J. OPERATOR THEORY
25(1991), 163–176

MAXIMAL TRIANGULAR SUBALGEBRAS OF AF ALGEBRAS

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1. INTRODUCTION

A subalgebra A of C^*-algebra B is triangular if $A \cap A^*$ is a maximal abelian self-adjoint subalgebra (masa) of B. It is maximal triangular if it is not contained in any larger triangular subalgebra of B. Maximal triangular algebras have been the subject of recent studies, for example that of P. S. Muhly, K.-S. Saito and B. Solel in [3]. Triangular subalgebras of AF algebras have also been studied, by S. C. Power in [11] and [12], by R. L. Baker in [2] and by J. R. Peters and B. H. Wagner in [9]. The combination of the two has been studied by Y. T. Poon and the last two named authors in [8]. In this paper some tools for studying these and other subalgebras of AF algebras are developed, including an AF version of the Spectral Theorem for Bimodules of [3], continuing the work in [10] and [11]. The main result, Theorem 3.3, is a characterisation of maximal triangular AF algebras having a special Cartan-type masa, called a canonical masa, as diagonal. It shows that symmetric "gaps" can occur in them in terms of partial isometries which are not contained in the algebra. This reveals that these algebras are not highly structured, contrasting with both the finite-dimensional case and the ultraweak theory of [3]. We give examples of maximal triangular algebras with various properties motivated by this theorem. Dilation theory is studied and an example of a contractive Hilbert space representation of a maximal triangular algebra without a Stinespring dilation is given.

Every closed subspace of an AF algebra which is a bimodule for a canonical masa defines a binary relation, called the fundamental relation as in Section 1. It follows from Theorem 2.2, the AF Spectral Theorem, that for a fixed canonical masa there is a 1–1 correspondence between bimodules and binary relations satisfying a topological condition. The properties of a relation for it to be a fundamental relation of a triangular or maximal triangular algebra can be identified. (Proposition 2.3 and Theorem 3.3.) This provides a new way of analysing these algebras directly instead of as the limit of finite-dimensional subalgebras. This approach is particularly helpful for subalgebras of the tensor product of an abelian AF algebra with a matrix algebra, the context of Examples 4.1 and 4.2. A variant of the technique is used in the more complicated Example 4.3. In this case the fundamental relation of a triangular bimodule is constructed from a chain of finite-dimensional subspaces, allowing any discussion of embeddings to be completely avoided. The methods used treat an AF algebra as the C^*-algebra of its fundamental relation (as a groupoid), and bimodule subalgebras as supported on a subset of the relation, rather than treating these subalgebras as inductive limit algebras.

The results introduced in the next two sections are the basics required for the study of non-self-adjoint subalgebras of AF algebras which are bimodules for a canonical masa.

This paper is taken from my thesis [13] and I would like to thank my supervisor, Professor S. C. Power, for all his help, encouragement and advice.

1. PRELIMINARIES

We first look at simple spaces of matrices defined by specifying which matrix entries must be zero. Let $M(n)$ denote the set of $n \times n$ complex matrices. Pick a set of matrix units. The algebra $D(n)$ spanned by the self-adjoint matrix units is an n-dimensional masa. From direct calculations, a subspace of $M(n)$ is spanned by the matrix units it contains if and only if it is a bimodule for $D(n)$. Every triangular subalgebra of $M(n)$ contains, and is a bimodule for, a masa generated in this way. A $D(n)$-bimodule N defines a directed graph with vertices the matrix units in $D(n)$ and arrows the matrix units in N, where e_{ij} is an arrow from e_{jj} to e_{ii}. The set of $D(n)$--bimodules in $M(n)$ is thus in 1–1 correspondence with the set of directed graphs on n vertices. Given a set of matrix units, all properties of a $D(n)$-bimodule can be read off its directed graph, for example if it is a triangular or maximal triangular algebra. This process allows large matrix spaces to be analysed easily, and is essential for Example 4.3. A general finite-dimensional C^*-algebra is isomorphic to the direct sum of full matrix algebras so similar statements hold in this case. This construction was generalised in [11] to some subspaces of AF algebras.

Let $B_1, B_2, \ldots$ be an increasing chain of finite-dimensional $*$-algebras with closed union the AF C^*-algebra B. For a subspace L of B write L_n for $L \cap B_n$ and L_∞ for the union of all these. The space L is *inductive* if L_∞ is dense in L. Let C be an inductive masa in B such that C_n is a masa in B_n for all n, called a *canonical* masa. Let M be a C-bimodule in B (a closed subspace with $CM \subset M$ and $MC \subset M$). It is a key result of Power [10, Lemma 1.3] that all C^*-bimodules are inductive. Each M_n has the properties described in the above paragraph since it is a C_n-bimodule in B_n. This is already good enough to characterise certain properties of C-bimodules but not more complex ones, which need the fundamental relation, introduced in [11, Chapter 6] to study tensor products of some triangular subalgebras of AF algebras.

