigma(\overline{T}) = \{0\}$. Then*

(1) *$\sigma(T)$ is connected;*

(2) *T has no nonzero eigenvalues;*

(3) *$\sigma(T) = W(T) = \omega(T) \cup \Phi^{-}(T)$.*

Proof. Let $\mathscr{B}$ be the algebra of all $T \in \mathscr{B}(Y)$ which have an extension $\overline{T} \in \mathscr{B}(X)$. $\mathscr{B}$ is a Banach algebra which is studied in [1] and [4]. Now assume $T \in \mathscr{B}$ and that $\sigma(\overline{T}) = \{0\}$. Consider $\psi: \mathscr{B} \to \mathscr{B}(X)$ given by $\psi(S) = \overline{S} \in \mathscr{B}(X)$. Then ψ is a continuous unital algebra monomorphism of $\mathscr{B}$ into $\mathscr{B}(X)$. By [3, Theorem 4.5] every component of $\sigma_{\mathscr{B}}(T)$ has nonempty intersection with $\sigma(\psi(T)) = \sigma(\overline{T}) = \{0\}$. Therefore $\sigma_{\mathscr{B}}(T)$ must be connected. By [1, Theorem 5.2] $\sigma_{\mathscr{B}}(T) = \sigma(T) \cup \sigma(\overline{T}) = \sigma(T) \cup \{0\}$. Since $\sigma_{\mathscr{B}}(T)$ is connected, it follows that $0 \in \sigma(T)$ and $\sigma(T)$ is connected.

The proof of (2) is clear; T has no nonzero eigenvalue since $\overline{T}$ has no nonzero eigenvalue.

To prove (3), note that it is always true that

$$\sigma(T) \supseteq W(T) \supseteq \omega(T) \cup \Phi^{-}(T).$$

If $\lambda \in \sigma(T)$, $\lambda \neq 0$, and $\lambda \notin \omega(T) \cup \Phi^{-}(T)$, then $\lambda - T \in \Phi(X)$ and $\mathrm{ind}(\lambda - T) \geqslant 0$. In this case λ must be an eigenvalue of T, contradicting (2). Therefore

$$\sigma(T) \setminus \{0\} = (\omega(T) \cup \Phi^{-}(T)) \setminus \{0\}. \tag{4}$$

As noted in the proof of (1), $0 \in \sigma(T)$. Suppose $0 \notin \omega(T) \cup \Phi^{-}(T)$. Then $T \in \Phi(X)$ and $\mathrm{ind}(T) \geqslant 0$. Therefore $\exists \varepsilon > 0$ such that for $|\lambda| < \varepsilon$, $\lambda - T \in \Phi(X)$ and $\mathrm{ind}(\lambda - T) \geqslant 0$ [12, Theorem 3.2, p. 115]. This fact and (4) imply that 0 is an isolated point of $\sigma(T)$. Therefore by (1) $\sigma(T) = \{0\}$, so $0 \in \omega(T)$ a contradiction.

Let μ be a fixed regular Borel measure on the interval $I = [0, \infty)$ or on $I = [0, 1]$. Also assume μ is continuous in the sense $\mu(\{b\}) = 0$ for all $b \in I$. Assume X is some Banach space of μ-measurable functions on I. For $a > 0$, let

$$M_a = \{f \in X : f = 0 \quad \mu\text{-a.e. on } [0, a]\}.$$

We assume that M_a is closed in X for all $a > 0$. Set

$$\mathscr{A} = \{T \in \mathscr{B}(X) : T(M_a) \subseteq M_a \text{ for all } a > 0\}.$$

Then $\mathscr{A}$ is a closed subalgebra of $\mathscr{B}(X)$. For $T \in \mathscr{A}$, $a > 0$, let $T_a \in \mathscr{B}(M_a)$ be the restriction of T to M_a.

PROPOSITION 3. *Assume* $T \in \mathscr{A}$.

(1) $\sigma_{\mathscr{A}}(T) = (\bigcup_{a>0} \sigma(T_a)) \cup \sigma(T)$;

(2) $\partial\sigma_{\mathscr{A}}(T) \subseteq \sigma(T) \subseteq \sigma_{\mathscr{A}}(T)$;

(3) *Assume* $\mathrm{span}\{M_a : a > 0\}$ *is dense in* X. *If* $\sigma_{\mathscr{A}}(T)$ *is disconnected, then* $\sigma(T_a)$ *is disconnected for some* $a > 0$.

Proof. To prove (1) we characterize when $T \in \mathscr{A}$ is invertible in $\mathscr{A}$. If $T \in \mathscr{A}$ is invertible in $\mathscr{A}$, then it follows easily that T is invertible in $\mathscr{B}(X)$ and T_a is invertible in $\mathscr{B}(M_a)$ for all $a > 0$. Conversely, assume $S = T^{-1}$ exists in $\mathscr{B}(X)$ and T_a is invertible for all $a > 0$. If $f \in M_a$, then $f = T_a(g) = T(g)$ for some $g \in M_a$. Therefore $S(f) = g \in M_a$. This proves $S(M_a) \subseteq M_a$ for all $a > 0$, so $S \in \mathscr{A}$ by definition. Then (1) follows directly from this characterization of invertibility in $\mathscr{A}$.

(2) follows from general Banach algebra theory [5, Proposition 12, p. 25].

