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POSITIVE COMPLETIONS OF MATRICES OVER C^*-ALGEBRAS

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1. INTRODUCTION

Let A be a C^*-algebra, and let $M_n(A)$ be the C^*-algebra of $n \times n$ matrices with entries in A. An element $a \in A$ is called *positive* (notation: $a \geqslant 0$) if $a = b^*b$ for some $b \in A$, and *strictly positive* if a is positive and invertible (in this case it is assumed that A is unital). Analogously one defines positive and strictly positive elements in $M_n(A)$.

A *partial Hermitian* $n \times n$ matrix Q over A is, by definition, an $n \times n$ matrix some of whose entries are specified to be elements in A in such a way that if the (j,i) entry in Q is $a \in A$ then (j,i) entry in Q is also specified and it is equal to a^*. The unspecified entries in Q are question marks. It will be always assumed that the entries on the diagonal are specified.

The pattern of specified entries in a partial Hermitian $n \times n$ matrix Q over A will be described by the undirected graph G having vertices $\{1, 2, \ldots, n\}$ and an edge (i,j) if and only if $i \neq j$ and the entry (i,j) in Q is specified. Conversely, given an undirected graph G with vertices $\{1, 2, \ldots, n\}$ and without edges from a vertex to itself (only such graphs will be considered in this paper), we say that a partial Hermitian $n \times n$ matrix Q over A is *subordinate* to G if the entry (i,j) in Q is specified precisely when either $i = j$ or $i \neq j$ and (i,j) is an edge in G.

A *completion* of a partial Hermitian matrix $Q = [Q_{i,j}]_{i,j=1}^{n}$ over A is any $Z = [Z_{i,j}]_{i,j=1}^{n} \in M_n(A)$ with the property that $Z_{ij} = Q_{ij}$ whenever the entry Q_{ij} is specified as an element in A.

We will be interested in positive and strictly positive completions. An obvious necessary condition for existence of a positive (or strictly positive) completion of a partial Hermitian matrix $Q = [Q_{ij}]_{i,j=1}^n$ over A subordinate to the graph G is that for any clique V of G the matrix $[Q_{ij}]_{i,j\in V} \in M_k(A)$ is positive (or strictly positive), where k is the number of elements in V. (Recall that a set of vertices V is called a *clique* of G if there is an edge in G between any two distinct vertices in V.) The main purpose of this paper is to characterize the patterns (or graphs) G for which this necessary condition is also sufficient for *all* partial Hermitian matrices subordinate to G.

In the case $A = \mathbb{C}$ or A is the algebra of all $n \times n$ complex matrices, the problem of positive and strictly positive completions was extensively studied (see, e.g., [5], [12], [8], [14]), and for more general algebras (but less general patterns) this problem was studied in [6], [13].

Throughout the paper we denote by H a (complex) Hilbert space, and by $B(H)$ the algebra of all bounded linear operators on H.

2. STRICTLY POSITIVE COMPLETIONS

An undirected graph G without loops from a vertex to itself is called *chordal*, or *triangulated*, if it has no minimal (simple) circuits with 4 or more edges.

THEOREM 2.1. *Let A be a unital C^*-algebra, and let G be a chordal graph. Then every partial Hermitian matrix $Q = [Q_{ij}]_{i,j=1}^n$ over A subordinate to G, with the property that for every clique V of G matrix $[Q_{ij}]_{i,j\in V} \in M_k(A)$ is strictly positive, admits a strictly positive completion. Conversely, assume that the graph G is not chordal. Then there exists a partial Hermitian matrix $Q^{(0)} = [Q_{ij}^{(0)}]_{i,j=1}^n$ over A subordinate to G, with the property that for every clique V of G the matrix $[Q_{ij}^{(0)}]_{i,j\in V} \in M_k(A)$ is strictly positive, which does not admit a strictly positive (even positive) completion. In fact, the specified entries $Q_{ij}^{(0)}$ can be chosen to be scalar multiples of e, where e is the unit in A.*

It should be noted that the condition on every clique V of G in Theorem 2.1 can be obviously replaced by the same condition on every *maximal* (in the sense of set-theoretic inclusion) clique of G.

The proof of the direct statement of Theorem 2.1 is based on the following well-known property of chordal graphs (see, e.g., [11]), which allows us to use the "one step at a time" approach.

LEMMA 2.2. *Let G be a chordal graph. Then there exists a sequence of chordal graphs $G_1 = G, G_2, G_3, \ldots, G_p$ with the following properties:*

(i) *G_j is obtained from G_{j-1} by adjoining precisely one edge, call it (r_j, s_j), to*

G_{j-1};

(ii) *there is exactly one maximal clique in* G_j *that contains both vertices* r_j *and* s_j;

(iii) *the last graph in the sequence* G_p *is the full graph,* i.e. *there is an edge between any two distinct vertices in* G_p.

For the proof of the converse statement of Theorem 2.1 the following lemma is needed.

LEMMA 2.3. *For a given* $n \geqslant 3$ *define the set*

$$D_n = \left\{Q = \left[Q_{ij}\right]_{i,j=1}^n \in M_n(A) : \ Q_{ij} = e \text{ if } |i-j| \leqslant 1; \ Q_{1,n} = Q_{n,1} = -e\right\}.$$

Then the distance between D_n *and the set of all positive elements in* $M_n(A)$ *is positive.*

Proof. Let $P = [a_{ij}]_{i,j=1}^n$ be a positive element in $M_n(A)$ such that $a_{ij} = e$ if $|i-j| \leqslant 1$. We claim that all elements in P are equal to e (thereby proving the lemma). Indeed, using Lemma 2.1 in [4] we have

$$0 \leqslant \begin{pmatrix} e & e & a_{13} \\ e & e & e \\ a_{13}^* & e & e \end{pmatrix} - \begin{pmatrix} e \\ e \\ a_{13}^* \end{pmatrix} [e \ e \ a_{13}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e - a_{13} \\ 0 & e - a_{13}^* & e - a_{13}^* a_{13} \end{pmatrix}.$$

