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# ENTROPY FOR $*$-ENDOMORPHISMS AND RELATIVE ENTROPY FOR SUBALGEBRAS

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## 1. INTRODUCTION

Connes and Størmer ([4]) extended the notion of the entropy from the classical ergodic theory to the frame of $\mathrm{II}_1$-von Neumann algebras and showed that the $n$-shift of the hyperfinite $\mathrm{II}_1$ factor is not conjugate to the $m$-shift for $n \neq m$. For each $n$, the $n$-shift is the automorphism corresponding to the translation of 1 in the infinite tensor product $\underset{i\in\mathbf{Z}}{\otimes}(M_i, \mathrm{tr}_i)$ of the algebra $M_i$ of $n \times n$ matrices with the normalized trace $\mathrm{tr}_i$ on $M_i$, for each $i$.

In the index theory ([8]) for finite factors, Jones defined the index $[M : N]$ for a subfactor $N$ of a finite factor $M$. In a subsequent work, Pimsner and Popa ([11]) introduced the Connes and Størmer's relative entropy $H(M|N)$ as an invariant of a subfactor $N$ of a finite factor $M$ up to conjugation. They obtained interesting relations between $H(M|N)$ and the Jones' index $[M : N]$. They also introduced the automorphism $\theta_\lambda$ of the hyperfinite $\mathrm{II}_1$ factor $R$ which shifts $e_i$ to $e_{i+1}, \{e_i\}_{i\in\mathbf{Z}}$ being the Jones' sequence of projections with the trace $\lambda$ which generates $R$ and they compute the entropy $H(\theta_\lambda)$ for $\lambda \neq 1/4$. However, the value of the entropy has not been obtained in the case $\lambda = 1/4$.

Then in [10] Powers initiated the study of another class of shifts on $R$ that he called binary shifts, which translate $u_i$ to $u_{i+1}, \{u_i\}_{i\geqslant 0}$ being a sequence of unitaries generating $R$. After that, in [2], [6], [1] and [12], the conjugacy problem of $*$-endomorphisms corresponding to sequences of more general unitaries is investigated. We call these $*$-endomorphisms unitary shifts. On the other hand, the author [2] treated $*$-endomorphisms of $R$ generated by the sequence $(p_i)_i$ of projections translating $p_i$ to $p_{i+1}$, which we call projection shifts. Those unitary shifts and projection shifts turn out to be ergodic automorphisms of the hyperfinite $\mathrm{II}_1$ factor.

We will try in this paper to bring together this circle of ideas. We will first define the entropy for $*$-endomorphisms of finite von Neumann algebras by essentially adapting the Connes-Størmer definitions. Then we shall obtain simple formulas for entropy of $*$-endomorphisms. Under some good conditions, the entropy $H(\sigma)$ of a $*$-endomorphism $\sigma$ of an approximately finite von Neumann algebra $M$ is determined by the entropy $H(A_n)$'s for an increasing sequence $(A_n)_n$ of finite dimensional subalgebras which generate $M$:

$$H(\sigma) = \lim_{n\to\infty} \frac{H(A_n)}{n}.$$

The above all $*$-endomorphisms satisfy the relevant conditions and similar formula are obtained in [5] and [11]. As an application of this result to the automorphism $\theta_\lambda$ treated by Pimsner and Popa, we show that $H(\theta_\lambda) = \log 2$ for $\lambda = 1/4$. We shall show that if the inclusion data for the above sequence $(A_n)_n$ is periodic in the sense described later, then

$$H(\sigma) = \frac{1}{p}\log\beta$$

where $p$ is the period of the data and $\beta$ is the Perron-Frobenius eigenvalue of the inclusion matrix.

Next we shall investigate some relations between the entropy for $*$-endomorphisms and the relative entropy for subalgebras. If $\sigma$ is not an automorphism but a $*$-endomorphism of a finite von Neumann algebra $M$, it is natural to discuss on relations between the subalgebra $\sigma(M)$ and $H(\sigma)$. We shall show that if $M$ is generated by an increasing sequence $(N_j)_j$ of finite dimensional subalgebras, the inclusion data of which satisfies the bounded growth condition defined in below, then

$$H(\sigma) = \frac{1}{2}H(M|\sigma(M)).$$

All automorphisms and shifts discussed above satisfy the bounded growth condition. Hence, as an application of this result, we can also obtain the value of the entropies $H(\theta_\lambda)$ for all $\lambda$.

The bounded growth condition is satisfied if the inclusion data is periodic. Under the periodic condition, the algebra $M$ is a factor, and we shall obtain that

$$H(\sigma) = \frac{1}{2}\log[M:\sigma(M)] = \frac{1}{2}H(M|\sigma(M)).$$

If a von Neumann algebra $N$ is generated by $\{\theta^i(P) : i \in \mathbb{Z}\}$ for an $*$-automorphism $\theta$ of $N$ and a subalgebra $P$ of $N$, we get a $*$-endomorphism $\sigma$ of the von Neumann subalgebra $M$ generated by $\{\theta^i(P) : i = 1, 2, \ldots\}$. Then we shall show