A partial isometry v in B is said to be *C-normalising* if $vCv^* \subset C$ and $v^*Cv \subset C$. It will be seen later that any C-bimodule is spanned by the C-normalising partial isometries which it contains. The *fundamental relation* of M is the binary relation $\mathrm{R}(M)$ defined on the Gelfand space $\Phi(C)$ of C by $x\mathrm{R}(M)y$ if and only if there is a C-normalising partial isometry v in M such that $y(c) = x(vcv^*)$ for all c in C, where x and y are (unit norm multiplicative linear functionals) in $\Phi(C)$. This relation also defines a directed graph with set of vertices $\Phi(C)$ and a subset $\{(x,y) : x\mathrm{R}(M)y\}$ of $\Phi(C) \times \Phi(C)$, we shall use these interchangeably for $\mathrm{R}(M)$. Notice that the relation is not dependent on the particular chain of $*$-subalgebras of B picked at the start, just the choice of masa.

A *system* of matrix units for the AF algebra B (with respect to $B_1, B_2, \ldots$) is a choice of matrix units for each B_n such that each one in B_n is a sum of matrix units from B_{n+1}. It is always possible to find a system of C-normalising matrix units and we shall fix such a system for B for the rest of this paper. By inductivity, any C-bimodule is the closed span of the matrix units it contains. From [11, Lemma 6.3], every C--normalising partial isometry is the product of partial isometries, one from C and the other a finite sum of matrix units. In particular the fundamental relation could have been defined using matrix units instead of partial isometries. Many properties of a C-bimodule can be found in its fundamental relation and in the next section it is shown that they will then be reflected in the matrix units contained. Often this will lead to the construction of a chain of finite-dimensional subspaces with the property and having dense union. For some properties there are examples to show that this does not happen.

2. BIMODULES

Recall that we have an AF algebra B, a canonical masa C, a C-bimodule M and a system of C-normalising matrix units. In this section we explore the relationship between M, its fundamental relation, the matrix units it contains and the chain $M_1, M_2, \ldots$ for some simple properties.

We construct a natural topology on $\mathrm{R}(B)$. If x is a functional in $\Phi(C)$ and v is a C-normalising partial isometry with $x(vv^*) = 1$ then a new functional, denoted x_v, is defined by $x_v(c) = x(vcv^*)$ for all c in C. For a partial isometry v let Γv be the set $\{(x, x_v): x(vv^*) = 1\}$. The topology we consider is that having base $\{\Gamma w: w$ is a matrix unit$\}$. Direct calculation shows that the base sets are clopen and the topology is Hausdorff. The following lemma shows that it is also locally compact. Notice that the locally compact space $\Phi(C)$ is topologically isomorphic to $\{(x, x): x$ is in $\Phi(C)\}$ with this topology.

LEMMA 2.1. *If w is a matrix unit then Γw is compact.*

Proof. Fix an open covering of Γw. For every functional x with $x(ww^*) = 1$, the point (x, x_w) is in Γw so there is a matrix unit, $v(x)$ say, such that $\Gamma v(x)$ is contained in an element of the open cover and $x_w = x_{v(x)}$.

The set $\{\Gamma v(x)v(x)^*: x(ww^*) = 1\}$ forms an open cover of Γww^* which is compact, since it is isomorphic to the compact space $\Phi(Cww^*)$, so there is a finite number of functionals $x_1, \ldots, x_r$ such that the set $\{\Gamma v(x_1)v(x_1)^*, \ldots, \Gamma v(x_r)v(x_r)^*\}$ covers Γww^*. Now Γw is covered by $\{\Gamma v(x_1), \ldots, \Gamma v(x_r)\}$. ■

THEOREM 2.2 (AF Spectral Theorem). *Let w be a matrix unit. If $\Gamma w \subset \mathrm{R}(M)$ then w is in M.*

Proof. Suppose that $\Gamma w \subset \mathrm{R}(M)$, so that $x\mathrm{R}(M)x_w$ for all x with $x(ww^*) = 1$. From the previous section, for all these x there is a matrix unit $v(x)$ in M with $\Gamma v(x) \subset \Gamma w$ such that $x_w = x_{v(x)}$. Now $\{\Gamma v(x): x(ww^*) = 1\}$ is an open cover of Γw so by the lemma there is a finite subcover $\{\Gamma v(x), \ldots, \Gamma v(x_r)\}$. These sets may overlap.

Take n large enough so that $v(x_1), \ldots, v(x_r)$ are all in M_n. We have $v(x_1)\cdot \cdot v(x_1)^* + \ldots + v(x_r)v(x_r)^* \geqslant ww^*$ by identifying both sides as functions on $\Phi(C)$. The left hand side is a sum of projections from C_n. Also each $v(x_i)v(x_i)^*$ is a sum of minimal projections in C_n, the duplicated projections can be eliminated by choosing mutually orthogonal projections $p_1, \ldots, p_r$ in C_r such that $p_1v(x_1)v(x_1)^* + \cdots + +p_rv(x_r)v(x_r)^* = ww^*$. The partial isometry $v = p_1v(x_1) + \cdots + p_rv(x_r)$ is a finite sum of matrix units from M_n and has the same final space projections as w. Let e be a matrix unit in C_n with $e \leqslant ww^*$ and x a functional such that $x(e) = 1$. Since $x(vv^*) = x(ww^*) = 1$, the functional x_v exists and (x, x_v) is in $\Gamma v \subset \Gamma w$ and therefore $x_v = x_w$. This forces $ew = ev$ and over all matrix units e it implies that w is equal to v. This finishes the proof since v is in M_n. ■

It folows that a subset of $\mathrm{R}(B)$ is the fundamental relation of a C-bimodule if and only if it is an open set. Also since from [11] every C-normalising partial isometry is the product of an element of C with a sum of matrix units the theorem holds for these partial isometries too.