Now assume $\sigma_{\mathscr{A}}(T)$ is disconnected, so

$$\sigma_{\mathscr{A}}(T) = \sigma_1 \cup \sigma_2 \subseteq U_1 \cup U_2$$

where σ_1 and σ_2 are nonempty disjoint compact sets and U_1 and U_2 are disjoint open sets with $\sigma_k \subseteq U_k$, $k = 1, 2$. For $k = 1, 2$, choose γ_k a cycle in U_k surrounding σ_k and satisfying the usual requirements so that

$$E_k = \frac{1}{2\pi i} \int_{\gamma_k} (\lambda - T)^{-1} d\lambda$$

is the nonzero spectral projection corresponding to the spectral set σ_k. Thus, $I = E_1 + E_2$, $E_1E_2 = E_2E_1 = 0$, and $E_1, E_2 \in \mathscr{A}$. Since $\mathrm{span}\{M_a : a > 0\}$ is dense in X, there exists $a > 0$ such that $(E_1)_a \neq 0$ and $(E_2)_a \neq 0$. Then $\sigma(T_a) \subseteq \sigma_{\mathscr{A}}(T) \subseteq U_1 \cup U_2$, and for $k = 1, 2$

$$(E_k)_a = \frac{1}{2\pi i} \int_{\gamma_k} (\lambda - T_a)^{-1} d\lambda.$$

It follows that $\sigma(T_a) \cap U_1$ and $\sigma(T_a) \cap U_2$ must both be nonempty.

COROLLARY 4. *Assume* $T \in \mathscr{A}$ *and* $\sigma(T_a) = \{0\}$ *for all* $a > 0$. *Assume* $\mathrm{span}\{M_a : a > 0\}$ *is dense in* X. *Then*

(1) $\sigma(T)$ *is connected;*

(2) *For all* $\lambda \neq 0$, $\mathscr{R}(\lambda - T)$ *is dense in* X. *Therefore if* $\lambda - T \in \Phi(X)$, *then* $\mathscr{R}(\lambda - T) = X$.

Proof. (1) follows directly from (1) and (3) of Proposition 3.

Now assume $\lambda \neq 0$. Since $(\lambda - T_a)(M_a) = M_a$ for all $a > 0$, it follows that $\operatorname{span}\{M_a : a > 0\} \subseteq \mathscr{R}(\lambda - T)$. This proves (2).

We need the following notation in the next theorem. Assume $I = [0, 1]$ or $I = [0, \infty)$, and let $X = L^p(I, \mu)$ where μ is a continuous measure on I. As before, for $a > 0$ let

$$M_a = \{f \in X : f(x) = 0 \quad \mu\text{-a.e. on } [0, a]\}.$$

Also, for $f \in X$ and $g \in L^q(I, \mu)$ where $p^{-1} + q^{-1} = 1$, set $\langle g, f\rangle = \int_I fg \, d\mu$. When $X = \ell^p(\gamma)$ where $\{\gamma(n)\}$ is a sequence of positive weights, for $n \geqslant 1$ let

$$J_n = \{\{b_k\} \in X : b_k = 0 \text{ for } 1 \leqslant k \leqslant n\}.$$

THEOREM 5. *Assume $T \in \mathscr{K}(X)$ and $T \in \mathscr{A}$ where either:*

(i) *$X = L^p(I, \mu)$ where μ is a continuous regular Borel measure, $1 \leqslant p < \infty$, and $\mathscr{A} = \{T \in \mathscr{B}(X) : T(M_a) \subseteq M_a$ for all $a > 0\}$; or*

(ii) *$X = \ell^p(\gamma)$ where $\{\gamma(n)\}$ is a sequence of positive weights, $1 \leqslant p < \infty$ and $\mathscr{A} = \{T \in \mathscr{B}(X) : T(J_n) \subseteq J_{n+1}$ for all $n \geqslant 1\}$.*

Then $\sigma(T) = \{0\}$.

Proof. We prove the theorem in the case where (i) holds; the proof when (ii) holds is similar.

Assume that (i) holds. Then it is easy to establish that X has the properties:

(1) $M_b = \bigcap\{M_a : 0 < a < b\}$ for all $b > 0$;

(2) if $g \in L^q(I, \mu)$, $b > 0$, and $\langle g, f\rangle = 0$ for all $f \in M_a$ whenever $b < a$, then $\langle g, f\rangle = 0$ for all $f \in M_b$.

Now suppose that $\lambda_0 \in \sigma(T)$ and $\lambda_0 \neq 0$. Let E be the spectral projection corresponding to the spectral set $\{\lambda_0\}$. Note that $\sigma_{\mathscr{A}}(T) = \sigma(T)$, so that $E \in \mathscr{A}$. Also $\mathscr{R}(E)$ is finite dimensional. When $0 < a < b$, then $\dim(\mathscr{R}(E) \cap M_a) \geqslant \dim(\mathscr{R}(E) \cap M_b)$. Let m be the smallest positive integer such that

$$m = \dim(\mathscr{R}(E) \cap M_a)$$

for some $a > 0$. Note that if $m = \dim(\mathscr{R}(E) \cap M_a)$ and $b > a$, then either $m = \dim(\mathscr{R}(E) \cap M_b)$ or $\mathscr{R}(E) \cap M_b = \{0\}$. Now $\{a > 0 : \dim(\mathscr{R}(E) \cap M_a) = m\}$ has an upper bound. Let d be the least upper bound of this set. Fix c such that $0 < c < d$ and $\dim(\mathscr{R}(E) \cap M_a) = m$ for $c \leqslant a < d$. Let $\{f_1, \ldots, f_m\}$ be a basis for $\mathscr{R}(E) \cap M_c$. Then $\{f_1, f_2, \ldots, f_m\} \subseteq \mathscr{R}(E) \cap M_a$ whenever $0 < a < d$, so by (1) $\{f_1, \ldots, f_m\} \subseteq M_d$. Also, by the definition of d, $\mathscr{R}(E) \cap M_a = \{0\}$ for $a > d$.

Let E_d be the restriction of E to M_d. Then there exist $g_k \in L^p(I, \mu)$, $1 \leqslant k \leqslant m$, such that

$$E_d(f) = \sum_{k=1}^{m} \langle g_k, f \rangle f_k \quad (f \in M_d).$$

If $f \in M_a$ with $a > d$, then

$$0 = E_d(f) = \langle g_1, f \rangle f_1 + \dots + \langle g_m, f \rangle f_m.$$

Therefore $\langle g_k, f \rangle = 0$ for $1 \leqslant k \leqslant m$. It follows from (2) that $\langle g_k, f \rangle = 0$ for all $f \in M_d$, $1 \leqslant k \leqslant m$. But then $\langle g_j, f_k \rangle = 0$ whenever $1 \leqslant k, j \leqslant m$. This implies $E_d^2 = 0$, a contradiction.