Consequently, $e - a_{13} = 0$. Analogously, we prove that $a_{ij} = e$ for $|i-j| = 2$. To prove that $a_{14} = e$ apply the same argument to the 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{14} \\ a_{12}^* & a_{22} & a_{24} \\ a_{14}^* & a_{24}^* & a_{44} \end{pmatrix} = \begin{pmatrix} e & e & a_{14} \\ e & e & e \\ a_{14}^* & e & e \end{pmatrix},$$

and so on. ■

Proof of Theorem 2.1. Assume G is chordal. Using Lemma 2.2, we can assume by induction that the result of Theorem 2.1 is already proved for the chordal graph G_2. Given a partial Hermitian matrix $Q = \left[Q_{ij}\right]_{i,j=1}^n$ over A subordinate to G as in Theorem 2.1, it remains to find $a \in A$ with the following property: Let $Q' = \left[Q'_{ij}\right]_{i,j=1}^n$ be the partial Hermitian matrix subordinate to G_2 obtained from Q by replacing the question marks in (r_2, s_2) and (s_2, r_2) entries by a and a^*, respectively. Then for every clique V' of G_2 the matrix $\left[Q'_{ij}\right]_{i,j \in V'}$ is strictly positive. Because of the property (ii) of Lemma 2.2 the only clique we have to worry about is the unique maximal clique of G_2 that contains both vertices r_2 and s_2. But then existence of the required $a \in A$ follows from the general result of [13].

For the sake of completeness, we outline here a proof of the existence of the required $a \in A$ which is sufficient for our case.

By the above discussion it is sufficient to show that if b is a $1 \times p$ matrix over A, C is a $p \times p$ matrix over A, d is a $p \times 1$ matrix over A and a and f are elements of A such that the matrices $\begin{pmatrix} a & b \\ b^* & C \end{pmatrix}$ and $\begin{pmatrix} C & d \\ d^* & f \end{pmatrix}$ are strictly positive matrices over A, then there exists x in A such that

$$\begin{pmatrix} a & b & x \\ b^* & C & d \\ x^* & d^* & f \end{pmatrix} \tag{2.1}$$

is a strictly positive matrix over A. To this end let $b_1 = a^{-\frac{1}{2}} b C^{-\frac{1}{2}}$, $d_1 = C^{-\frac{1}{2}} d f^{-\frac{1}{2}}$ and note that $\|b_1\|$, $\|d_1\|$ are both strictly less than 1. We claim that setting $x = a^{\frac{1}{2}} b_1 d_1 C^{\frac{1}{2}}$ suffices. To see this, note that

$$\begin{pmatrix} a^{-\frac{1}{2}} & 0 & 0 \\ 0 & C^{-\frac{1}{2}} & 0 \\ 0 & 0 & f^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} a & b & x \\ b^* & C & d \\ x^* & d^* & f \end{pmatrix} \begin{pmatrix} a^{-\frac{1}{2}} & 0 & 0 \\ 0 & C^{-\frac{1}{2}} & 0 \\ 0 & 0 & f^{-\frac{1}{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} e & b_1 & b_1 d_1 \\ b_1^* & e & d_1 \\ d_1^* b_1^* & d_1^* & e \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & d_1^* & e \end{pmatrix} \begin{pmatrix} e & b_1 & 0 \\ b_1^* & e & 0 \\ 0 & 0 & e - d_1^* d_1 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & e & d_1 \\ 0 & 0 & e \end{pmatrix}$$

which is easily seen to be strictly positive.

Conversely, assume G is not chordal, and let $\{1, \dots, k\}$ be the set of vertices ($k \geqslant 4$) that form a minimal circuit. Thus, the only edges in G between the vertices $1, \dots, k$ are $(1,2)$,$(2,3)$,$\dots$,$(k-1,k)$,$(k,1)$. For $\delta > 0$ sufficiently small let $Q = [Q_{ij}]_{i,j=1}^n$ be the partial Hermitian matrix subordinate to G defined by $Q_{ii} = (1+\delta)e$ ($i = 1, \dots, n$); $Q_{ij} = e$ for $|i - j| = 1$, $1 \leqslant i,j \leqslant k$; $Q_{1k} = Q_{k1} = -e$; and $Q_{ij} = 0$ in all other specified positions. Clearly, for every clique V of G the matrix $[Q_{ij}]_{i,j \in V}$ is strictly positive. However, by Lemma 2.3, if $\delta > 0$ is sufficiently small, there is no strictly positive completion for Q.

The partial hermitian matrix Q constructed above proves also the last sentence of Theorem 2.1. ∎

3. POSITIVE COMPLETIONS

We turn now to the patterns that allow positive completions for any partial Hermitian matrix over A provided the obvious necessary condition is satisfied.

A partial Hermitian matrix $[p_{ij}]_{i,j=1}^n$ over C^*-algebra A subordinate to graph G will be called *partially positive* if for every clique V in G the matrix $[p_{ij}]_{i,j \in V}$ is positive.

We start with an example showing that Theorem 2.1 cannot be generally carried over to positive extensions:

EXAMPLE 3.1. Let $z = re^{i\theta}$ be a complex number in polar form. Consider

$$\begin{pmatrix} 1 & x & z \\ \overline{x} & 1 & |z| \\ \overline{z} & |z| & |z|^2 \end{pmatrix}, \tag{3.1}$$

and x unknown. It is easy to show that this a partially positive, chordal pattern and that for $z \neq 0$ the only value of x for which there is a positive completion is $x = e^{i\theta}$. Hence the completion is discontinuous as $|z| \to 0$. So in the C^*-algebra of continuous functions on the closed unit disc $\overline{\mathbb{D}}$, with z the coordinate function, the partial matrix (3.1) has no positive completion. ■

This example can be fairly easily modified to $C([0,1])$: replace z by $t \cdot g(t)$ where $|g(t)| = 1$ for all t, g is continuous on $(0,1]$ but $\lim_{t \to 0+} g(t)$ is dense on the unit circle. Again $x = g(t)$ is the unique completion, but is discontinuous at 0.