Suppose that x and y are functionals with $x\mathrm{R}(M)y$. From the last section there is a matrix unit w in M such that $x_w = y$. Let n be a large integer such that w is in M_n. Take $r > n$. There is a matrix unit e of B_n in C_r such that $x(e) = 1$, so $x_{ew} = y$ and ew is a matrix unit of B_n in M_n.

In summary, $x\mathrm{R}(M)y$ if and only if for all n large enough there is a matrix unit w of B_n in M_n such that $y = x_w$.

The knowledge of the topology of the fundamental relation gained above can now be applied to investigate properties of bimodules reflected in it.

PROPOSITION 2.3. *The relation* $\mathrm{R}(M)$ *is transitive, reflexive, symmetric, anti-symmetric, anti-reflexive if and only if* M *is an algebra,* $C \subset M$, $M = M^*$, $M \cap M^* \subset C$, $M \cap C = \emptyset$ *respectively.*

Proof. All the parts have similar proofs, we show one.

Suppose that M is an algebra, $x\mathrm{R}(M)y$ and $y\mathrm{R}(M)z$. For large enough n there are matrix units v and w of B_n in M_n such that $y = x_v$ and $z = y_w$. For c in C, $z(c) = y(wcw^*) = x(vwcw^*v^*)$ and so $z = x_{vw}$. Conversely, suppose that $\mathrm{R}(M)$ is transitive. Let v and w be matrix units of B_n in M_n with vw non-zero. If x is a functional with $x(vww^*v^*) = 1$ then $x\mathrm{R}(M)x_v$ and $x_v\mathrm{R}(M)x_{vw}$ so $x\mathrm{R}(M)x_{vw}$. By the AF Spectral Theorem, vw is in M_n. Now M_n is an algebra for all n and by inductivity M is too. ■

Every property of M mentioned in the above proposition is hereditary in the sense that M has it if and only if every M_n has the corresponding property. Similar techniques to the above can be used to investigate ideals of subalgebras of AF C^*--algebras which are also bimodules.

PROPOSITION 2.4. *If* M *is an algebra and* I *a* C*-bimodule then the following are equivalent.*

(i) I *is an ideal of* M.

(ii) I_n *is an ideal of* M_n *for all* n.

(iii) $\mathrm{R}(I) \subset \mathrm{R}(M)$ *and* $x\mathrm{R}(M)y$, $y\mathrm{R}(I)z \Rightarrow x\mathrm{R}(I)z$ *and* $x\mathrm{R}(I)y$, $y\mathrm{R}(M)z \Rightarrow$ $\Rightarrow x\mathrm{R}(I)z$.

Proof. (i)$\Leftrightarrow$(ii) One way is a result of the inductivity of M, the other is a direct calculation. (ii)$\Rightarrow$(iii) Suppose (ii) holds. It follows from the definition that $\mathrm{R}(I) \subset \mathrm{R}(M)$. Suppose that $x\mathrm{R}(I)y$ and $y\mathrm{R}(M)z$. For large enough n there are matrix units v and w of B_n in I_n and M_n respectively such that $y = x_v$ and $z = y_w$. Since the matrix unit vw is in I_n and $z = x_{vw}$ we have $x\mathrm{R}(I)z$. The final part is proven in a similar way.

(iii)$\Rightarrow$(ii) Suppose (iii) holds. It follows from the AF Spectral Theorem that I

is a subspace of M. Let v and w be matrix units of B_n in I_n and M_n respectively and suppose that vw is non-zero. If $x(vww^*v^*) = 1$ then $x\mathrm{R}(I)x_v$ and $x_v\mathrm{R}(M)x_{vw}$ so $x\mathrm{R}(I)x_{vw}$. Again using the AF Spectral Theorem vw is in I_n. Similarly wv is in I_n so this is an ideal of M_n. ∎

If M is an algebra containing C then any ideal of M is a C-bimodule and so the proposition classifies the fundamental relation of its ideals.

In the next section we shall investigate some more complex properties of M, ones which are semi-hereditary in some sense and ones which are not hereditary at all but can still be classified in terms of the fundamental relation.