Theorem 5 may be derived from the more general result [11, Theorem 4.3.10]. However, the proof given here has the virtue of being completely elementary.

By the previous theorem compact Volterra operators have zero spectrum. One of the basic results we prove in the next two sections is a generalization of this: there are large classes of Volterra operators for which the spectrum is connected. In the noncompact case there are numerous examples of Volterra operators of the types I and II described just before Theorem 8 which do not have zero spectrum. The same is true for the strictly lower triangular operators on sequence spaces considered in Section 3 (the unilateral shift on ℓ^2, for example).

2. VOLTERRA OPERATORS ON $L^p(\gamma)$

Throughout this section $I = [0, 1]$ or $I = [0, \infty)$ and $\gamma(t)$ is a nonnegative continuous weight function on $[0, 1)$ when $I = [0, 1]$, and on all of $I = [0, \infty)$ in that case. Denote by $\mathscr{U}_p(\gamma)$ the set of all measurable kernels $K(x, t)$ on $I \times I$ such that $K(x, t) = 0$ whenever $x > t$ and the integral operator

$$\operatorname{Int}(K)(f)(x) = \int_I K(x, t) f(t) \gamma(t)\, dt$$

is a bounded operator on $L^p(\gamma)$.

Many of the results of this section involve a weak boundedness condition on K which we now define. When J is a subset of I, we use the notation χ_J for the characteristic function of J.

Definition. A kernel $K(x, t)$ in $I \times I$ is *locally bounded* if the kernel $K(x, t)\chi_{[0,a]}(x)\chi_{[0,a]}(t)$ is bounded whenever $0 < a < 1$ when $I = [0, 1]$, and for all $a > 0$ when $I = [0, \infty)$.

It is not difficult to verify that when $K(x, t)$ is locally bounded on I, then there exist $k(x)$ and $h(t)$ positive continuous functions on $I = [0, \infty)$, or on $[0, 1)$ when $I = [0, 1]$, such that

$$|K(x, t)| \leqslant k(x)h(t) \quad \text{for all } x, t \in [0, \infty), \text{ or for all } x, t \in [0, 1) \text{ when } I = [0, 1].$$

The next two theorems are the main results of this section. It is always assumed that $p \geqslant 1$ and $p^{-1} + q^{-1} = 1$. If Ω is a topological space, then $C(\Omega)$ denotes the space of all complex-valued *bounded* continuous functions on Ω.

THEOREM 6. *Fix* p, $1 < p \leqslant \infty$, *and assume* $K \in \mathscr{U}_p(\gamma)$. *Set* $T = \mathrm{Int}(K) \in \mathscr{B}(L^p(\gamma))$.

(1) *If* K *is locally bounded, then:*

(i) $\sigma(T)$ *is connected;*

(ii) T *has no nonzero eignevalues;*

(iii) $\sigma(T) = W(T) = \omega(T) \cup \Phi^-(T)$.

(2) *If* $\{K_n\} \subseteq \mathscr{U}_p(\gamma)$, *each* K_n *is locally bounded, and* $\|T - \mathrm{Int}(K_n)\| \to 0$, *then parts* (i) *and* (iii) *above hold for* T.

Proof. We do the proof in the case where $I = [0, \infty)$; the proof when $I = [0, 1]$ is the same. Assume K is locally bounded on I. As noted above there exist $k(x)$ and $h(t)$ continuous positive functions on I such that

(a) $|K(x, t)| \leqslant k(x)h(t)$ for all $x, t \in I$.

First assume $1 < p < \infty$. Define $v(x)$ on I by

$$v(x) = [(1 + k^p(x))(1 + \gamma(x))(1 + x^2)]^{-1}.$$

Note that $v(x)$ has the following properties:

(b) $v(x) \in C(I)$;

(c) $v(t)^{-1} \in C([0, x])$ for all $x \in I$;

(d) $v(x)k^p(x) \in L^1(\gamma)$.

Also, define

$$w(x) = \int_0^x h(\ )^q v(t)^{1-q} \gamma(t)\, \mathrm{d}t$$

for all $x \in I$. Finally, let

$$\rho(x) = v(x)\mathrm{e}^{-w(x)} \quad (x \in I).$$

Then $\rho(x)$ is a bounded continuous weight function on I.

Next we prove

$$\text{(e)} \quad J = \int_0^\infty \left[\int_0^\infty |K(x, t)|^q \rho(t)^{1-q} \gamma(t)\, \mathrm{d}t \right]^{p/q} \rho(x)\gamma(x)\, \mathrm{d}x < \infty.$$

Using (a) and the definition of w, we have

$$J \leqslant \int_0^\infty \left[\int_0^x k^q(x) h^q(t) \rho^{1-q}(t) \gamma(t)\, \mathrm{d}t \right]^{p/q} \rho(x)\gamma(x)\, \mathrm{d}x =$$

$$= \int_0^\infty k^p(x) v(x) \mathrm{e}^{-w(x)} \left[\int_0^x h^q(t) v^{1-q}(t) \mathrm{e}^{(q-1)w(t)} \gamma(t)\, \mathrm{d}t \right]^{p/q} \gamma(x)\, \mathrm{d}x =$$

$$= \int_0^\infty k^p(x) v(x) \mathrm{e}^{-w(x)} [(q-1)^{-1}(\mathrm{e}^{(q-1)w(x)} - 1)]^{p/q} \gamma(x)\, \mathrm{d}x \leqslant$$

$$\leqslant (q-1)^{-p/q} \int_0^\infty k^p(x) v(x) \mathrm{e}^{-w(x)} \mathrm{e}^{(q-1)(p/q)w(x)} \gamma(x)\, \mathrm{d}x =$$

$$= (q-1)^{-p/q} \int_0^\infty k^p(x) v(x) \gamma(x)\, \mathrm{d}x.$$

This last integral is finite by (d). This proves $J < \infty$ as claimed in (e).