$$H(\theta) = H(\sigma).$$

## 2. DEFINITION OF ENTROPY FOR $*$-ENDOMORPHISMS

In this section, we shall define the entropy for $*$-endomorphisms and state some of its properties. Throughout this section, $M$ will be a finite von Neumann algebra with a faithful normal trace $\tau$, $\tau(1) = 1$. For a von Neumann subalgebra $N$ of $M$, we denote by $E_N$ the unique faithful normal conditional expectation of $M$ onto $N$ defined by $\tau$. The letter $\eta$ designates the function on $[0,\infty)$ defined by $\eta(t) = -t\log t$. For each $k \in \mathbf{N}$, we let $S_k$ be the set of all families $(x_{i_1,i_2,\ldots,i_k})_{i_j\in\mathbf{N}}$ of positive elements of $M$, zero except for a finite number of indices and satisfying

$$\sum_{i_1,\ldots,i_j,\ldots,i_k} x_{i_1,\ldots,i_k} = 1.$$

For $x \in S_k, j \in \{1,2,\ldots,k\}$, and $i_j \in \mathbf{N}$, put

$$x^j_{i_j} = \sum_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_k} x_{i_1,i_2,\ldots,i_k}.$$

Let $N_1, N_2, \ldots, N_k$ be finite dimensional von Neumann subalgebras of $M$. Then Connes-Størmer defined:

$$H(N_1,\ldots,N_k) = \sup_{x\in S_k}\left[\sum_{i_1,\ldots,i_k} \eta\tau(x_{i_1,\ldots,i_k}) - \sum_j\sum_{i_j} \tau\eta E_{N_j}(x^j_{i_j})\right].$$

We recall the basic properties concerning the function $H(\cdot)$ which will be used later.

(A) $H(N_1,\ldots,N_k)\leqslant H(P_1,\ldots,P_k)$ if $N_j \subseteq P_j$ for all $j$.

(B) $H(N_1,\ldots,N_k,N_{k+1},\ldots,N_p)\leqslant H(N_1,\ldots,N_k) + H(N_{k+1},\ldots,N_p)$.

(C) If $P_i \subset P$ for $i = 1,2,\ldots,n$, then

$$H(P_1,\ldots,P_n,P_{n+1},\ldots,P_m)\leqslant H(P,P_{n+1},\ldots,P_m).$$

(D) Let $(e_\alpha)_{\alpha\in I}$ be a familiy of minimal projections of $N$ such that $\sum_{\alpha\in I} e_\alpha = 1$. Then $H(N) = \sum_{\alpha\in I}\eta\tau(e_\alpha)$.

(E) If $(N_1,N_2,\ldots,N_k)''$ is generated by pairwise commuting subalgebras $P_j$ of $N_j$, then $H(N_1,\ldots,N_k) = H((N_1,\ldots N_k)'')$.

2.1. Definition and properties of $H(\sigma)$. Let $\sigma$ be a $\tau$-preserving $*$-endomorphism of $M$. Following [4], we shall define the entropy for such a $\sigma$.

Remark 1. If a $*$-endomorphism of $M$ preserves a faithful normal trace $\tau$, $\tau(1) = 1$, then $\sigma$ is one to one , $\sigma(1) = 1$ and $\sigma$ is weakly continuous.

Lemma 2. *Let $\sigma$ be a $\tau$-preserving $*$-endomorphism of $M$. Then*

$$H(\sigma(N_1),\sigma(N_2),\ldots,\sigma(N_k))\leqslant H(N_1,N_2,\ldots,N_k).$$

*Equality holds if $\sigma$ is an automorphism.*

*Proof.* Although the inequality is obtained by [5, Proposition III.6], we shall give a proof of it for the sake of completeness. By Remark 1, $\sigma(M)$ is a von Neumann subalgebra of $M$. Let $E$ be the conditional expectation of $M$ onto $\sigma(M)$. Then

$$H(\sigma(N_1),\ldots,\sigma(N_k)) = \sup_{x\in S_k}\left[\sum_{i_1,\ldots,i_k}\eta\tau(E(x_{i_1,\ldots,i_k})) - \sum_j\sum_{i_j}\tau\eta E_{\sigma(N_j)}(E(x^j_{i_j}))\right].$$

Let $S_k(\sigma(M))$ be the subset of $S_k$ such that all $x_{i_1,\ldots,i_k}$ are contained in $\sigma(M)$. Since

$$E_{\sigma(N)}(\sigma(y)) = \sigma(E_N(y)) \text{ and } \tau\eta\sigma = \tau\eta,$$

$$H(\sigma(N_1),\ldots,\sigma(N_k)) =$$

$$= \sup_{x\in S_k(\sigma(M))}\left[\sum\eta\tau(x_{i_1,\ldots,i_k}) - \sum_j\sum_{i_j}\tau\eta E_{\sigma(N_j)}(x^j_{i_j})\right] =$$

$$= \sup_{y\in\sigma^{-1}(S_k(\sigma(M)))}\left[\eta\tau(y_{i_1,\ldots,i_k}) - \sum_j\sum_{i_j}\tau\eta\sigma E_{N_j}(y^j_{i_j})\right] \leqslant$$

$$\leqslant H(N_1,N_2,\ldots,N_k).$$

If $\sigma$ is an automorphism, then $\sigma^{-1}(S_k(\sigma(M))) = S_k$, so that the equality holds. ∎

For each $\tau$-preserving $*$-endomorphism $\sigma$ of $M$ and a finite dimensional von Neumann subalgebra $N$ of $M$, the following limit exists by Lemma 2 and properties (B), (D):

$$H(N,\sigma) = \lim_{k\to\infty}\frac{1}{k}H(N,\sigma(N),\ldots,\sigma^{k-1}(N)).$$

DEFINITION 3. The *entropy* $H(\sigma)$ for $\sigma$ is the supremum of $H(N,\sigma)$ for all finite dimensional subalgebra $N$ of $M$.