3. MAXIMAL TRIANGULAR ALGEBRAS

We shall restrict our attention to triangular subalgebras of B which have diagonal C, such algebras can have very different properties from triangular subalgebras of finite-dimensional C^*-algebras. A triangular subalgebra A of B is *strongly maximal* if there is an increasing chain $G_1, G_2, \ldots$ of finite-dimensional $*$-algebras with closed union B such that $A \cap G_n$ is maximal triangular in G_n for all n. In [8] it is shown that this is a strictly stronger condition than maximal triangularity. Proposition 3.2 classifies the fundamental relation of such an algebra in terms similar to Corollary 3.6 of [3].

A study of the fundamental relation of a general maximal triangular algebra brings out interesting properties not found in the strongly maximal case. Theorem 3.3 classifies such relations using some topological condition which are suficiently loose to allow the examples of the next section and Section 5.

From Proposition 2.3, the C-bimodule M is a triangular algebra if and only if its (unique) fundamental relation is reflexive, transitive and anti-symmetric. We introduce a definition to help analyse strongly maximal algebras. The relation $\mathrm{R}(M)$ is called *full* (in $\mathrm{R}(B)$) if whenever $x\mathrm{R}(B)y$ then $x = y$ or $x\mathrm{R}(M)y$ or $y\mathrm{R}(M)x$. This property is more complicated than those in the last section, it is not hereditary but has a related property (ii) with the matrix units.

LEMMA 3.1. *The following are equivalent.*

(i) $\mathrm{R}(M)$ *is full in* $\mathrm{R}(B)$.

(ii) *All matrix units of B are contained in* $M_\infty + C_\infty + M_\infty^*$.

(iii) $M + M^* + C$ *is dense in* B.

Proof. (i)⇒(ii) Suppose that $\mathrm{R}(M)$ is full. Let w be a matrix unit and let x be a functional in $\Phi(C)$ with $x(ww^*) = 1$. Since $x\mathrm{R}(B)x_w$, there is a matrix unit $v(x)$

in M, C or M^* such that $x_w = x_{v(x)}$. As in the proof of the AF Spectral Theorem, using the compactness of Γw, there is a finite sequence $v_1, \ldots, v_n$ of matrix units in $M_\infty \cup C_\infty \cup M_\infty^*$ which sum to w.

(ii)$\Rightarrow$(i) Let x and y be functionals in $\Phi(C)$ with $x\mathrm{R}(B)y$, and let v be a matrix unit with $y = x_v$. Although v^* is not necessarily in M, it can be expressed as the sum of matrix units v_1 in M_∞, v_2 in C_∞ and v_3 in M_∞^*. The functional x evaluated at $v_1v_1^*, v_2v_2^*$ and $v_3v_3^*$ gives the value 1 at one of them, but each one corresponds to one of the three conditions for $\mathrm{R}(M)$ to be full.

(ii)$\Rightarrow$(iii) From (ii), B_∞ is contained in $M_\infty + C_\infty + M_\infty^*$.

(iii)$\Rightarrow$(ii) Let v be a matrix unit of B_m. If $M + C + M^*$ is dense in B then so is $M_\infty + C_\infty + M_\infty^*$ and there is a sequence of a_n in $M_n + C_n + M_n^*$ with $a_n \to v$. If v is not in the C_n-bimodule $M_n + C_n + M_n^*$ for $n > m$ then for some projections p_n and q_n in C_n the product $q_n(v - a_n)p_n$ is a matrix unit q_nvp_n so $\|v - a_n\| \geqslant 1$. The convergence of the sequence shows that v is in $M_\infty + C_\infty + M_\infty^*$. ∎

PROPOSITION 3.2. *The C-bimodule M is strongly maximal triangular if and only if its fundamental relation totally orders each equivalence class of* $\mathrm{R}(B)$.

Proof. From Proposition 2.3, $\mathrm{R}(M)$ is a partial order on $\Phi(C)$ if and only if M is a triangular algebra containing C. This partial order on $\Phi(C)$ is a total order on each equivalence class of $\mathrm{R}(B)$ if and only if $\mathrm{R}(M)$ is full.

If M is strongly maximal then $M + M^*$ is dense in B so $\mathrm{R}(M)$ is full. Conversely, suppose that $\mathrm{R}(M)$ is full. From the lemma, for every integer n and for large enough $k > n$ there is a smallest finite set of matrix units of B_k in M_k such that each matrix unit in B_n is a sum of elements and adjoints from this set. The C_k-bimodule generated by this matrix units is triangular and is by construction maximal triangular in the C_k-bimodule G_n generated by B_n. Some subsequence of $G_1, G_2, \ldots$ is increasing and so M is strongly maximal. ∎

We now study more general maximal triangular algebras. The condition on M equivalent to it being a maximal triangular algebra is that if S is an open, transitive, reflexive, anti-symmetric subset of $\mathrm{R}(B)$ containing $\mathrm{R}(M)$ then the two are equal. A more useful characterisation can be obtained by studying the topology on the relation.