By (b) ρ is a bounded continuous function on I. Therefore $L^p(\gamma)$ is continuously and densely embedded as a subspace of $L^p(\rho\gamma)$. Define a kernel $\bar{K}$ by

$$\bar{K}(x,\ t) = K(x,\ t)\rho(t)^{-1}.$$

Then

$$J = \int_0^\infty \left[\int_0^\infty |\bar{K}(x,\ t)|^q \rho(t)\gamma(t)\, \mathrm{d}t \right]^{p/q} \rho(x)\gamma(x)\, \mathrm{d}x.$$

Therefore by (e), $\bar{K}$ determines a Hellinger-Tamarkin operator on $L^p(\gamma\rho)$. Let $\bar{T} = \mathrm{Int}(\bar{K})$ on $L^p(\gamma\rho)$. Then $\bar{T}$ is a Volterra operator and $\bar{T}$ is compact [9, Theorem 11.6, p. 275], so $\sigma(\bar{T}) = \{0\}$ by Theorem 5. Furthermore, $\bar{T}$ is an extension of T since for $f \in L^p(\gamma)$,

$$\bar{T}(f)(x) = \int_0^x \bar{K}(x, t) f(t)\rho(t)\gamma(t)\, \mathrm{d}t = \int_0^x K(x, t) f(t)\gamma(t)\, \mathrm{d}t = T(f)(x).$$

Therefore part (1) follows from Theorem 2.

Now assume $p = \infty$. Repeat the proof of the construction of $\rho(x)$ and $\bar{K}(x, t)$ just as above in the case where $p = q = 2$. Note that $\rho(x)\gamma(x)\,dx$ is a finite measure on $[0, \infty)$. Therefore $L^\infty[0,\infty)$ is continuously and densely embedded in $L^2(\rho\gamma)$. Also, $\bar{T} = \mathrm{Int}(\bar{K})$ is Hilbert-Schmidt on $L^2(\rho\gamma)$. Thus, (i)—(iii) follow just as before.

The properties in (2) follow from (1) and Proposition 1.

Theorem 6 holds when K is a Volterra kernel with $T = \mathrm{Int}(K) \in \mathscr{B}(C(I))$. For the construction in the proof of Theorem 6 for the case $p = \infty$ works equally as well with $C(I)$ as underlying Banach space in place of $L^\infty(I)$.

THEOREM 7. *Let* $I = [0,\infty)$. *Assume* $K \in \mathscr{U}_1(\gamma)$, *and set* $T = \mathrm{Int}(K) \in \mathscr{B}(L^1(\gamma))$. *Then*

(1) *If K is essentially bounded, then*

(i) $\sigma(T)$ *is connected;*

(ii) T *has no nonzero eigenvalues;*

(iii) $\sigma(T) = W(T) = \omega(T) \cup \Phi^-(T)$.

(2) *If* $\{K_n\} \subseteq \mathscr{U}_1(\gamma)$, *each* K_n *is essentially bounded, and* $\|T - \mathrm{Int}(K_n)\| \to 0$, *then parts* (i) *and* (iii) *above hold for* T.

Proof. First assume that K is essentially bounded. In this case

$$T(L^1(\gamma)) \subseteq L^\infty \cap L^1(\gamma) \subseteq L^2(\gamma).$$

Set $Y = L^\infty \cap L^1(\gamma)$, and let T_r be the restriction of T to Y, so $T_r \in \mathscr{B}(Y)$. By [2, Theorem 4] if T_r has the properties (i) — (iii), then T does also. Let $k(x) = h(t) = \|K\|_\infty^{1/2}$. Now repeat the same construction of ρ and $\bar{K}$ as in the proof of Theorem 6 using $p = q = 2$. Then $\bar{K}$ is a Hilbert-Schmidt kernel on $L^2(\rho\gamma)$. Also, Y is a continuously and densely embedded subspace of $L^2(\rho\gamma)$, and $\bar{T}_r = \mathrm{Int}(\bar{K})$ on $L^2(\rho\gamma)$ is an extension of T_r. Thus, Theorem 2 implies that T_r satisfies (i)—(iii), so as noted above, the same is true of T. This proves (1).

Part (2) follows from (1) and Proposition 1.

There are two types of Volterra kernels of special interest to which we shall apply the ideas and results of this section. One of these classes of kernels are those of the form:

(I) $K(x, t) = H(x, t)\, k(x - t)\chi_{[0,x]}(t)$ where $k \in L^1(I)$ and $H \in L^\infty(I \times I)$.

The Volterra operators determined by kernels of type (I) are considered in Theorem 11. At present we prove several results concerning operators with kernels of the following type:

(II) $K(x, t) = \left(\sum_{j=1}^{m} \psi_j(x)\varphi_j(t)\right)\chi_{[0,x]}(t)$ where ψ_j and φ_j are measurable functions on I, $1 \leqslant j \leqslant m$, and $K \in \mathscr{U}_p(I)$.

When for each j, ψ_j and φ_j satisfy some weak boundedness properties, then K will be locally bounded (or essentially bounded) so Theorem 6 (or Theorem 7) will apply..

In the next two results we consider kernels of type (II) where the functions φ_j and ψ_j satisfy some integrability conditions.

THEOREM 8. *Let* $I = [0, \infty)$. *Fix* p, $1 < p < \infty$. *Assume* $\psi_j \in L^p(\gamma)$ *and* $\varphi_j \in L^q([0, x], \gamma)$ *whenever* $x > 0$, *for* $1 \leqslant j \leqslant m$. *Assume*

$$K(x, t) = \left(\sum_{j=1}^{m} \psi_j(x)\varphi_j(t)\right)\chi_{[0,x]}(t) \in \mathscr{U}_p(\gamma).$$

Set $T = \mathrm{Int}(K) \in \mathscr{B}(L^p(\gamma))$.

(i) $\sigma(T)$ *is connected;*

(ii) T *has no nonzero eigenvalues;*

(iii) $\sigma(T) = W(T) = \omega(T) \cup \Phi^-(T)$.