PROPOSITION 4. *Let $M$ be an approximately finite dimensional finite von Neumann algebra, $\tau$ a faithful normal trace of $M$ with $\tau(1) = 1$ and $\sigma$ a $\tau$-preserving $*$-endomorphism of $M$. Let $(N_j)_{j\in\mathbb{N}}$ be an increasing sequence of finite dimensional subalgebras which generates $M$. Then*

$$H(\sigma) = \lim_{j\to\infty} H(N_j,\sigma).$$

*Proof.* This proposition is proved by the same method as [4, Theorem 2] applying property (F) and Lemma 6 below. ∎

Using Lemma 2 and Proposition 4, the following proposition is obtained by the same proof as [4]:

PROPOSITION 5. (1) *Entropy for $*$-endomorphisms is conjugacy invariant, that is,*

$$H(\sigma) = H(\theta^{-1}\sigma\theta)$$

*if $\theta$ is a $\tau$–preserving $*$-automorphism of $M$.*

(2)

$$H(\sigma^n) \leqslant nH(\sigma)$$

*for all $n \in \mathbb{N}$, and equality holds if $M$ is approximately finite dimensional.*

2.2. RELATIVE ENTROPY. The relative entropy $H(A|B)$ for arbitrary von Neumann subalgebras $A$ and $B$ of $M$ is defined ([4], [11]) by

$$H(A|B) = \sup_{x \in S_1} \sum_i [\tau\eta E_B(x_i) - \tau\eta E_A(x_i)]$$

and it is allowed $H(A|B) = \infty$.

(F) $H(N_1, N_2, \ldots, N_k) \leqslant H(P_1, P_2, \ldots, P_k) + \sum_j H(N_j|P_j)$.

LEMMA 6. *If $\sigma$ is a $\tau$-preserving $*$-endomotphism of $M$, then*

$$H(\sigma(A|\sigma(B)) \leqslant H(A|B).$$

*Equality holds if $\sigma$is an automorphism.*

*Proof.* Immediate by a similar argument as Lemma 2. ∎

2.3. RELATIVE ENTROPY FOR FINITE DIMENSIONAL ALGEBRAS. If $A \subset B$ are finite dimensional subalgebras of $M$, then $H(B|A)$ is given by a more concrete formula. We decompose such $A$ and $B$ as follows:

$$A = \bigoplus_{l \in L} A_l \quad \text{and} \quad B = \bigoplus_{k \in K} B_k,$$

where $L, K$ are finite sets and $A_l$ or $B_k$ are the algebras of $a_l \times a_l$ or $b_k \times b_k$ matrices, respectively. Then row vectors $a = (a_l)_l$ and $b = (b_k)_k$ are called the dimension vectors of $A$ and $B$. The inclusion matrix $[A \hookrightarrow B] = (m_{lk})_{l \in L, k \in K}$ is given by the number $m_{lk}$ of simple components of a simple $B_k$ module viewed as an $A_l$ module. The trace (column) vectors $t = (t_l)_l$ and $s = (s_k)_k$ has $t_l$ respectively $s_k$ as the traces of the minimal projections in $A_l$ respectively $B_k$. Then

$$a[A \hookrightarrow B] = b, \quad [A \hookrightarrow B]s = t, \quad \sum_l a_l t_l = 1 = \sum_k b_k s_k.$$

Pimsner and Popa showed the following equality:

$$H(B|A) = \sum_l a_l t_l \log t_l - \sum_l a_l t_l \log a_l -$$

$$- \sum_k b_k s_k \log s_k + \sum_k b_k s_k \log b_k +$$

$$+ \sum_{kl} a_l m_{lk} s_k \log \min(\frac{a_l}{m_{lk}}, 1).$$

In order to simplify the notation we put

$$I(A) = \sum_{l \in L} a_l t_l \log \frac{a_l}{t_l}.$$

LEMMA 7. *If $m_{lk} \leqslant a_l$, then*

$$H(B|A) = I(B) - I(A).$$

## 3. APPROXIMATELY FINITE DIMENSIONAL ALGEBRAS

In this section, we shall obtain some formulas in the case of approximately finite dimensional algebras. Throughout this section, $M$ is a von Neumann algebra with a faithfull normal trace $\tau$, $\tau(1) = 1$ and $(N_j)_{j \in \mathbf{N}}$ is an increasing sequence of $M$ which generates $M$, also $\sigma$ is a $\tau$-preserving $*$-endomorphism of $M$.

3.1. SIMPLE FORMULA FOR $H(\sigma)$. If the inclusion data in (2.3) for the sequence $(N_j)_{j \in \mathbf{N}}$ satisfies the following two conditions, then the sequence $(N_j)_j$ is said to be periodic with a period $p$ ([13]).