If M_0 is a triangular subalgebra of a finite-dimensional C^*-algebra B_0 and v is a partial isometry sum of non-self-adjoint matrix units from B_0 then there are matrix units u_i in M_0 such that $u_1vu_2\ldots vu_n$ is a non-zero projection if and only if the algebra generated by M_0 and v is not triangular. In graph theory terms this means that the $M_0 \cap M_0^*$-bimodule generated by M_0 and v has a directed cycle in its directed graph. A directed cycle is a finite, non-zero sequence of arrows, each arrow coming from the vertex which the previous arrow went to and the last arrow going to the vertex which the first arrow started from. None of the arrows may start and finish at the same vertex.

THEOREM 3.3. *The C-bimodule M is maximal triangular if and only if all of the following conditions hold.*

(i) $\mathrm{R}(M)$ *is transitive, reflexive and anti-symmetric.*

(ii) *If U is a non-empty open subset of* $\mathrm{R}(B)\backslash(\mathrm{R}(M)\cap\mathrm{R}(M^*))$ *then* $\mathrm{R}(M)\cap U$ *contains a directed cycle.*

(iii) *If U is a open subset of* $\mathrm{R}(M)$ *and its closure is open then this is also in* $\mathrm{R}(M)$.

Proof. Suppose that M is maximal triangular. By Proposition 2.3, (i) holds. Let U be an open subset of $\mathrm{R}(B)\backslash(\mathrm{R}(M)\cap\mathrm{R}(M^*))$. Since $\mathrm{R}(M)\cap U$ is open it is the fundamental relation of a bimodule, N say, which contains M. If $\mathrm{R}(M)\cap U$ contains no directed cycle then the algebra generated by each N_n is triangular by definition, so the algebra generated by N is triangular. Since M is maximal, $N = M$ and so U is an empty set, (ii) holds.

Let v be a matrix unit not in M and let U be an open subset of Γv. It will be shown that U is not dense in Γv. Since Γv is not in $\mathrm{R}(M)$ we can take an integer n such that the algebra generated by M_n and v is not triangular and v is in B_n. Since v is a sum of non-self-adjoint matrix units of B_n, there are matrix units $u_1,\ldots,u_r$ of B_n in M_n with $r>1$ such that $u_1vu_2v\ldots vu_r = p$, a non-zero projection. If x is a functional with $x(p)=1$ then a directed cycle in $\mathrm{R}(M)\cap\Gamma v$ can be defined from x as follows. Let $x_0 = x$ and let

$$x_{2i}(c) = x(u_1vu_2v\ldots vu_ivc(u_1vu_2v\ldots vu_iv)^*) \quad \text{for } i = 1,\ldots,r-1,$$

$$x_{2i-1}(c) = x(u_1vu_2v\ldots vu_ic(u_1vu_2v\ldots vu_i)^*) \quad \text{for } i = 1,\ldots,r$$

define functionals $x_0,\ldots,x_{2r-1}$ with $x_{2r-1} = x_0$ and (x_i, x_{i+1}) in $\mathrm{R}(M)\cap\Gamma v$ for all i. This is a directed cycle and so for some positive integer i the element (x_{2i-1}, x_{2i}) is not in $\mathrm{R}(M)$. Let $X(j)$ be the set of functionals y with $y(p)=1$ and (y_{2j-1}, y_{2j}) not in $\mathrm{R}(M)$. If z is a functional with $z(p)=1$ and (z_{2j-1}, z_{2j}) in $\mathrm{R}(M)$ then there is a matrix unit w in M such that $\Gamma w \subset \Gamma v$ and w is small enough not to be a sum of more than one matrix unit in B_n. Also we can choose w so that Γw is a subset of the complement of $X(j)$ in Γv, thus this complement is open and $X(j)$ is closed. Since $X(1)\cup\ldots\cup X(r) = \Gamma p$, the finite version of Baire's Theorem (true for all topological spaces) forces some $X(j)$ to have non-empty interior. This implies that a non-empty open subset of Γv is in $\mathrm{R}(M)$ so U is not dense in Γv. Thus if the closure of an open subset of $\mathrm{R}(M)$ is an element of the base then this element is also in $\mathrm{R}(M)$.

For a general open set U with open closure, the closure of U is the union of countably many elements $W_1, W_2, \dots$ of the base and $U \cap W_i$ is an open set with closure W_i so by the above W_i is in $\mathrm{R}(M)$ and hence the whole of the closure of U is in $\mathrm{R}(M)$ and (iii) holds.

For the converse, suppose that M is a C-bimodule satisfying (i), (ii) and (iii). By (i), M is a triangular algebra.

Suppose that W is an open set and that the C-bimodule with fundamental relation $\mathrm{R}(M) \cup W$ is also a triangular algebra. It will be shown that W is contained in $\mathrm{R}(M)$. If the set $W \cap \mathrm{R}(M^*)$ is not empty then since it is open it contains Γw for some matrix unit w. For large enough n both w and w^* are in M_n so it is not triangular. Thus $W \cap \mathrm{R}(M^*)$ must be the empty set.