In view of this example we introduce the following definition. Let (PC) stand for the algebras A with the property that all partially positive matrices over A with chordal patterns have positive completions. Example 3.1 shows that $C(\overline{\mathbb{D}})$ and $C([0,1])$ are not (PC). On the other hand, Theorem 6 of [12] shows that the algebra of all $n \times n$ matrices is (PC). Actually, every finite dimensional C^*-algebra is (PC), and we will see more classes of (PC) algebra later on. Also $B(H)$ is (PC) (Theorem 4.3 in [18]).

A C^*-algebra A is called *injective* (see [17]) if every completely positive map into A has a completely positive extension, or, equivalently, if for any representation of A as a C^*-algebra of operators on $B(H)$ there exists a completely positive projection $\Phi: B(H) \to A$ with $\Phi(a) = a$ for all a in A. The equivalence follows easily from the fact that $B(H)$ is injective (by the Arveson extension theorem [1], [17]).

THEOREM 3.1. *Every injective C^*-algebra and every W^*-algebra (= weakly closed $*$-algebra of operators containing I on a Hilbert space H) is* (PC).

Proof. Assume that A is injective, and represent A as a C^*-algebra of operators on some Hilbert space H. Let $Q = [Q_{ij}]_{i,j=1}^n$ be a partially positive matrix over A subordinate to G. By [18, Theorem 4.3] there exists a positive completion of Q, call it $P = [P_{ij}]_{i,j=1}^n$, over $B(H)$. Let $\Phi: B(H) \to A$ be a completely positive projection onto A, as in the definition of an injective C^*-algebra. Then $[\Phi(P_{ij})]_{i,j=1}^n$ is a positive completion of Q over A.

Assume now that A is a W^*-algebra, and let $Q = [Q_{ij}]_{i,j=1}^n$ be as in Theorem 3.1.

For $m = 1, 2, \ldots$ the partial Hermitian matrix $Q_m = Q + \frac{1}{m}I$ is strictly positive, hence by Theorem 2.1 Q_m admits a strictly positive completion $Z_m = [Z_{ij}^{(m)}] \in M_n(A)$. It is easy to see that the norms $\|Z_{ij}^{(m)}\|$ $(m = 1, 2, \ldots)$ are uniformly bounded. Now we use the fact that the unit ball in $B(H)$ is weakly* compact. So there is a subnet (m_λ) such that the weak* limits $Z_{ij}^{(m_\lambda)} \to Z_{ij}$ exist, and $Z_{ij} \in A$. One verifies without difficulty that $Z = [Z_{ij}]_{i,j=1}^{n}$ is a positive completion of Q. ∎

We remark that every weakly closed $*$-algebra of operators on H contains its own identity (which need not be the identity operator); see, e.g., [19]. Because of this fact, the proof of Theorem 3.1 extends to all weakly closed $*$-algebras of operators on H. Furthermore, by Sakai's theorem, every C^*-algebra which is a dual (as a Banach space) can be identified with such algebra. So every dual C^*-algebra is (PC).

It turns out that the (PC) property is preserved under passing to ideals and quotients.

THEOREM 3.2. *Let A be a C^*-algebra with the* (PC)*-property, and let J be a two-sided $*$-ideal in A. Then J and A/J are* (PC).

Proof. Let $P = [p_{ij}]_{i,j=1}^n$ be a partially defined matrix with specified entries in J subordinate to a chordal pattern G, and assume that for every clique V of G the matrix $[p_{i,j}]_{i,j\in V}$ is positive. As A is (PC), there is a positive completion $\tilde{P} = [\tilde{p}_{ij}]_{i,j=1}^n$ of P with entries in A. Passing to the factor algebra A/J we see that $\tilde{P} + J = [\tilde{p}_{ij} + J]_{i,j=1}^n$ is positive in A/J. However, $\tilde{p}_{ii} + J = \tilde{p}_{jj} + J = 0$, hence the positivity of $\tilde{P} + J$ implies that $\tilde{p}_{ij} + J = 0$ as well for all $i \neq j$. So actually all entries in $\tilde{P}$ belong to J.

For the proof of the second part of Theorem 3.2 it is convenient to prove first two lemmas.

LEMMA 3.3. *Let $[p_{ij}]_{i,j=1}^n \geqslant 0$ then* $n\begin{pmatrix} p_{11} & & 0 \\ & \ddots & \\ 0 & & p_{nn}\end{pmatrix} - [p_{ij}]_{i,j=1}^n \geqslant 0.$

Proof. Let λ be a primitive n-th root of unity, and let D_k be the diagonal unitary whose i-th entry is $(\lambda^k)^i$. Then $D_k[p_{ij}]_{i,j=1}^n D_k^* = [(\lambda^k)^{i-j}p_{ij}]_{i,j=1}^n \geqslant 0$. Hence

$$0 \leqslant \sum_{k=1}^{n-1} D_k[p_{ij}]_{i,j=1}^n D_k^* = \begin{pmatrix} (n-1)p_{11} & -p_{12} & \ldots & -p_{1n} \\ -p_{21} & (n-1)p_{22} & \ldots & -p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{n1} & -p_{n2} & \ldots & (n-1)p_{nn} \end{pmatrix} =$$

$$= n\begin{pmatrix} p_{11} & & 0 \\ & \ddots & \\ 0 & & p_{nn}\end{pmatrix} - [p_{ij}]_{i,j=1}^n,$$

since

$$\sum_{k=1}^{n-1}(\lambda^{i-j})^k = \begin{cases} n-1 & i=j \\ -1 & i\neq j,\ |i-j|<n \end{cases}.$$

LEMMA 3.4. *Let $\mathcal{A}$ be a C^*-algebra with a two-sided $*$-ideal J. Let $p_{ij} \in \mathcal{A}/J$ so that $P = [p_{ij}]_{i,j=1}^n$ is a partially positive Hermitian matrix over $\mathcal{A}/J$ subordinate to a chordal pattern. Then there exists a partially positive Hermitian matrix $A = [a_{ij}]_{i,j=1}^n$ over $\mathcal{A}$ which lifts P.*

Proof. We use a property of chordal graphs that they have a perfect vertex elimination scheme (see, e.g., [11]). Namely, there exists an ordering of vertices $S_n, S_{n-1}, \dots, S_1$ so that S_n is a simplicial vertex in G (i.e. the set of vertices adjacent to s_n is a clique), and if we remove s_n from G (toghether with all the adjacent edges) then s_{n-1} is a simplicial vertex in the remaining graph, etc. Without loss of generality we assume that in this perfect vertex elimination scheme $s_n = n$, $s_{n-1} = n-1, \dots, s_1 = 1$.