There are $n_0 \geqslant 0$ and $p \geqslant 1$ such that for all $j \geqslant n_0$ :

(i) $[N_j \hookrightarrow N_{j+1}] = [N_{j+p} \hookrightarrow N_{j+p+1}]$,

(ii) The matrix $[N_j \hookrightarrow N_{j+p}]$ is primitive.

REMARK. If $M$ is generated by a periodic sequemce $(N_j)_{j \in \mathbf{N}}$, then $M$ is a factor ([7],[13]).

LEMMA 8. *Assume that $(N_j)_{j \in \mathbf{N}}$ is a periodic sequence with a period $p$. Let $\beta_j$ be the Perron-Frobenius eigenvalue of the matrix $[N_j \hookrightarrow N_{j+p}]$ for a $j \geqslant n_0$. Then*

$$t_j = \beta_j t_{j+p}$$

*and*

$$\lim_{n \to \infty} \frac{H(N_n)}{n} = \frac{1}{p} \log \beta_j.$$

*Proof.* Let fix a $j \geqslant n_0$. We denote by $T$ the matrix $[N_j \hookrightarrow N_{j+p}]$ and by $\beta$ the Perron-Frobenius eigenvalue of $T$. Since

$$t_{j+p} = T^k t_{j+(k+1)p}$$

for all $k \geqslant 0$, and $T$ is primitive, there is a positive real number $\alpha$ so that $t_{j+p} = \alpha\xi$, where $\xi$ is a Perron-Frobenius eigenvector of $T$ for $\beta$. Hence

$$t_j = Tt_{j+p} = \alpha T\xi = \beta t_{j+p} = \ldots = \beta^n t_{j+np}$$

for all $n \geqslant 0$. By the property (D),

$$H(N_{j+np}) = -\sum_k d_{j+np}(k) t_{j+np}(k) \log \beta^{-n} t_j(k) =$$

$$= n \log \beta - \sum_k d_{j+np}(k) t_{j+np}(k) \log t_j(k).$$

Since $\{t_j(k); k\}$ is a finite set and $\sum_k d_l(k) t_l(k) = 1$,

$$\lim_{n\to\infty} \frac{H(N_n)}{n} = \lim_n \frac{n}{j+np} \log \beta = \frac{1}{p} \log \beta.$$

■

Assumption (*) for $\sigma$. (1) For $j$ and $m$, there is a $\tau$-preserving *-endomorphism $\alpha$ so that $\alpha(N_{j+m})$ contains the von Neumann algebra generated by $\{N_j, \sigma(N_j), \ldots, \sigma^m(N_j)\}$.

(2) There exists a sequence $(n_j)_{j\in\mathbf{N}}$ with the properties

$$x\sigma^m(y) = \sigma^m(y)x, \quad (x, y \in N_j,\ m \in n_j\mathbf{N})$$

$$\tau(x\sigma^{kn_j}(y)) = \tau(x)\tau(y) \quad (x \in (N_j, \sigma^{n_j}(N_j), \ldots, \sigma^{(k-1)n_j}(N_j))'', y \in N_j)$$

and

$$\lim_{j\to\infty} \frac{n_j - j}{j} = 0.$$

Theorem 9. *Under the Assumption* (*),

$$H(\sigma) = \lim_{j\to\infty} \frac{H(N_j)}{j}.$$

*Proof.* By Proposition 4 and the Lemma 2, we have

$$H(\sigma) = \lim_{j\to\infty} H(N_j, \sigma) = \lim_j \lim_k H(N_j, \sigma(N_j), \ldots, \sigma^{k-1}(N_j))/k \leqslant$$

$$\leqslant \lim_j \liminf_k H((N_j, \dots, \sigma^{k-j}(N_j))'', (\sigma^{k-j+1}(N_j), \dots, \sigma^{k-1}(N_j))'')/k \leqslant$$

$$\leqslant \lim_j \liminf_k H(\alpha(N_k), \sigma^{k-j+1}(\alpha(N_{2j-1})))/k \leqslant$$

$$\leqslant \lim_j \liminf_k [H(N_k) + H(N_{2j-1})]/k \leqslant$$

$$\leqslant \liminf_k H(N_k)/k.$$

On the other hand, by properties (D) and (E),

$$n_j H(\sigma) = H(\sigma^{n_j}) \geqslant$$

$$\geqslant \lim_k H(N_j, \sigma^{n_j}(N_j), \dots, \sigma^{kn_j}(N_j))/k =$$

$$= H(N_j),$$

because $(N_j, \sigma^{n_j}(N_j), \dots, \sigma^{kn_j}(N_j))$ are pairwise commuting and $\tau$ satisfies a kind of multiplicative property. Hence

$$H(\sigma) \geqslant \frac{H(N_j)}{n_j} = \frac{H(N_j)}{j} - \frac{n_j - j}{jn_j} H(N_j)$$

which implies

$$H(\sigma) \geqslant \limsup_j \frac{H(N_j)}{j}.$$

Therefore

$$H(\sigma) = \lim_{j \to \infty} \frac{1}{j} H(N_j).$$

■

COROLLARY 10. *If the sequence $(N_j)_j$ is periodic with a period $p$, under the Assumption* $(*)$,

$$H(\sigma) = \frac{1}{p} \log \beta_n$$

*for a large enough $n$, where $\beta_n$ is the Perron-Frobenius eigenvalue of $[N_n \hookrightarrow N_{n+p}]$.*