Suppose that U is an open subset of W with $U \cap \mathrm{R}(M)$ empty. From the above paragraph $U \cap \mathrm{R}(M^*)$ is also empty. Now U must be empty by (ii) as the fundamental relation of a triangular algebra is transitive and anti-symmetric hence it contains no directed cycles. Every open subset of W has non-empty intersection with $\mathrm{R}(M)$, so $W \cap \mathrm{R}(M)$ is dense in W. It follows that $\mathrm{R}(M)$ is dense in $\mathrm{R}(M) \cup W$. Let v be a matrix unit with Γv in $\mathrm{R}(M) \cup W$. Now Γv is clopen and $\mathrm{R}(M) \cap \Gamma v$ is dense in Γv so that by (iii), Γv is in $\mathrm{R}(M)$. This finishes off the proof since $\mathrm{R}(M) \cup W$ is the union of the base sets it contains. ∎

4. EXAMPLES

The following three examples will illustrate both the conditions of Theorem 3.3 and the properties which maximal triangular subalgebras of AF algebras may have. For the first two examples the AF algebra will be the tensor product of an abelian AF algebra with a matrix algebra. This form is chosen for its simplicity and for the ease of visualising the fundamental relation in this context.

Let D be the abelian AF algebra which is the limit of the sequence $D_1, D_2, \dots$ where D_n is the direct sum of 2^n copies of $\mathbb{C}$ and the i^{th} copy of $\mathbb{C}$ in D_n is embedded in the $2i^{\text{th}}$ and $2i+1^{\text{th}}$ copy of $\mathbb{C}$ in D_{n+1}. As matrix units for each D_n, use the projections onto each copy of $\mathbb{C}$. This forms a D-normalising system.

Identifying the i^{th} matrix unit of D_n with the characteristic function of the interval $[(i-1)2^{-n}, i2^{-n}]$ the algebra D is isomorphic to a subalgebra of $L^\infty[0,1]$, the algebra of multiplication operators on $L^2[0,1]$. A unital multiplicative linear functional on D corresponds otherwise to pointwise evaluation at a non-dyadic point or to left or right limit evaluation at a dyadic point in $[0,1]$. Thus the fundamental relation of a D-bimodule can be identified with a subset of $[0,1]^2$, with each dyadic rational in (0,1) counting twice.

Let e_i be the penultimate matrix unit of D_{2i-1} and let f_i be the penultimate matrix unit of D_{2i} for all i. The set $E = \Gamma e_1 \cap \Gamma e_2 \cap \ldots$ is open. Let x be the functional on D defined on each D_n by $x(\lambda_1, \ldots, \lambda_{2^n}) = \lambda_{2^n}$, so that x is not in E since $x(e_i) = 0$ for all i. However if v is a matrix unit with Γv containing (x, x) then it contains Γe_n for all large enough n so (x, x) is a limit point for E. In fact (x, x) is the only limit point for E using the same argument. Let $F = \Gamma f_1 \cup \Gamma f_2 \cup \ldots$ which is the complement of E so $E \cup F$ is an open set with closure the maximal ideal space of D. In fact $E \cup F$ corresponds to the set [0,1), missing just the functional of left limit evaluation at 1.

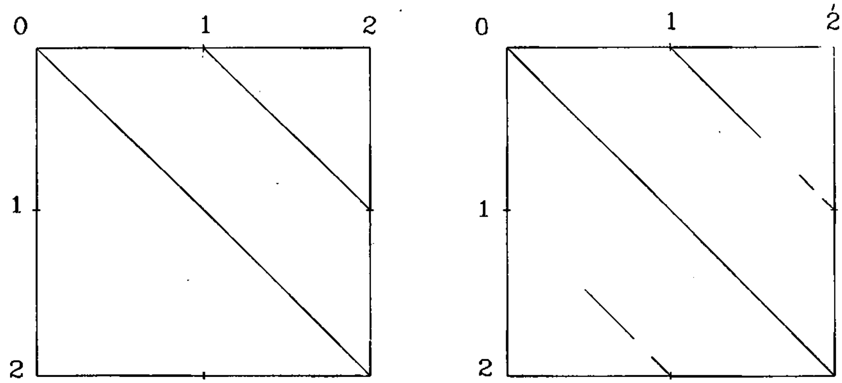
For any integer k the algebra of continuous functions from the maximal ideal space of D to the matrix algebra $M(k)$ is a unital AF algebra isomorphic to the tensor product $D \otimes M(k)$ and is the limit of the finite-dimensional $*$-algebras $D_n \otimes M(k)$. With respect to this sequence the space of continuous functions into the diagonal is a canonical masa. We shall denote the standard matrix units of $M(k)$ by v_{ij} for $i, j = 1, \ldots, k$ in the usual way. We tensor the matrix units of D with all the v_{ij} to give a system of matrix units for $D \otimes M(k)$.

We can identify $D \otimes M(k)$ as a subalgebra of $L(L^2[0, k])$. Identify the operator $1 \otimes v_{ij}$ with the identity map from $L^2[j-1, j]$ to $L^2[i-1, i]$ and $D \otimes v_{ij}$ with a subspace of $L^\infty[i-1, i]$ as above. From this the fundamental relation of a bimodule is isomorphic to a subset of the set of (α, β) in $[0, k] \times [0, k]$ such that $\alpha - \beta$ is an integer, with pairs of dyadic points counting twice.