Starting with s_1 we may lift p_{11} to a positive element a_{11} in $\mathcal{A}$. Assume that we have defined a_{ij} in $\mathcal{A}$ for all (i,j) in the graph with $1 \leqslant i,j \leqslant k-1$, such that $(a_{ij})_{i,j=1}^{k-1}$ is partially positive and $\pi(a_{ij}) = p_{ij}$ where $\pi\colon \mathcal{A} \to \mathcal{A}/J$ is the natural homomorphism. We wish to define inductively $a_{i,k}$ and $a_{k,j}$ for $1 \leqslant i,j \leqslant k$ so that $[a_{ij}]_{i,j=1}^k$ is partially positive, $\pi(a_{ij}) = p_{ij}$ $(1 \leqslant i,j \leqslant k)$.

Since s_k is simplicial in the graph spanned by $s_1, \dots, s_k$, we have the following picture:

$$[p_{ij}]_{i,j=1}^k = \left(\begin{array}{c|ccc|c} & & & & ? \\ \hline & p_{\ell\ell} & \cdots & p_{\ell,k-1} & p_{\ell k} \\ & \vdots & & \vdots & \vdots \\ & p_{k-1,\ell} & \cdots & p_{k-1,k-1} & p_{k-1,k} \\ \hline ? & p_{k\ell} & \cdots & p_{k,k-1} & p_{kk} \end{array}\right).$$

Here $\ell < k$; $p_{ij} \in \mathcal{A}/J$ for $\ell \leqslant i,j \leqslant k$; in the shaded region anything could be specified or not specified, and in the $(1,3)$ and $(3,1)$ blocks nothing is specified. As $[p_{ij}]_{i,j=\ell}^k$ is fully specified over $\mathcal{A}/J$ and is positive, there exist $b_{ij} \in \mathcal{A}$ such that $\pi(b_{ij}) = p_{ij}$ $(\ell \leqslant i,j \leqslant k)$, and $[b_{ij}]_{i,j=\ell}^k$ is positive over $\mathcal{A}$. Let $a_{ik} = b_{ik}$, $a_{ki} = b_{ki}$ $(\ell \leqslant i \leqslant k)$, $c_{ij} = b_{ij} - a_{ij}$ $(\ell \leqslant i,j \leqslant k)$. Observe that $c_{ij} \in J$. Also, $c_{ik} = c_{ki} = 0$ $(\ell \leqslant i \leqslant k)$. Write

$$[c_{ij}]_{i,j=\ell}^k = [d_{ij}]_{i,j=\ell}^k - [f_{ij}]_{i,j=\ell}^k,$$

where $[d_{ij}]_{i,j=\ell}^k$ and $[f_{ij}]_{i,j=\ell}^k$ are positive matrices over J. As $[b_{ij}]_{i,j=\ell}^k$ is positive. By

Lemma 3.3

$$(k-\ell)\begin{pmatrix} d_{\ell\ell} & & 0 \\ & \ddots & \\ 0 & & d_{kk} \end{pmatrix} - [d_{ij}]_{i,j=\ell}^{k} \geqslant 0,$$

hence if we define

$$a'_{ij} = \begin{cases} a_{ij} + (k-\ell)d_{ij}; & \ell \leqslant i = j \leqslant k \\ a_{ij}; & \ell \leqslant i,j \leqslant k;\ i \neq j \end{cases}$$

then since we have only altered some diagonal elements by adding positives, $[a'_{ij}]_{i,j=1}^{k}$ is partially positive and $\pi(a'_{ij}) = p_{ij}$, $1 \leqslant i,j \leqslant k$. This completes the inductive step.

Observe that in the inductive step one cannot generally define a'_{ij} by the formulas $a'_{ij} = a_{ij} + d_{ij}$ if $\ell \leqslant i,j \leqslant k$; and $a'_{ij} = a_{ij}$ if at least one of the indices i,j is less than ℓ. The reason is that the specified entries in the shaded areas of $[p_{ij}]_{i,j=1}^{k}$ may be such that the matrix $[a'_{ij}]_{i,j=1}^{k}$ will not be partially positive with this definition of a'_{ij}. ∎

The proof of Theorem 3.2 can be now easily completed using Lemma 3.4.

In particular, both the ideal $K(H)$ of the compact operators on H and the Calkin algebra are (PC).

4. UMF AND QF ALGEBRAS

In this section we explore the connections between the (PC) class and the classes of UMF and QF algebras.

Recall that a C^*-algebra A is called MF if given $a, b \in A$ such that $aa^* \leqslant bb^*$ there is $c \in A$ such that $a = bc$. If, in addition, we may take $\|c\| \leqslant 1$, then A is called UMF (uniform majorization-factorization). This class was introduced by Fialkow [9]. A C^*-algebra A is called *completely UMF* if the defining property holds for matrices over A, i.e. given $m \times p$ and $m \times n$ matrices $a = [a_{ij}]$ and $b = [b_{ij}]$ over A, respectively, such that $aa^* \leqslant bb^*$ there is an $n \times p$ matrix $c = [c_{ij}]$ over A such that $\|C\| \leqslant 1$ and $a = bc$.