Proof. Let

$$\varphi(t) = \sup\{|\varphi_j(t)| : 1 \leqslant j \leqslant m\}, \quad \text{and} \quad \psi(x) = \sum_{j=1}^{m} |\psi_j(x)|.$$

Note that $\psi \in L^p(\gamma)$ and $\varphi \in L^q([0, x], \gamma)$ whenever $x > 0$.

Assume $1 < p < \infty$. The proof of the result proceeds along the same lines as the proof of Theorem 6. Let

$$w(x) = \int_0^x \varphi(t)^q \gamma(t)\mathrm{d}t,$$

and let $\rho(x) = \mathrm{e}^{-w(x)}$. Define J just as in the proof of Theorem 6. Then

$$J \leqslant \int_0^\infty \left[\int_0^x \psi(x)^q \varphi(t)^q \rho(t)^{1-q} \gamma(t)\,\mathrm{d}t\right]^{p/q} \rho(x)\gamma(x)\,\mathrm{d}x =$$

$$= \int_0^\infty \psi(x)^p \left[\int_0^x \varphi(t)^q \mathrm{e}^{(q-1)w(t)} \gamma(t)\,\mathrm{d}t\right]^{p/q} \rho(x)\gamma(x)\,\mathrm{d}x.$$

Applying [13, 11.54, p. 361]

$$\int_0^x \varphi(t)^q e^{(q-1)w(t)}\gamma(t)\, dt \leqslant (q-1)^{-1}(e^{(q-1)w(x)} - 1).$$

Therefore

$$J \leqslant \int_0^\infty \psi(x)^p[(q - 1)^{-1}(e^{(q-1)w(x)} - 1)]^{p/q} e^{-w(x)}\gamma(x)\, dx \leqslant$$

$$\leqslant (q - 1)^{-p/q} \int_0^\infty \psi(x)^p \gamma(x)\, dx < \infty.$$

From this point the proof proceeds exactly as the proof of Theorem 6.

THEOREM 9. *Fix p, $1 \leqslant p < \infty$. Assume for $1 \leqslant j \leqslant m$ that*

(1) $\varphi_j \in L^q(\gamma)$; *and*

(2) $\psi_j \in L^p([a, \infty), \gamma)$ *for all* $a > 0$ *when* $I = [0, \infty)$, *or* $\psi_j \in L^p([a, 1], \gamma)$ *for all* $a \in (0, 1)$ *when* $I = [0, 1]$.

Further, assume

$$K(x, t) = \left(\sum_{j=1}^m \psi_j(x)\varphi_j(t)\right)\chi_{[0,x]}(t) \in \mathscr{U}_p(\gamma).$$

Set $T = \mathrm{Int}(K) \in \mathscr{B}(L^p(\gamma))$. Then $\sigma(T)$ is connected, and for all $\lambda \neq 0$, $\mathscr{R}(\lambda - T)$ is dense in $L^p(\gamma)$.

Proof. Let M_a and T_a be as in the set-up for Theorem 3. Hypotheses (1) and (2) imply that T_a is a compact Volterra operator on $L^p(M_a, \gamma)$ for all a. Therefore $\sigma(T_a) = \{0\}$ for all a, and the result follows from Corollary 4.

Before considering kernels of type (I) we need a result concerning some special kernels. The space of weakly compact linear operators on a Banach space X will be denoted by $\mathscr{W}(X)$. Two important facts we use concerning this space are first, $\mathscr{W}(X)$ is a closed ideal in $\mathscr{B}(X)$ [7, Corollary 4, p. 483, and Theorem 5, p. 484], and secondly, in the case where $T \in \mathscr{W}(X)$ with $X = L^p$ for $p = 1$ or $p = \infty$, $T^2 \in \mathscr{K}(X)$ [7, Corollary 13, p. 510, and Corollary 5, p. 494].

PROPOSITION 10. *Let (Ω, μ) be a σ-finite measure space. Assume $K \in L^\infty(\Omega \times \Omega)$ and A and B are subsets of Ω having finite measure such that*

$$K(x, t) = 0 \quad \text{if } (x, t) \notin A \times B.$$

For $1 \leqslant p \leqslant \infty$, *let* $T_p = \operatorname{Int}(K)$ *acting on* $L^p(\Omega)$.

(1) $T_p \in \mathscr{K}(L^p)$, $1 < p < \infty$;

(2) $T_p \in \mathscr{W}(L^p)$, $p = 1, \infty$.

Proof. If $1 < p < \infty$, then T_p is a Hellinger-Tamarkin operator and hence compact.

Now assume $p = 1$. Let S_1 be the unit ball of $L^1(\Omega)$, and set $J = T_1(S_1)$. For all $f \in J$, $E \subseteq \Omega$, E measurable,

$$\left|\int_E f\,d\mu\right| \leqslant \int_{E\cap A}\left[\int_B |K(x,t)|\,|f(t)|\,d\mu(t)\right]d\mu(x) \leqslant \left(\int_{E\cap A} d\mu\right)\|K\|_\infty .$$

Therefore

$$\lim_{\mu(E\cap A)\to 0}\left(\int_{E\cap A} f\,d\mu\right) = 0 \quad \text{uniformly for } f\in J.$$

Noting that all the functions $f \in J$ are zero off A, it follows that J is a weakly sequentially compact subset of $L^1(\Omega)$ by [7, Corollary 11, p. 294]. Therefore $T_1 \in \mathscr{W}(L^1(\Omega))$.

For the case $p = \infty$, let $\tilde{K}(x, t) = K(t, x)$ for $x, t \in \Omega$. Set $(\tilde{T})_1 = \operatorname{Int}(\tilde{K}) \in \mathscr{W}(L^1(\Omega))$ by the previous argument. But T_∞ is the adjoint of $(\tilde{T})_1$, so by [7, Theorem 8, p. 485], $T_\infty \in \mathscr{W}(L^\infty(\Omega))$.