*Proof.* By Theorem 9 and Lemma 8,

$$H(\sigma) = \lim_{j \to \infty} \frac{1}{j} H(N_j) = \frac{1}{p} \log \beta_n,$$

for a large enough $n$. ■

3.2. APPLICATION TO $\theta_\lambda$ AND $n$-SHIFTS. Let $(e_i)_{i \in \mathbb{Z}}$ be a sided sequence of projections satisfying the axioms:

a) $e_i e_{i \pm 1} e_i = \lambda e_i$ for a $\lambda \in (0, 1/4] \cup \{1/4 \sec^2 \pi/m; m \geqslant 3\}$,

b) $e_i e_j = e_j e_i$ for $|i - j| \geqslant 2$,

c) the von Neumann algebra $P$ generated by $(e_i)_{i\in\mathbb{Z}}$ is a hyperfinite $\mathrm{II}_1$ factor with the trace $\tau$.

d) $\tau(we_i) = \lambda\tau(w)$ for the trace $\tau$ of $P$ if $w$ is a word on 1 and $\{e_j; j < i\}$.

We define the sequence $(A_j)_{j\in\mathbb{N}}$ of finite dimensional subalgebras of $P$ which generates $P$ by

$$A_{2j} = \{e_i; |i| \leqslant j-1\}'', \quad A_{2j+1} = \{A_{2j}, e_j\}''$$

Then we proved in [3] that the inclusion data for $(A_j)_{j\in\mathbb{N}}$ is the same as the data in [8]. In case $\lambda = 1/4$, the sequence of trace vectors has beautiful values as follows.

LEMMA 11. *Let $t_j$ be the trace vector for the restriction of $\tau$ to $A_j$. If $\lambda = 1/4$, then for all $k \in \mathbb{N}$,*

$$t_{2k} = (1/4^k, 3/4^k, 5/4^k, \ldots, (2k+1)/4^k)$$

*and*

$$t_{2k+1} = (1/4^k, 2/4^k, 3/4^k, \ldots, (k+1)/4^k).$$

*Proof.* We shall prove that by the induction on $k$. It is obvious for $k = 1$. Assume that lemma holds for $k = m$. It is shown by Jones [8] that $t_{2(m+1)}(j) = (1/4)t_{2m}(j)$ for $j = 1, 2, \ldots, m+1$. Hence we just have to show $t_{2(m+1)}(m+2) = (2m+3)/4^{2m+1}$. The dimension vector $d_{2k}$ of $A_{2k}$ satisfies $d_{2k}(i) = \binom{2k}{k-1+i} - \binom{2k}{k-i}$, where $\binom{n}{i}$ is the binomial symbol with the convention $\binom{n}{-1} = 0$. Since $\sum_i d_j(i)t_j(i) = 1$, we have $\alpha = t_{2p+1}(p+1) = (2p+3)/4^{2p+1}$ by the equality:

$$4^{p+1} = \binom{2(p+1)}{p+1} - (2p+1) + \sum_{j=0}^{p-1} \binom{2(p+1)}{p-j} + 4^{p+1}\alpha.$$

Similary, we have the values for $t_{2k+1}$. ■

THEOREM 12.

$$H(\theta_\lambda) = \log 2, \qquad \text{for } \lambda = 1/4.$$

*Proof.* We denote $\theta_\lambda$ by $\theta$. It is easy to check that Assumption (*) for $\theta$ is satisfied. Hence $H(\theta) = \lim_{j\to\infty} H(A_j)/j$. On the other hand,

$$H(A_{2k}) = -\sum_{j=1}^{k+1} d_{2k}(j)t_{2k} \log t_{2k}(j) =$$

$$= -\sum_j d_{2k}(j)t_{2k}(j)(\log(2j+1) - \log 4^k) =$$

$$= \log 4^k - \sum_j d_{2k}(j)t_{2k}(j)\log(2j+1),$$

for $k = 1, 2, \ldots$ .

Similarly,

$$H(A_{2k+1}) = \log 4^k - \sum_j d_{2k+1}(j)t_{2k+1}(j)\log j \quad \text{for } k = 1,2,3,\ldots .$$

On the other hand,

$$0 \leqslant \lim_{k\to\infty}(1/2k)\sum_j d_{2k}(j)t_{2k}(j)\log(2j+1) \leqslant \lim_{k\to\infty}(1/2k)\log(2k+1) = 0$$

and

$$\lim_{k\to\infty}\frac{1}{2k+1}\sum_j d_{2k+1}(j)t_{2k+1}(j)\log j = 0.$$

Hence we have

$$H(\theta) = \lim_{k\to\infty}\frac{\log 4^k}{2k} = \log 2.$$

■

After I had known for this paper to be accepted, I received the preprint "Entropy of certain noncomutative shifts" from H. S. Yin in which he had the same result as Theorem 12.