THEOREM 4.1. *Any unital* (PC) *algebra is completely* UMF.

Proof. Let a, b be as in the definition of completely UMF. Then the partial matrix

$$\begin{pmatrix} I & b^* & ? \\ b & bb^* & a \\ ? & a^* & I \end{pmatrix} \tag{4.1}$$

is partially positive (indeed,

$$\begin{pmatrix} I & -a \\ 0 & I \end{pmatrix} \begin{pmatrix} bb^* & a \\ a^* & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -a^* & I \end{pmatrix} = \begin{pmatrix} bb^* - aa^* & 0 \\ 0 & I \end{pmatrix},$$

which is positive). As (4.1) is subordinate to a chordal pattern, there is X such that

$$\begin{pmatrix} I & b^* & X \\ b & bb^* & a \\ X^* & a^* & I \end{pmatrix} \geqslant 0.$$

Now by [4] we have

$$0 \leqslant \begin{pmatrix} I & b^* & X \\ b & bb^* & a \\ X^* & a^* & I \end{pmatrix} - \begin{pmatrix} I \\ b \\ X^* \end{pmatrix} (\, I \;\; b^* \;\; X\,) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a - bX \\ 0 & a^* - X^*b^* & I - X^*X \end{pmatrix}.$$

Consequently, $a - bX = 0$ and $I - X^*X \geqslant 0$, and we are done. ∎

The compact operators are a (PC) algebra which is not UMF. Indeed, assuming for simplicity that H is separable infinite dimensional, let D be any compact diagonal operator with strictly positive diagonal entries. Setting $a = b = D$, we find that $aa^* \leqslant bb^*$ but there is no compact c such that $a = bc$. We do not know if every completely UMF is (PC).

It was communicated to us by L. Fialkow [10] that for a separably acting C^*--algebra A the conditions

(a) A is MF;

(b) A is completely UMF;

(c) A is AW*

are equivalent. For the definition and theory of AW*-algebras see [2]; in particular, every AW*-algebra is unital. In connection with that the following question arises naturally: Is every separably acting AW*-algebra (PC)? The affirmative answer would imply that the classes of separably acting AW*-algebras and separably acting unital (PC)-algebras are the same.

We introduce now the quasifactorization algebras (*QF algebras*). Given a C^*--algebra $A \subseteq B(H)$ (non-degenerately), its multiplier algebra $M(A)$ is defined by $M(A) = \{X \in B(H) \mid XA \subseteq A, AX \subseteq A\}$. When A is unital $M(A) = A$, but for A non-unital $M(A)$ is in some appropiate sense the universal unital C^*-algebra which contains A as an ideal. If $A = C_0(X)$, the C_0-functions on a locally compact Hausdorff space, then $M(A) = C_b(X)$, the continuous bounded functions on X, which can be identified with $C(\beta X)$, where βX denotes the Stone-Čech compactification of X. A C^*-algebra $A \subseteq B(H)$ will be called QF algebra if for any triple of matrices

P, B, C of sizes 1×1, $1 \times p$, $p \times p$, respectively over A such that $\begin{pmatrix} P & B \\ B^* & C \end{pmatrix} \geqslant 0$ there is $D \in M_{1\times p}(M(A))$ such that $\|D\| \leqslant 1$ and $B = P^{\frac{1}{2}}DC^{\frac{1}{2}}$. Thus, for unital algebras the QF property is merely a factorizaton property, inside the algebra, while for non-unital algebras it is a factorization property relative to this universal algebra.

For examples, von Neumann algebras are QF. To see this, note that for any $\xi > 0$, $(P+\xi I)^{\frac{-1}{2}}B(C+\xi I_p)^{\frac{-1}{2}}$ is a contraction. If D denotes any weak*-limit point as ξ tends to 0, then $B = P^{\frac{1}{2}}DC^{\frac{1}{2}}$. On the other hand, algebras of continuous or C_0-functions are only QF when the underlying space is very disconnected. To see this, let f be a continuous (respectively, C_0-function) on X and set $p = 1$, $B = f$, $P = C = |f|$ in the above definition. If $C(X)$ (respectively, $C_0(X)$) were QF then we could find d in $C(X)$ (respectively, $C_b(X)$) such that $f = d|f|$ for every function f.

The following results is our main motivation for introducing this concept.

THEOREM 4.2. *Any* QF *algebra is* (PC).

Proof. Let A be a QF algebra. Using Lemma 2.2 it is sufficient to show that given matrices a, b, c, d, e over A of sizes 1×1, $1 \times p$, $p \times p$, $p \times 1$, 1×1, respectively such that $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \geqslant 0$, $\begin{pmatrix} c & d \\ d^* & e \end{pmatrix} \geqslant 0$, there exists $x \in A$ such that

$$\begin{pmatrix} a & b & x \\ b^* & c & d \\ x^* & d^* & e \end{pmatrix} \geqslant 0. \tag{4.2}$$

Using the definition of a QF algebra find $b_1, d_1 \in M_{1\times p}(M(A))$ with $\|b_1\|, \|d_1\| \leqslant 1$, $b = a^{\frac{1}{2}}b_1c^{\frac{1}{2}}$, $d = c^{\frac{1}{2}}d_1e^{\frac{1}{2}}$. It is easy to check that $x = a^{\frac{1}{2}}b_1d_1e^{\frac{1}{2}}$ is in A and makes (4.2) positive. ∎

A C^*-algebra A is called *σ-unital* (see [19]) if there is a countable approximate unit $[E_\lambda]_{\lambda\in\Lambda}$, i.e., a countable net Λ and an element $E_\lambda \in A$ for every $\lambda \in \Lambda$ such that the following conditions are satisfied:

(i) $E_\lambda \geqslant 0, \|E_\lambda\| \leqslant 1$ for all $\lambda \in \Lambda$;

(ii) $\lambda < \mu \Rightarrow E_\mu - E_\lambda \geqslant 0$;

(iii) for every $a \in A$ we have $\|a - E_\lambda a\| \to 0, \|a - aE_\lambda\| \to 0$.