EXAMPLE 4.1. This example will be explained in detail, the next one is similar. Let S be the union of the sets $\Gamma(e_i \otimes v_{12}) \cup \Gamma(f_i \otimes v_{12})$ for all i together with $\Gamma(D \otimes v_{11})$ and $\Gamma(D \otimes v_{22})$. As a set, this is an open subset of $\mathrm{R}(D \otimes M(2))$ which is reflexive, transitive and anti-symmetric but it is not the fundamental relation of a maximal triangular algebra since the closure $S \cup \Gamma(1 \otimes v_{12})$ of S is also open, reflexive, anti-symmetric and transitive, it contains one more point than S. Let N be the bimodule with fundamental relation S. Every $N_n = N \cap (D_n \otimes M(2))$ is isomorphic to the direct sum of $2^n - 1$ copies of the upper triangular part of $M(2)$ and one copy of the diagonal of $M(2)$. The diagram below left shows the fundamental relation. Every closed interval in a line corresponds to a sum of matrix units.

Let S' be the union of the sets $\Gamma(e_i \otimes v_{12}) \cup \Gamma(f_i \otimes v_{21})$ for all i together with the sets $\Gamma(D \otimes v_{11})$ and $\Gamma(D \otimes v_{22})$. This is an open subset of $\mathrm{R}(D \otimes M(2))$ which is reflexive, transitive and anti-symmetric, it is the fundamental relation of N' say. This is a maximal triangular algebra but not strongly maximal since if v is a matrix unit not in N' or N'^* then $\Gamma v \cap \mathrm{R}(N')$ and $\Gamma v \cap \mathrm{R}(N'^*)$ are both non-empty. The closure of

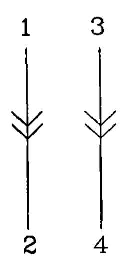
S' is not open, it contains two more points than S'. Note that each $N' \cap (D_n \otimes M(2))$ is isomorphic to N_n! The diagram below right shows the fundamental relation.



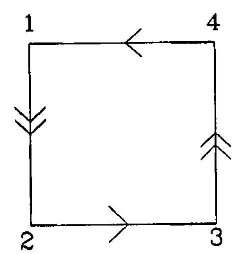
EXAMPLE 4.2. Let S be the union of the sets $\Gamma(1 \otimes v_{ij})$ for $i \leqslant j$ and $(i,j) \neq (3,4)$ together with the union of all the sets $\Gamma(e_i \otimes v_{34}) \cup \Gamma(f_i \otimes v_{43})$ for all i. This is an open subset of $\mathrm{R}(D \otimes M(4))$ which is reflexive, transitive and anti-symmetric with the bimodule N having fundamental relation S maximal triangular, but not strongly maximal as in the second example above. In fact the C^*-algebra generated by N is the entire AF algebra since $1 \otimes v_{34}$ is the product $(1 \otimes v_{32})(1 \otimes v_{24})$, so N is an example of a maximal triangular algebra in the C^*-algebra it generates.

EXAMPLE 4.3. We shall see that it is possible to have a maximal triangular algebra A and a matrix unit w such that $\Gamma w \cap \mathrm{R}(A) = \Gamma w \cap \mathrm{R}(A^*) = \{\ \}$, justifying part (ii) of Theorem 3.3. The containing algebra will be the CAR algebra, so that B_n is isomorphic to $M(2^n)$ for all n. The example will be constructed in terms of the directed graph of a bimodule M at each stage M_n, with the labelling of the vertices, which is important to keep track of the bimodule, from the standard embedding.

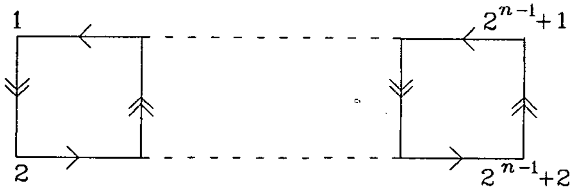
We use single arrows to denote matrix units in M and double arrows for those which are not in M but sum with others to w. Let M_1 have the digraph $1 \twoheadrightarrow 2$ this double arrow represents all of w. The C_2-bimodule generated by M_1 in B_2 is :



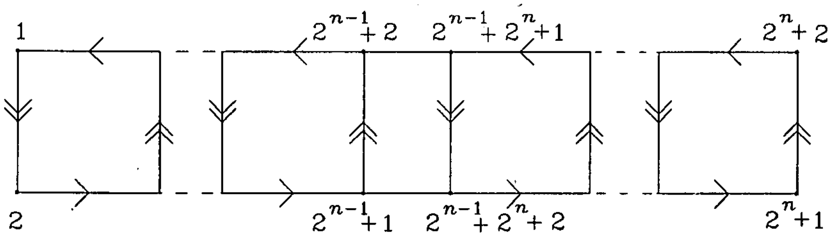
and M_2 is



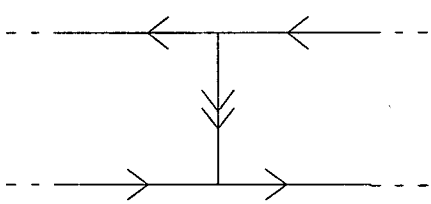
If the digraph



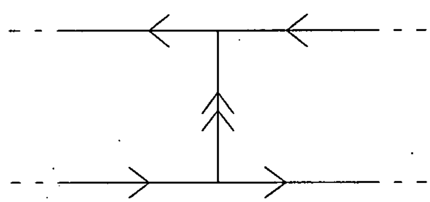
is of M_n then the digraph of M_{n+1} is obtained by taking two copies of the digraph of M_n and joining them together after rotating the second by 180 degrees:



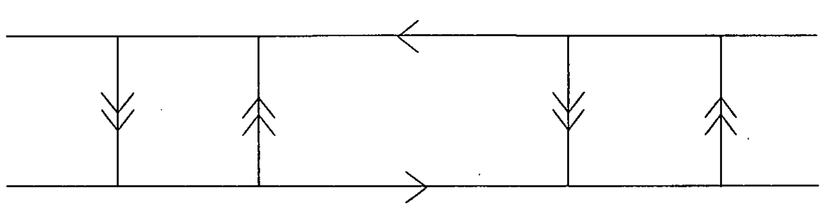
Now if v is a matrix unit of B_n with $\Gamma v \subset \Gamma w$ then in the digraph of M_n the double arrow representing v is



or



In the first case, from the construction of the bimodule, the four matrix units of B_{n+2} which sum to v are represented on the digraph of M_{n+2}, with the direction of their arrows coming from that of v, by



so either v or v^* added to M_n, will result in the algebra generated by M_{n+2} not being triangular. The same happens in the second case.

The same argument as above shows that the algebra A generated by M is maximal triangular with $A_n = \operatorname{alg} M_n$ which is isomorphic to the direct sum of two copies of the upper triangular part of $M(2^{n-1})$. In fact A is very small to be maximal triangular, for example the ratio of the number of matrix units in A_n to the number in the upper triangular part of $M(2^n)$ tends to 0.5 as n increases.

REMARK 4.4. The example above of a maximal but no strongly maximal triangular algebra can be embedded in the hyperfinite II_1 factor. The binary relation P on [0,1] defined by $x\mathrm{P}y$ if and only if $x-y$ is dyadic and together with measure ν defined on it from Lebesgue measure and counting measure can be used with the techniques of Feldman and Moore to construct this factor. Our example can be embedded in this algebra but the property of not totally ordering the euqivalence classes of the diagonal, which ultraweakly closed maximal triangular algebras must have as shown in [3], is lost in the containing maximal triangular algebra. The partial isometries $\chi_S w$ and $w'\chi_{S'}$ can be added, where S is a Borel subset of [0,1], and S' its complement with the property that for all $0 \leqslant \alpha < \beta \leqslant 1$ the measures of both $S \cap (\alpha,\beta)$ and $S' \cap (\alpha,\beta)$ are non-zero. The first of these is supported on a subset of the support of w.

5. REPRESENTATIONS

A representation of a subalgebra of a C^*-algebra admits a Stinespring dilation if and only if it is completely contractive [1]. Every contractive representation of a finite-dimensional nest algebra is completely contractive from [7]. Thus, by density, every contractive representation of a strongly maximal triangular algebra is completely contractive and so has a Stinespring dilation. In fact it follows in a similar way from [6] that this is also true for the tensor product of two strongly maximal triangular algebras. For general maximal triangular algebras this fails, as in the folowing example.

In [6] a representation ρ of the algebra

$$A = \begin{pmatrix} D(3) & M(3) \\ 0 & D(3) \end{pmatrix}$$

(where $D(3)$ is the diagonal of $M(3)$) which is contractive but not completely contractive was given. The methods of the first two examples of the last section will be used to construct a maximal triangular algebra as the sum of the algebra A with another, A'. The representation $0+\rho$ of $A'+A$ will then be shown to be contractive without having a Stinespring dilation.

Let S be the union of the sets $\Gamma(1 \otimes v_{ii})$ for $i = 1,\ldots,6$, $\Gamma(1 \otimes v_{ij})$ for $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 6$, $\Gamma(e_n \otimes v_{ij})$ for all n and $1 \leqslant i < j \leqslant 6$, and $\Gamma(f_n \otimes v_{ij})$ for $(i,j) =$

$= (2,1),(3,1),(3,2),(6,5),(6,4),(5,4)$. The bimodule N in $D \otimes M(6)$ with fundamental relation S has been constructed so that $N \cap (D_n \otimes M(6))$ is isomorphic to the direct sum of $2^n - 1$ copies of the upper triangular part of $M(6)$ with one copy of the algebra A. From Theorem 3.3 it is maximal triangular. Writing N as $A' + A$, if b is in A' and a is in A then $\|a + b\| \geqslant \|a\|$ so the representation $0 + \rho$ is contractive.

Supported at Lancaster University by the Science and Engineering Research Council.

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Received September 4, 1989 ; revised February 10, 1990.