Now we consider spectral properties of $\operatorname{Int}(K)$ when K is a kernel of type (I)

THEOREM 11. *Assume I has Lebesgue measure ($\gamma(t) \equiv 1$). Assume*

$$K(x, t) = H(x, t)k(x - t)\chi_{[0,x]}(t)$$

where $H \in L^\infty(I \times I)$ *and* $k \in L^1(I)$. *For* $1 \leqslant p \leqslant \infty$, *set* $T_p = \operatorname{Int}(K) \in \mathscr{B}(L^p)$.

(1) *If* $I = [0, 1]$, *then* $\sigma(T_p) = \{0\}$.

(2) *If* $I = [0, \infty)$ *and* $1 \leqslant p \leqslant \infty$, *then* $\sigma(T_p)$ *is connected and* $\sigma(T_p) = \omega(T_p) \cup \Phi^-(T_p)$.

Proof. For each p the operator $T_p \in \mathscr{B}(L^p)$ and

$$\|T_p\| \leqslant \|H\|_\infty \|k\|_1$$

choose a sequence $\{k_n\} \subseteq L^1(I) \cap L^\infty$ such that $\|k_n - k\|_1 \to 0$. Set

$$K_n(x, t) = H(x, t)k_n(x - t)\chi_{[0,x]}(t), \text{ and } T_{n,p} = \operatorname{Int}(K_n) \in \mathscr{B}(L^p).$$

Then

$$\|T_p - T_{n,p}\| \leqslant \|H\|_\infty \|k_n - k\|_1 \to 0.$$

First assume $I = [0, 1]$. For $1 < p < \infty$ the operators $T_{n,p}$ are compact since the kernels K_n are essentially bounded. For $p = 1, \infty$, by Proposition 10 the operators $T_{n,p}$ are weakly compact. Therefore $T_p \in \mathscr{K}(L^p)$ when $1 < p < \infty$ and $T_p^2 \in \mathscr{K}(L^p)$ for $p = 1, \infty$, as noted before Proposition 10. It follows that $\sigma(T_p) = \{0\}$ for all p.

Now assume $I = [0, \infty)$. Since for each n, K_n is an essentially bounded kernel, (2) follows from Theorem 6, Theorem 7, and Proposition 1.

The next two theorems are based on the fact that the form of certain Volterra kernels forces the corresponding Volterra operator to be nilpotent. If $K(x, t)$ is a kernel with $|K| \in \mathscr{U}_p(\gamma)$, then let $K^{(*n)}$ be the nth interated convolution of K computed in the usual way. Thus, for $n \geqslant 1$

$$(\mathrm{Int}(K))^n = \mathrm{Int}(K^{(*n)}).$$

If $|K| \in \mathscr{U}_p(\gamma)$ and $\varepsilon > 0$, then define:

$$K_\varepsilon(x, t) = \begin{cases} 0 & \text{if } 0 \leqslant x < \varepsilon; \\ K(x, t) & \text{if } x \geqslant \varepsilon \text{ and } t \leqslant x - \varepsilon; \\ 0 & \text{if } x \geqslant \varepsilon \text{ and } t > x - \varepsilon. \end{cases}$$

Since $|K_\varepsilon| \leqslant |K| \in \mathscr{U}_p(\gamma)$, we have $K_\varepsilon \in \mathscr{U}_p(\gamma)$.

LEMMA 12. *Fix* $\varepsilon > 0$. *For* $n \geqslant 1$ *if either* $0 \leqslant x < n\varepsilon$, *or* $x \geqslant n\varepsilon$ *and* $> x - n\varepsilon$, *then* $K_\varepsilon^{(*n)}(x, t) = 0$.

Proof. For $n = 1$ the statement holds by the definition of K_ε. Assume it holds for some n. Assume $0 \leqslant x < (n + 1)\varepsilon$. We prove that

(1) $K_\varepsilon(x, z)K_\varepsilon^{(*n)}(z, t) = 0$.

When $0 \leqslant x < \varepsilon$, then $K_\varepsilon(x, z) = 0$. Suppose $\varepsilon \leqslant x < (n + 1)\varepsilon$. Then $K_\varepsilon(x, z) = 0$ when $z > x - \varepsilon$, and $K_\varepsilon^{(*n)}(z, t) = 0$ when $0 \leqslant z < n\varepsilon$ by the induction hypothesis. This proves (1) in this case.

Now assume $x \geqslant (n + 1)\varepsilon$ and $t > x - (n + 1)\varepsilon$. Then $K_\varepsilon(x, z) = 0$ when $z > x - \varepsilon$, and $K_\varepsilon^{(*n)}(z, t) = 0$ when either $0 \leqslant z < n\varepsilon$ or when $z \geqslant n\varepsilon$ and $t > z - n\varepsilon$, again, by the induction hypothesis. Suppose $z \leqslant x - \varepsilon$. Then $t > x - (n + 1)\varepsilon = (x - \varepsilon) - n\varepsilon \geqslant z - n\varepsilon$. This proves (1) in this case also.

Finally, note that since (1) holds when either $0 \leqslant x < (n + 1)\varepsilon$ or when $x \geqslant (n + 1)\varepsilon$ and $t > x - (n + 1)\varepsilon$, then

$$K_\varepsilon^{(*(n+1))}(x, t) = \int_I K_\varepsilon(x, z)K_\varepsilon^{(*n)}(z, t)\gamma(z)\mathrm{d}z = 0$$

under the same conditions.

THEOREM 13. *Let* $I = [0, 1]$ *and* $1 \leqslant p \leqslant \infty$. *Assume* $K \in \mathscr{U}_p(\gamma)$, *and set* $T_p = \mathrm{Int}(K) \in \mathscr{B}(L^p(\gamma))$. *Also, for* $\varepsilon > 0$ *let* $(T_\varepsilon)_p = \mathrm{Int}(K_\varepsilon)$ *acting on* $L^p(\gamma)$.

(1) *If for some* $\varepsilon > 0$, $K - K_\varepsilon$ *is essentially bounded, then* $\sigma(T_p) = \{0\}$.