For example, all separable C^*-algebras are σ-unital.

Our next result concerns *corona algebras*, that is, algebras of the form $C(A) = M(A)/A$. If $A = C_0(X)$ for some locally compact Hausdorff space X, then $M(A) = C_b(X) = C(\beta X)$ and $C(A) = C(\beta X)/C_0(X) = C(\beta X/X)$. Our next result shows that corona algebrs are QF, so in particular we see that $\beta X/X$ is very disconnected. Many of the ideas for this proof can be found in [20].

THEOREM 4.3. *Let A be a σ-unital C^*-algebra, $A \subseteq B(H)$. Then the corona algebra $C(A) = M(A)/A$ is* QF *(and hence* (PC)*).*

Proof. Let $C_0 = [c_{ij}]_{i,j=1}^p$, $B_0 = (b_{11}, \dots, b_{1p})$ and a_0 be in $C(A)$ so that $P_0 = \begin{pmatrix} a_0 & B_0 \\ B_0^* & C_0 \end{pmatrix}$ is a positive $(p+1) \times (p+1)$-matrix over $C(A)$. Choose an arbitrary positive lifting of this to elements $C = [C_{ij}]_{i,j=1}^p$, $B = (B_{11}, \dots, B_{1p})$ and a in $M(A)$. So that, $P = \begin{pmatrix} 1 & B \\ B^* & C \end{pmatrix}$ is a positive $(p+1) \times (p+1)$-matrix over $M(A)$ and has image P_0 in $C(A)$.

Set $T_k = a^{\frac{1}{2}}(a + k^{-3})^{\frac{1}{2}} B (C + k^{-3})^{-\frac{1}{2}} C^{\frac{1}{2}} - B$, which is $1 \times p$. We may by [20] choose an approximate unit $\{E_n\}_{n=1}^{\infty}$ for A such that for all n.

a) $E_{n+1}E_n = E_n$; $\|E_n\| \leqslant 1$;

b) $\|E_n B - B E_n^{(p)}\| \leqslant 2^{-n}$, where $E_n^{(p)}$ denotes the diagonal $p \times p$ matrix with diagonal entries E_n;

c) $\|E_n a^{\frac{1}{2}} - a^{\frac{1}{2}} E_n\| \leqslant 2^{-n}$;

d) $\|E_n^{(p)} C^{\frac{1}{2}} - C^{\frac{1}{2}} E_n^{(p)}\| \leqslant 2^{-n}$;

e) $\|E_n T_k - T_k E_n^{(p)}\| \leqslant 2^{-n}$ for all $k \leqslant n+1$.

Formally we put $E_0 = 0$. Since $\|(a + n^{-3})^{\frac{1}{2}} B (C + n^{-3})^{\frac{-1}{2}}\| \leqslant 1$, we have that $\sum_{n=1}^{\infty} (E_n - E_{n-1})^{\frac{1}{2}} (a + n^{-3})^{\frac{-1}{2}} B (C + n^{-3})^{\frac{-1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}}$ converges strongly in $M(A)$ to an element D in $M(A)$, see [20]. (Here by strong convergence in $M(A)$ of a sequence $\{X_m\}_{m=1}^{\infty}$ to D in $M(A)$ we mean that $X_m Y \to DY$ and $YX_m \to YD$ in the norm topology for every $Y \in A$.) We claim that $a^{\frac{1}{2}} D C^{\frac{1}{2}} - B$ is in A. To see this we use the fact that for $P \geqslant 0$, $\|PX - XP\| < \varepsilon$ implies that $\|P^{\frac{1}{2}} X - X P^{\frac{1}{2}}\| \leqslant (2\|X\|\varepsilon)^{\frac{1}{2}}$, see [20].

Thus,

$$\|(E_n - E_{n-1})^{\frac{1}{2}} B - B (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}}\| \leqslant (2\|B\|(2^{-n} + 2^{-n+1}))^{\frac{1}{2}},$$

and hence,

$$\|(E_n - E_{n-1})^{\frac{1}{2}} B (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} - B (E_n^{(p)} - E_{n-1}^{(p)})\| \leqslant \delta_n$$

where $\sum_{n=1}^{\infty} \delta_n$ is finite.

We therefore have that the series

$$\sum_{n=1}^{\infty} \left[(E_n - E_{n-1})^{\frac{1}{2}} B (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} - B (E_n^{(p)} - E_{n-1}^{(p)}) \right]$$

is norm convergent to a $1 \times p$ matrix over A. We have

$$\sum_{n=1}^{\infty}(E_n - E_{n-1})^{\frac{1}{2}} B(E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} - B =$$

$$(4.3) \qquad = \sum_{n=1}^{\infty} \left[(E_n - E_{n-1})^{\frac{1}{2}} B(E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} - B(E_n^{(p)} - E_{n-1}^{(p)})\right] + \\ + \left[\sum_{n=1}^{\infty} B(E_n^{(p)} - E_{n-1}^{(p)}) - B\right].$$

Since $\sum_{n=1}^{\infty} B(E_n^{(p)} - E_{n-1}^{(p)})$ converges strongly in $M(A)$ to B, the second bracketed term in the right-hand side of (4.3) goes strongly to 0. The first series in the right-hand side of (4.3) is norm convergent and each term belongs to A. We obtain therefore that the left-hand side of (4.3) converges strongly in $M(A)$ to a $1 \times p$ matrix, call it Y, over A. Note that

$$\sum_{n=1}^{\infty} \left[a^{\frac{1}{2}}(E_n - E_{n-1})^{\frac{1}{2}}(a + n^{-3})^{\frac{-1}{2}} B(C + n^{-3})^{\frac{-1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} C^{\frac{1}{2}} - \right. \\ \left. -(E_n - E_{n-1})^{\frac{1}{2}} a^{\frac{1}{2}} (a + n^{-3})^{\frac{1}{2}} B(C + n^{-3})^{\frac{-1}{2}} C^{\frac{1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})\right] =$$

$$(4.4) \qquad = \sum_{n=1}^{\infty} \left[a^{\frac{1}{2}}(E_n - E_{n-1})^{\frac{1}{2}} - (E_n - E_{n-1})^{\frac{1}{2}} a^{\frac{1}{2}}\right] \cdot \\ \cdot \left[(a + n^{-3})^{\frac{-1}{2}} B(C - n^{-3})^{\frac{-1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} C^{\frac{1}{2}}\right] + \\ + \sum_{n=1}^{\infty} \left[(E_n - E_{n-1})^{\frac{1}{2}} a^{\frac{1}{2}} (a + n^{-3})^{\frac{-1}{2}} B(C + n^{-3})^{\frac{-1}{2}}\right] \cdot \\ \cdot \left[(E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} C^{\frac{1}{2}} - C^{\frac{1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}}\right].$$