By applying Corollary 10 to $\theta_\lambda$, we have the following results by Pimsner-Popa:

COROLARY 13. *Let $\lambda > 1/4$. Then $H(\theta_\lambda) = -(1/2)\log\lambda$.*

*Proof.* Since the inclusion data for the sequence $(A_j)_{j\in\mathbb{N}}$ is the same as one obtained by Jones [8] ([3]), the sequence $(A_j)_{j\in\mathbb{N}}$ is periodic and the period of it is 2 [8]. For a sufficiently large $j$, the Perron-Frobenius eigenvalue of the inclusion matrix $[A_j \hookrightarrow A_{j+2}]$ is $1/\lambda$ if $\lambda > 1/4$. Hence $H(\theta_\lambda) = -(1/2)\log\lambda$ by Corollary 10. ■

As another application of Corrolary 10, we have the following result of Connes--Størmer.

COROLLARY 14. *Let $S_n$ be the n-shift of the hyperfinite $\mathrm{II}_1$ factor, then $H(S_n) = \log n$.*

*Proof.* Let $M$ be the algebra of $n \times n$-matrices. For an integer $j$, let

$$x_j = \ldots \otimes 1 \otimes \ldots \otimes 1 \otimes \underbrace{x}_{j} \otimes 1 \otimes \ldots \quad (x \in M).$$

Then $S_n(x_j) = x_{j+1}$ for all $x \in M$ and $j \in \mathbb{Z}$. For an integer $j \geqslant 0$, let

$$N_{2j+1} = \{\cup M_k; |k| \leqslant j\}'', \quad N_{2j} = \{M_j \cup N_{2j-1}\}''.$$

Then $S_n$ satisfies Assumption (∗). The sequence $(N_j)_{j\in\mathbb{N}}$ is a periodic sequence which generates the hyperfinite $\mathrm{II}_1$ factor. The period of $(N_j)_{j\in\mathbb{N}}$ is 1 and the inclusion matrix $[N_j \hookrightarrow N_{j+1}]$ is the number $n$ for all $j \geqslant 0$. Hence we have $H(S_n) = \log n$. ∎

3.3. Bounded growth condition for $(N_j)_{j\in\mathbb{N}}$. Let

$$N_j = \bigoplus_{k\in K_j} N_j(k)$$

be such a descomposition as in (2.3), $d_j = (d_j(k))_{k\in K_j}$ the dimension vector of $N_j$ and $t_j = (t_j(k))_k$ the trace vector of $\tau$ on $N_j$.

Definition 15. We shall say that the increasing sequence $(N_j)_{j\in\mathbb{N}}$ satisfies the bounded growth condition if

(1) $\sup_j \sharp(K_j)/j < \infty$ and

(2) For some $m_0$, $N_{j+1}(l)$ contains at most $d_j(k)$ $N_j(k)$-components for a $j \geqslant m_0$, where $\sharp(K_j)$ is the cardinal number of $K_j$.

Proposition 16. *If an increasing sequence $(N_j)_j$ satisfies the bounded growth condition, then*

$$I(N_n) - I(N_{m_0-1}) = \sum_{j=m_0}^{n} H(N_j|N_{j-1})$$

*for $n > m_0$ and*

$$\lim_{j\to\infty} \frac{\sum_k t_j(k)d_j(k)\log t_j(k)d_j(k)}{j} = 0.$$

*Proof.* Since the condition (2) in the bounded growth condition implies the condition in Lemma 7, we have

$$H(N_n|N_{n-1}) = I(N_n) - I(N_{n-1}) = I(N_n) - I(N_{n-2}) - [I(N_{n-1}) - I(N_{n-2})] =$$

$$= I(N_n) - I(N_{m_0-1}) - \sum_{j=m_0}^{n-1} H(N_j|N_{j-1}).$$

Hence

$$I(N_n) - I(N_{m_0-1}) = \sum_{j=m_0}^{n} H(N_j|N_{j-1}).$$

Put $r_j(k) = t_j(k)d_j(k)$ and $\sharp(K_j) = k_j$. Then $\sum_{k \in K_j} r_j(k) = 1$ and $r_j(k) \geqslant 0$. Since the function $\eta$ is concave, we have

$$\sum_k \eta(r_j(k)) = k_j \sum_k \frac{\eta(r_j(k))}{k_j} \leqslant$$

$$\leqslant k_j \eta \left( \sum_k \frac{r_j(k)}{k_j} \right) = \log k_j.$$

Hence by the condition (1),

$$0 \leqslant \lim_j \frac{\sum_k \eta(r_j(k))}{j} \leqslant \lim_j \frac{\log k_j}{j} = 0,$$

which implies the conclusion. ∎

## 4. ENTROPY $H(\sigma)$ AND RELATIVE ENTROPY $H(M|\sigma(M))$

In this section we shall obtain relations among the entropy for $*$-automorphisms, the entropy for $*$-endomorphisms (which are not automorphisms) and the relative entropy for subalgebras.

DEFINITION 17. Let $M$ be a von Neumann algebra with a faithful normal trace $\tau$, $\tau(1) = 1$. A $\tau$-preserving $*$-endomorphism $\sigma$ of $M$ is said to have a finite dimensional $\tau$-independent generator $P$ if the following conditions are satisfied:

(1) $M$ is generated by the family $\{\sigma^i(P);$ all considerable $i\}$ (that is $M = \{\sigma^i(P); i \in \mathbb{Z}\}''$ if $\sigma^{-1}(P)$ is defined, otherwise $M = \{\sigma^i(P); i \in \mathbb{N}\}''$),

(2)
$$\tau(\sigma^{i_1}(x_{i_1})\sigma^{i_2}(x_{i_2}) \ldots \sigma^{i_n}(x_{i_n})) = \prod_{j=1}^{n} \tau(x_{i_j}),$$

for $x_{i_j} \in P$ and $i_l \neq i_k$ if $l \neq k$, and

(3) The algebra generated by $\{\sigma^i(P); i = 0, 1, \ldots, n\}$ is finite dimensional for all $n$ and there is such an $m$ that $P$ and $\sigma^i(P)$ commute for all $i \geqslant m$.