(2) *If* $\lim_{\varepsilon \to 0^+} \|T_p - (T_\varepsilon)_p\| = 0$, *then* $\sigma(T_p)$ *is connected*, $\omega(T_p)$ *is connected, and* $\omega(T_p) = W(T_p)$.

Proof. The key fact here is that $(T_\varepsilon)_p$ is nilpotent by Lemma 12. Therefore if $\|T_p - (T_\varepsilon)_p\| \to 0$ as $\varepsilon \to 0^+$, then, T_p is a limit of nilpotent operators. Thus, Proposition 1(4) applies. This proves (2).

Now assume as in (1) that $K - K_\varepsilon$ is essentially bounded for some $\varepsilon > 0$. Applying Proposition 10, we have $T_p - (T_\varepsilon)_p$ is in $\mathscr{K}(L^p)$ if $1 < p < \infty$ and is in $\mathscr{W}(L^p)$ if $p = 1, \infty$. Now $T_p = (T_p - (T_\varepsilon)_p) + (T_\varepsilon)_p$, and $(T_\varepsilon)_p^n = 0$ for some n by Lemma 12. It follows that $T_p^n \in \mathscr{K}(L^p)$ for $1 < p < \infty$, and $T_p^n \in \mathscr{W}(L^p)$ for $p = 1, \infty$. Therefore $\sigma(T_p) = \{0\}$ by Theorem 5.

THEOREM 14. *Let* $I = [0, \infty)$. Assume $K \in \mathscr{U}_p(\gamma)$, *and let* $T_p = \mathrm{Int}(K) \in \mathscr{B}(L^p(\gamma))$. *Set* $P_\lambda(x, t) = K(x, t)\chi_{[0,\lambda)}(t)$, *and* $Q_{\lambda,p} = \mathrm{Int}(P_\lambda) \in \mathscr{B}(L^p(\gamma))$. *If K is ocally bounded and* $\|T_p - Q_{\lambda,p}\| \to 0$ *as* $\lambda \to \infty$, *then* $\sigma(T_p) = \{0\}$.

Proof. Let

$$N_\lambda(x, t) = K(x, t)\chi_{[\lambda,\infty)}(x)\chi_{[0,\lambda)}(t);$$

$$M_\lambda(x, t) = K(x, t)\chi_{[0,\lambda)}(x)\chi_{[0,\lambda)}(t).$$

Since M_λ is essentially bounded by hypothesis, Proposition 10 implies that $S_{\lambda,p} = \mathrm{Int}(M_\lambda)$ is in $\mathscr{K}(L^p)$ when $1 < p < \infty$, and is in $\mathscr{W}(L^p)$ when $p = 1, \infty$. Also, from the definition,

$$N_\lambda(x, z)N_\lambda(z, t) = 0 \quad \text{for all } z \geqslant 0.$$

Therefore $R_{\lambda,p} = \mathrm{Int}(N_\lambda)$ acting on L^p has the property $(R_{\lambda,p})^2 = 0$. Note that

$$Q_{\lambda,p} = S_{\lambda,p} + R_{\lambda,p},$$

so $(Q_{\lambda,p})^2$ is in $\mathscr{K}(L^p)$ for $1 < p < \infty$, and is in $\mathscr{W}(L^p)$ for $p = 1, \infty$. By hypothesis $\|T_p^2 - Q_{\lambda,p}^2\| \to 0$ as $\lambda \to \infty$, and therefore $T_p^2 \in \mathscr{K}(L^p)$ for $1 < p < \infty$, and $T_p^2 \in \mathscr{W}(L^p)$ for $p = 1, \infty$. It follows that $\sigma(T_p) = \{0\}$ by Theorem 5.

3. VOLTERRA OPERATORS ON SEQUENCE SPACES

In this section we consider spectral properties of linear operators on sequence spaces where the matrix of the operator is strictly lower triangular. D. A. Herrero has proved many interesting results concerning operators with a lower triangular matrix when the underlying space is a Hilbert space. One of his theorems implies that when such an operator has zero diagonal, then the spectrum is connected. This result (and much more) can be found in his preprint [8].

Throughout this section $\{\gamma(k)\}$ is a sequence of positive weights. For $1 \leqslant p \leqslant \infty$, a matrix kernel $\{K(n, m)\}, n, m \geqslant 1$, is in $\mathscr{U}_p(\gamma)$ if $K(n, m) = 0$ whenever $m \geqslant n$ and $\operatorname{Int}(K) \in \mathscr{B}(\ell^p(\gamma))$. Here

$$\operatorname{Int}(K)(b)(n) = \sum_{k=1}^{\infty} K(n, k)b(k)\gamma(k) \quad (b = \{b(k)\} \in \ell^p(\gamma)).$$

When $K \in \mathscr{U}_p(\gamma)$, then K is automatically "locally bounded", so perhaps it is not surprising that a result for $T = \operatorname{Int}(K)$ analogous to Theorem 6 can be proved without further restriction on K.

To carry out the construction we need the following fact.

LEMMA 15. *Let $\{b(k)\}_{k \geqslant 1}$ be a sequence of nonnegative numbers. Set $w(1) = 0$ and $w(n) = \sum_{j=1}^{n-1} b(j)$ for $n \geqslant 2$. Then*

$$\sum_{k=0}^{n} b(k+1)e^{w(k+1)} \leqslant e^{w(n+2)}.$$

Proof. Define $b(t)$ on $(0, \infty)$ by

$$b(t) = b(k) \quad \text{for } t \in (k-1, k],\ k \geqslant 1.$$

Let $[t]$ denote the largest integer m such that $m \leqslant t$. Define $B(t) = \int_0^t b(u)\,du$, and note

$$B(t) = \sum_{k=1}^{[t]} b(k) + (t - [t])b([t] + 1]).$$

Therefore whenever t is not an integer, $B'(t) = b([t] + 1)$. By [13, 11.54, p. 361]

$$\int_0^x B'(t)e^{B(t)}\,dt \leqslant e^{B(x)} - 1 \leqslant e^{B(x)}.$$

Now $w([t] + 1) \leqslant B(t) \leqslant w([t] + 2)$ for $t \geqslant 0$. Therefore

$$\int_0^x b([t] + 1)e^{w([t]+1)}\,dt \leqslant \int_0^x B'(t)e^{B(t)}\,dt \leqslant e^{B(x)} \leqslant e^{w([x]+2)}.$$

Taking $x = n \geqslant 0$ in this inequality, we have exactly the inequality in the statement of the lemma.