An argument similar to the one used above shows that

$$\sum_{n=1}^{\infty} \left[a_n^{\frac{1}{2}}(E_n - E_{n-1})^{\frac{1}{2}} - (E_n - E_{n-1})^{\frac{1}{2}} a_n\right]$$

and

$$\sum_{n=1}^{\infty} \left[C_n^{\frac{1}{2}} (E_n^{(p)} - E_{n-1}^{(p)})^{\frac{1}{2}} - (E_n^{(p)} - E_{n-1}^{(p)}) C_n^{\frac{1}{2}}\right]$$

are norm convergent series. However, since the terms

$$\left\| \left[(a + n^{-3})^{\frac{-1}{2}} B(C + n^{-3})^{\frac{-1}{2}} \cdot (E_n^{(p)} - E_{n-1}^{(p)}) C^{\frac{1}{2}}\right] \right\|$$

and

$$\left\| \left[(E_n - E_{n-1})^{\frac{1}{2}} a^{\frac{1}{2}} (a + n^{-3})^{\frac{-1}{2}} B(C + n^{-3})^{\frac{-1}{2}}\right] \right\|$$

are uniformly bounded, each of the two sums in the right hand side of (4.4) is norm convergent.

Thus, the sum (4.4) converges in norm to a $1 \times p$ matrix X over A. Since the series

$$\sum_{n=1}^{\infty} \left[a^{\frac{1}{2}}(E_n - E_{n-1})^{\frac{1}{2}}(a+n^{-3})^{\frac{-1}{2}} B(C+n^{-3})^{\frac{-1}{2}} \cdot \left(E_n^{(p)} - E_{n-1}^{(p)}\right)^{\frac{1}{2}} C^{\frac{1}{2}}\right]$$

converges strongly in $M(A)$ to $a^{\frac{1}{2}} D C^{\frac{1}{2}}$ we have that

$$a^{\frac{1}{2}} D C^{\frac{1}{2}} - B = X + \sum_{n=1}^{\infty}(E_n - E_{n-1})^{\frac{1}{2}} T_n \left(E_n^{(p)} - E_{n-1}^{(p)}\right)^{\frac{1}{2}} + \tag{4.5}$$
$$+ \sum_{n=1}^{\infty}(E_n - E_{n-1})^{\frac{1}{2}} B\left(E_n^{(p)} - E_{n-1}^{(p)}\right)^{\frac{1}{2}} - B.$$

Next, we shall show that $\sum_{n=1}^{\infty} \|T_n\|$ is a convergent series, from which it follows that $\sum_{n=1}^{\infty}(E_n - E_{n-1})^{\frac{1}{2}} T_n \left(E_n^{(p)} - E_{n-1}^{(p)}\right)^{\frac{1}{2}}$ is norm convergent to a $1 \times p$ matrix, call it Z, over A.

To estimate $\|T_n\|$, note that there exists a $1 \times p$ matrix F over $B(H)$ with $\|F\| \leqslant 1$ such that $B = a^{\frac{1}{2}} F C^{\frac{1}{2}}$. Hence,

$$T_n = a(a+n^{-3})^{\frac{-1}{2}} F (C+n^{-3})^{\frac{-1}{2}} C - a^{\frac{1}{2}} F C^{\frac{1}{2}} =$$

$$= a(a+n^{-3})^{\frac{-1}{2}} F\left[(C+n^{-3})^{\frac{-1}{2}} C - C^{\frac{1}{2}}\right] + \left[a(a+n^{-3})^{\frac{-1}{2}} - a^{\frac{1}{2}}\right] F C^{\frac{1}{2}}.$$

Note that the function $\left|(t+n^{-3})^{\frac{-1}{2}} t - t^{\frac{1}{2}}\right| \leqslant n^{-3/2}$ for $t \geqslant 0$, so that by the functional calculus,

$$\|T_n\| \leqslant n^{-3/2}\left[\left\|a(a+n^{-3})^{\frac{-1}{2}} F\right\| + \left\|F C^{\frac{1}{2}}\right\|\right] \leqslant n^{-3/2}\left[\left\|a^{\frac{1}{2}}\right\| + \left\|C^{\frac{1}{2}}\right\|\right]$$

which is a summable series as claimed.

We return now to (4.5). As we have shown above,

$$\sum_{n=1}^{\infty}(E_n - E_{n-1})^{\frac{1}{2}} B\left(E_n^{(p)} - E_{n-1}^{(p)}\right)^{\frac{1}{2}} - B$$

converges strongly in $M(A)$ to a matrix over A. It follows that the entries of $a^{\frac{1}{2}} D C^{\frac{1}{2}} - B$ are in A. Now we have $a_0^{\frac{1}{2}} D_0 C_0^{\frac{1}{2}} - B_0 = 0$ in $C(A)$, where D_0 is the $1 \times p$ matrix over $C(A)$ whose entries are the images in $C(A)$ of the entries of D. ■

5. CONTRACTION COMPLETIONS

A partial $m \times n$ matrix over a C^*-algebra A is, by definiton, an $m \times n$ matrix whose entries are either specified to be elements in A or are unspecified and designated by a question mark. We define a *completion* of the partial $m \times n$ matrix $Q = [Q_{ij}]_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ over A to be any $m \times n$ matrix $Z = [Z_{ij}]_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ with entries in A such that $Z_{ij} = Q_{ij}$ for every specified entry Q_{ij} of Q.