THEOREM 18. *Let $M$ be a von Neumann algebra with a faithful normal trace $\tau$, $\tau(1) = 1$ and $\sigma$ a $\tau$-preserving $*$-endomorphism of $M$ which has a finite dimensional $\tau$-independent generator $P$. Let $N_j$ be the algebra generated by $\{\sigma^i(P); i = 0, 1, \ldots \ldots, j\}$. Assume that the sequence $(N_j)_j$ satisfies the bounded growth condition. If $\sigma$ is not onto, then*

$$H(\sigma) = \frac{1}{2} H(M|\sigma(M)).$$

*Proof.* The algebra $M$ (resp. $\sigma(M)$) is generated by an increasing sequence $(N_j)_j$ (resp. $(\sigma(N_j)_j)$). By the property of $\tau$ or $\sigma$, we have for all $j$,

$$E_{N_j}E_{\sigma(N_j)} = E_{\sigma(N_{j-1})}.$$

Hence we have by [11, Proposition 3.4],

$$H(M|\sigma(M)) = \lim_{j\to\infty}(N_j|\sigma(N_{j-1})).$$

It is obvious that there is a $\tau$-preserving $*$-automorphism $\alpha$ of $M$ which satisfies $\alpha(N_j) = N_j$ and $\alpha(N_{j-1}) = \sigma(N_{j-1})$. Hence $H(N_j|\sigma(N_{j-1})) = H(N_j|N_{j-1})$. Let $d_j$ be the dimension vector of $N_j$ and $t_j$ the trace vector of $\tau$ on $N_j$. Then by property (D) and Theorem 9,

$$H(\sigma) = \lim_{j\to\infty} H(N_j)/j = -\lim(1/j)\sum_k t_j(k)d_j(k)\log t_j(k) =$$

$$= \lim_j(1/j)\left[I(N_j) - \sum_k t_j(k)d_j(k)\log d_j(k)\right].$$

Since

$$\lim\frac{1}{j}\sum_k t_j(k)d_j(k)\log t_j(k)d_j(k) = 0$$

by Proposition 16, we have

$$H(\sigma) = \lim\frac{1}{j}\sum_k t_j(k)d_j(k)\log d_j(k).$$

Hence

$$2H(\sigma) = \lim(I(N_j))/j = \lim_j[I(N_j) - I(N_{m-1})]/j =$$

$$= \lim(1/j)\sum_{i=m}^{j} H(N_i|N_{i-1}) = H(M|\sigma(M)).$$

■

COROLLARY 19. *Let $M$, $\sigma$ and $(N_j)_j$ be the same as in Theorem 18. If the sequence $(N_j)_j$ is periodic, then*

$$H(\sigma) = \frac{1}{2}\log[M:\sigma(M)]$$

*and*

$$H(M|\sigma(M)) = \log[M:\sigma(M)].$$

*Proof.* By Remark in (3.1), $M$ is the hyperfinite $\mathrm{II}_1$ factor. Let $p$ be the period of the sequence $(N_j)_j$ and $\beta_j$ the Perron-Frobenius eigenvalue of the inclusion matrix

$[N_j \hookrightarrow N_{j+p}]$. The algebra $M$ is generated by $(N_j)_j$ and $\sigma(M)$ is generated by $\sigma(N_j)_j$. By the property of $\sigma$ for $\tau$, we have $E_{\sigma(N_j)}E_{N_j} = E_{\sigma(N_{j-1})}$. Hence the sequences $(N_j)_j$ and $(\sigma(N_j))_j$ have the property $N_j \supset \sigma(N_j)$ for all $j$ and satisfy the periodic condition due to Wenzl [13]. Denote by $t_j$ the trace vector of $\tau$ on $N_j$, then by ([13], [7])

$$[M : \sigma(M)] = \frac{\| t_j \|_2^2}{\| t_{j+1} \|_2^2}.$$

On the other hand, by Lemma 8,

$$\beta_j^2 = \frac{\| t_j \|_2^2}{\| t_{j+p} \|_2^2}.$$

Hence we have

$$\beta_j^2 = [M : \sigma(M)]^p,$$

which implies by Lemma 8,

$$H(\sigma) = \frac{1}{p}\log \beta_j = \frac{1}{2}\log[M : \sigma(M)].$$

On the other hand, a periodic sequence satisfies the bounded growth condition. Hence by Theorem 18, we have

$$H(M|\sigma(M)) = 2H(\sigma) = \log[M : \sigma(M)]$$

■

Next we shall discuss the relation between the entropy of automorphisms and the entropy of $*$-endomorphisms.