THEOREM 16. *Fix p, $1 < p \leqslant \infty$, and assume $K \in \mathcal{U}_p(\gamma)$. Set $T = \operatorname{Int}(K) \in \mathcal{B}(\ell^p(\gamma))$. Then*

(i) *$\sigma(T)$ is connected.*

When $p = 1$, assuming $\{\gamma(k)\}$ is bounded away from zero, then (i) *holds. In any case:*

(ii) *T has no nonzero eigenvalue;*

(iii) *$\sigma(T) = W(T) = \omega(T) \cup \Phi^-(T)$.*

Proof. Since K is a strictly lower triangular matrix, in this case an elementary computation shows that T has no nonzero eigenvalue. Therefore (ii) and (iii) hold.

The proof that $\sigma(T)$ is connected is analogous to the proof of Theorem 6. First choose sequences of positive real numbers $\{k(n)\}_{n \geqslant 1}$ and $\{h(j)\}_{j \geqslant 1}$ with the property

$$|K(n, j)| \leqslant k(n)h(j) \quad (n \geqslant 1, j \geqslant 1).$$

Assume $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Make the following definitions:

$$v(n) = (1 + k(n))^{-p}(1 + \gamma(n))^{-1}n^{-2} \quad (n \geqslant 1);$$

$$b(k) = h(k)^q v(k)^{1-q}\gamma(k) \quad (k \geqslant 1);$$

$$w(1) = 0, \quad w(m) = \sum_{k=1}^{m-1} b(k) \quad (m \geqslant 2);$$

$$\rho(n) = v(n)e^{-w(n)} \quad (n \geqslant 1).$$

Now we prove that

$$J = \sum_{n=1}^{\infty} \left[\sum_{j=1}^{\infty} |K(n, j)|^q \rho(j)^{1-q}\gamma(j) \right]^{p/q} \rho(n)\gamma(n)$$

is finite. From Lemma 15 it follows that

$$\sum_{j=1}^{n-1} h(j)^q v(j)^{1-q}\gamma(j)e^{(q-1)w(j)} \leqslant (q-1)^{-1}e^{(q-1)w(n)}$$

for $n \geqslant 2$. Using this ineqaulity we have

$$J \leqslant \sum_{n=2}^{\infty} \left[\sum_{j=1}^{n-1} k(n)^q h(j)^q \rho(j)^{1-q} \gamma(j) \right]^{p/q} \rho(n)\gamma(n) =$$

$$= \sum_{n=2}^{\infty} k(n)^p \gamma(n) v(n) e^{-w(n)} \left[\sum_{j=1}^{n-1} h(j)^q v(j)^{1-q} \gamma(j) e^{(q-1)w(j)} \right]^{p/q} \leqslant$$

$$\leqslant \sum_{n=2}^{\infty} k(n)^p \gamma(n) v(n) e^{-w(n)} [(q-1)^{-(p/q)} e^{(p/q)(q-1)w(n)}] =$$

$$= (q-1)^{-(p/q)} \sum_{n=2}^{\infty} k(n)^p \gamma(n) v(n) \leqslant (q-1)^{-(p/q)} \sum_{n=2}^{\infty} n^{-2}.$$

By definition $\{\rho(n)\} \subseteq \ell^\infty$, so $\ell^p(\gamma)$ is continuously and densely embedded as a subspace of $\ell^p(\gamma\rho)$. Define a kernel $\bar{K}$ by

$$\bar{K}(n, j) = K(n, j)\rho(j)^{-1} \quad (n \geqslant 1,\ j \geqslant 1).$$

Then by definition

$$J = \sum_{n=1}^{\infty} \left[\sum_{j=1}^{\infty} |\bar{K}(n, j)|^q \rho(j)\gamma(j) \right]^{p/q} \rho(n)\gamma(n).$$

Since J is finite, $\bar{K}$ determines a Hellinger-Tamarkin operator $\bar{T} = \operatorname{Int}(\bar{K})$ on $\ell^p(\gamma\rho)$. It follows that $\bar{T}$ is compact, and since $\bar{K} \in \mathscr{U}_p(\gamma\rho)$, $\sigma(\bar{T}) = \{0\}$. Furthermore, just as in the proof of Theorem 6, $\bar{T}$ is an extension of T. Therefore (i) follows by applying Theorem 2.

Now assume $p = 1$ or $p = \infty$. Using $p = q = 2$, repeat the construction of ρ, $\bar{K}$, and $\bar{T}$ just as above. In this case $\bar{T}$ is a Hilbert-Schmidt operator on $\ell^2(\rho\gamma)$. Note that $\sum_{n=1}^{\infty} \rho(n)\gamma(n) < \infty$. Therefore in the case $p = \infty$, ℓ^∞ is continuously and densely embedded in $\ell^2(\rho\gamma)$. Also, $\bar{T}$ is an extension of T so (i) holds by Theorem 2. If $p = 1$ and $\{\gamma(n)\}$ is bounded away from zero, then $\ell^1(\gamma)$ is continuously and densely embedded in $\ell^2(\rho\gamma)$. Therefore as before (i) holds for T.

We note that when K is a matrix kernel with $K(n, m) = 0$ whenever $m \geqslant n$ and $T = \operatorname{Int}(K) \in \mathscr{B}(c_0)$, then the conclusions of Theorem 16 hold for T.

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BRUCE A. BARNES
Department of Mathematics,
University of Oregon,
Eugene, Oregon 97403,
U.S.A.

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