We consider completions which are *contractions* (i.e. with norm less than or equal to 1) or *strict contractions* (i.e. with norm less than 1). Problems concerning contraction completions and strict contraction completion have been studied a lot in the literature (see, e.g., [7], [15], [13], [5], [3], [16], [21]).

An obvious necessary condition for existence of a contraction (resp. strict contraction) completion for a given partial $m \times n$ matrix Q over A is that every fully specified rectangular submatrix of Q is contraction (resp. strict contraction). Partial matrices with this property will be called *partial contractions* (resp. *partial strict contractions*). We characterize the patterns for which every partial contraction (resp. partial strict contraction) admits a contraction (resp. strict contraction) completion. Here, a *pattern* is, by definition, an $m \times n$ matrix each entry of which is either $*$ or ?, and a partial $m \times n$ matrix Q over A is said to be *subordinate* to the pattern P if the (i, j) entry of Q is specified precisely when the (i, j) entry of P is $*$.

Two partial $m \times n$ matrices (or patterns) Q_1 and Q_2 over A are said to be *permutation equivalent* if one may be obtained from another by independent reordering of rows and columns.

THEOREM 5.1. *Assume A is a unital C^*-algebra. Let P be an $m \times n$ pattern which is permutation equivalent to the following block diagonal form*

$$\begin{pmatrix} B_1 & ? & \dots & ? \\ ? & B_2 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & B_r \end{pmatrix} \tag{5.1}$$

possibly bordered by rows and/or columns of question marks, in which

$$B_j = \begin{pmatrix} B_{j11} & ? & ? & \dots & ? \\ B_{j21} & B_{j22} & ? & \dots & ? \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ B_{jp1} & B_{jp2} & \dots & \dots & B_{jpp} \end{pmatrix}, \quad j = 1, \dots, r$$

and the (possibly rectangular) blocks B_{jst} consist entirely of $*$'s. Then every strict partial contraction with specified entries from A subordinate to P admits a strict *contraction completion. Conversely, if P is not permutation equivalent to a block diagonal form* (5.1), *then there exists a strict partial contraction subordinate to P which does not admit any contraction completions (let alone strict constraction completions).*

Proof. Without loss of generality we assume that P is indecomposable, i.e. is not permutation equivalent to any pattern of the form (5.1) with $r \geqslant 2$. Let Q be a strict partial contraction subordinate to P. Form the $(m+n) \times (m+n)$ partial Hermitian matrix $\tilde{Q} = \begin{pmatrix} I_m & Q \\ Q^* & I_n \end{pmatrix}$. Clearly, $\tilde{Q}$ is strictly partially positive. Since P is permutation equivalent to (5.1) it is easy to verify that the graph G to which $\tilde{Q}$ is subordinate is chordal. Now apply Theorem 2.1.

For the proof of the converse statement apply the argument used in the proof of Theorem 1 in [14] together with the following lemma.

LEMMA 5.2. *Let P be the 2×3 pattern $\begin{pmatrix} ? & * & * \\ * & ? & * \end{pmatrix}$. Then there exists a partial strict contraction Q subordinate to P which does not admit contradiction completion.*

Proof. Take

$$Q = \begin{pmatrix} ? & \left(\frac{1}{\sqrt{2}} - \delta\right) e & \left(\frac{1}{\sqrt{2}} - \delta\right) e \\ \left(\frac{1}{\sqrt{2}} - \delta\right) e & ? & \left(\frac{1}{\sqrt{2}} - \delta\right) e \end{pmatrix},$$

where $\delta > 0$ is sufficiently small. Arguing by contradiction, assume that

$$\tilde{Q} = \begin{pmatrix} x & \left(\frac{1}{\sqrt{2}} - \delta\right) e & \left(\frac{1}{\sqrt{2}} - \delta\right) e \\ \left(\frac{1}{\sqrt{2}} - \delta\right) e & ? & \left(\frac{1}{\sqrt{2}} - \delta\right) e \end{pmatrix}$$

is a partial contraction for some $x \in A$. The first row then gives

$$\|x\|^2 = \|xx^*\| \leqslant \left\| \left(1 - 2\left(\frac{1}{\sqrt{2}} - \delta\right) e^2\right) e \right\| = \frac{4}{\sqrt{2}}\delta - 2\delta^2. \tag{5.2}$$

On the other hand, the matrix

$$\begin{pmatrix} 0 & \left(\frac{1}{\sqrt{2}} - \delta\right) e \\ \left(\frac{1}{\sqrt{2}} - \delta\right) e & \left(\frac{1}{\sqrt{2}} - \delta\right) e \end{pmatrix}$$

is not a contraction. Hence for $\|x\|$ sufficiently small the matrix

$$\begin{pmatrix} x & \left(\frac{1}{\sqrt{2}} - \delta\right) e \\ \left(\frac{1}{\sqrt{2}} - \delta\right) e & \left(\frac{1}{\sqrt{2}} - \delta\right) e \end{pmatrix}$$

is also not a contraction. Taking $\delta > 0$ small enough, in view of (5.2) we obtain a contradiction with $\tilde{Q}$ being a partial contraction. ■

For the case when A is the algebra of all $n \times n$ matrices, or, more generally, the algebra of all linear bounded operators in a Hilbert space Theorem 4.1 (as well as Theorem 5.3 below) was proved in [15], see also [16].

For contraction completions we have the following result.

THEOREM 5.3. *Assume that A is a unital* (PC) *algebra. Let P be an $m \times n$ pattern which is permutation equivalent to a block diagonal form* (5.1). *Then every partial contraction with specified entries from A subordinate to P admits a contraction completion.*

The proof is analogous to the proof of Theorem 5.1.

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