THEOREM 20. *Let $N$ be a finite von Neumann algebra with a faithful normal trace $\tau$, $\tau(1) = 1$ and $\theta$ a $\tau$-preserving $*$-automorphism of $N$, which has a finite dimensional $\tau$-independent generator $P$. We let $M$ be the von Neumann algebra generated by $\{\theta^i(P); i \geqslant 0\}$ and $\sigma$ the restriction of $\theta$ to $M$. Then*

$$H(\sigma) = H(\theta).$$

*Proof.* Let $N_j$ be the algebra generated by $\{\theta^i(P); i = 0, 1, \ldots, j\}$. Then the sequence $(N_j)_j$ staisfies the Assumption $(*)$ for $M$ and $\sigma$. Next we define the sequence $(A_j)_j$ of finite dimensional subalgebra of $N$ by

$$A_{2j} = \{\theta^i(P); |i| \leqslant j-1\}'', \quad A_{2j+1} = \{A_{2j}, \theta^j(P)\}''.$$

Then the sequence $(A_j)_j$ also satisfies the Assumption $(*)$ for $N$ and $\theta$. Since there exists an $\tau$-preserving $*$-automorphism $\gamma$ of $N$ such that $\gamma(N_j) = A_j$, we have

$$H(\theta) = \lim_j \frac{H(A_j)}{j} = \lim_j \frac{H(N_j)}{j} = H(\sigma).$$

REMARK. Let $N, \theta, M$ and $\sigma$ be as in Theorem 20. Then $\sigma$ is a shift of $N$ in the sense of Powers, that is $\bigcap_{i=0}^{\infty} \sigma^i(M) = \mathbb{C}1$ by the property of $\tau$. If $\sigma$ is a shift, then $\theta$ is ergodic.

To conclude we shall show some examples which satisfy the conditions discussed in Section 4.

EXAMPLE 1. Let $\theta$ be the $n$-shift and $\sigma$ the restriction of $\theta$ to the hyperfinite $\mathrm{II}_1$-factor $M = \bigoplus_{i \in \mathbf{N}} (M_i, \mathrm{Tr}_i)$, where $M_i$ is the $n \times n$ matrix algebra and $\mathrm{Tr}_i$ is the trace of $M_i$. Then $\theta$ has a finite dimensional independent generator $M_0$ and a periodic sequence $(N_j)_j$ defined as above. Hence

$$H(\theta) = H(\sigma) = \frac{1}{2} H(M|\sigma(M)) = \frac{1}{2} \log[M : \sigma(M)] = \log n.$$

EXAMPLE 2. Let $(e_j)_j$ be the two sided sequence of projections with the properties denoted in (3.2), but a) and b) are exchanged to a′) and b′) as follows: For a $k \in \mathbf{N}$

a′) $e_i e_j e_i = \lambda e_i$ if $| i - j | = k$,

b′) $e_i e_j = e_j e_i$ if $| i - j | \neq k$.

Then the automorphism $\theta$ (resp. $*$-endomorphism $\sigma$) of $\{e_i\}''_{i \in \mathbf{Z}}$ (resp. $M = \{e_i\}''_{i \in \mathbf{N}}$) has a finite dimensional independent generator, which is the algebra generated by $e_0$. The sequence $(A_j)_j$ defined by a similar method in (3.2) (resp. $(N_j)_j$) satisfies the bounded growth condition and if $\lambda > (1/4)$ then the sequence $(N_j)_j$ is periodic. Hence if $\lambda \leqslant 1/4$ then using the computations due to Pimsner and Popa [11]

$$H(\theta) = H(\sigma) = \frac{1}{2} H(M|\sigma(M)) = \eta t + \eta(1-t)$$

where $\lambda = t(1-t)$, and if $\lambda > (1/4)$, then

$$H(\theta) = H(\sigma) = \frac{1}{2} H(M|\sigma(M)) = \frac{1}{2} \log[M : \sigma(M)] = -\frac{\log \lambda}{2}.$$

Although the entropy of the $*$-endomorphism $\sigma$ depends only on $\lambda$, the conjugacy classes of $\sigma$ depend on $\lambda$ and $k$ ([2]).

EXAMPLE 3. Let $S$ be a finite subset of $\mathbf{N}$. Let $\gamma = \exp(2\pi \mathrm{i}/n)$ for some positive integer $n$. Then there exists a family $(u_i)_{i \in \mathbf{Z}}$ of unitaries which generates the hyperfinite $\mathrm{II}_1$-factor and satisfy the following conditions;

(i) $u_i^n = 1$

(ii) $u_i u_j = \gamma u_j u_i$ if $i - j \in S$ $(i \geqslant j)$

(iii) $u_i u_j = u_j u_i$ if $| i - j | \notin S$.

Let $\theta$ (resp. $\sigma$) be the $*$-automorphism (resp. the restriction of $\theta$) of $\{u_i\}''_{i\in\mathbf{Z}}$ (resp. to $M = \{u_i\}''_{i\in\mathbf{N}}$) defined by $\theta(u_i) = u_{i+1}$. Then $\theta$ has a finite dimensional independent generator $P$ which is generated by $u_0$. The algebra $M$ is generated by a periodic sequence. Hence

$$H(\theta) = H(\sigma) = \frac{1}{2}H(M|\sigma(M)) = \frac{1}{2}\log[M : \sigma(M)] = \frac{\log n}{2}.$$

Although the entropy of the $*$-endomorphism $\sigma$ depends only on $n$, the conjugacy classes of such $\sigma$'s depend on $n$ and $S$.

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