J. OPERATOR THEORY
25(1991), 283–298

IDEALS IN THE MULTIPLIER ALGEBRA OF A STABLE C^*-ALGEBRA

MIKAEL RØRDAM

1. INTRODUCTION

G. Elliott proved in [6] that if B is a stable non-elementary matroid C^*-algebra, then $\mathcal{M}(B)$ contains exactly one non-trivial ideal apart from B: the norm closure of the set for all X in $\mathcal{M}(B)$ where X^*X is of finite trace. H. Lin [8] extends Elliott's result to more general simple AF-algebras B, and he proves for example that if B is stable and has exactly n distinct extremal traces (with respect to some normalization), then $\mathcal{M}(B)$ has precisely $2^n - 1$ non-trivial ideals apart from B, each being the norm closure of the set of X in $\mathcal{M}(B)$ with $\tau(X^*X) < \infty$ for some trace τ on B.

Elliott extends these results further in [7], where he proves that the lattice of ideals of $\mathcal{M}(B)$ is isomorphic to the lattice of ideals in $D(\mathcal{M}(B))$, the abelian local semigroup of equivalence classes of projections in $\mathcal{M}(B)$, for all separable AF-algebras B. For simple B, this again is related to traces on B (however, in general in a more complicated way than above, where the set of extreme traces was assumed to be finite). These results have been further generalized by S. Zhang [10] to include all C^*-algebras B with property FS.

This paper concerns the ideal structure of $\mathcal{M}(A \otimes K)$ for unital C^*-algebras A. Given a bounded sequence $\{x_n\}_{n=1}^{\infty}$ in (a matrix algebra over) A, one can in a natural way associate a "diagonal" element $X = \mathrm{Diag}(x_1, x_2, \ldots)$ in $\mathcal{M}(A \otimes K)$. Given another diagonal element $Y = \mathrm{Diag}(y_1, y_2, \ldots)$ in $\mathcal{M}(A \otimes K)$, it is determined when Y belongs to $I(X)$, the ideal in $\mathcal{M}(A \otimes K)$ generated by X, in terms of asymptotic comparison in $A \otimes K$ of the sequences $\{x_n\}_{n=1}^{\infty}$, and $\{y_n\}_{n=1}^{\infty}$ (see Theorem 2.6).

In Section 3, it is proved that for unital C^*-algebras A, the corona algebra $\mathcal{M}(A \otimes K)/A \otimes K$ is simple if, and only if, either $A \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ or

A is a purely infinite simple C^*-algebra. (Recall that a unital simple C^*-algebra A, different from $\mathbb{C}$, is said to be purely infinite if for all non-zero $x \in A^+$ there is $r \in A$ such that $r^*xr = 1$.). In Section 4 it is proved that two ideals of $\mathcal{M}(A \otimes K)$ both arising from traces, as described above, are equal if and only if the traces are majorized by a multiple of each other. Moreover, it is proved that all non-trivial ideals in $\mathcal{M}(A \otimes K)$ apart from $A \otimes K$ arise from a trace, under the assumptions that A has only finitely many extremal traces and the comparison theory of A is determined by the traces on A.

I thank J. Cuntz and U. Haagerup for helpfull discussions.

2. IDEALS OF $\mathcal{M}(A \otimes K)$ AND ASYMPTOTIC COMPARISON THEORY

Following the lines of J. Cuntz [3], [4], [5] and B. Blackadar [1] we consider below comparison theory of positive elements (rather than the projections) in a C^*-algebra.

For each $\varepsilon > 0$ define continuous functions $f_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ and $g_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$f_\varepsilon(t) = \begin{cases} 0 & t \leqslant \varepsilon \\ \varepsilon^{-1}(t-\varepsilon) & \varepsilon \leqslant t \leqslant 2\varepsilon, \\ 1 & t \geqslant 2\varepsilon \end{cases} \qquad g_\varepsilon(t) = \begin{cases} \varepsilon^{-1} & t \leqslant \varepsilon \\ t^{-1} & t \geqslant \varepsilon \end{cases}.$$

In the following, let A be a unital C^*-algebra. Let $x, y \in A^+$. Write $y \precsim_n x$ if

$$y \leqslant \sum_{j=1}^{n} r_j x r_j^*$$

for some $r_j \in A$, and let $y \precsim x$ denote that $y \precsim_1 x$. Let further $y \precapprox x$ and $y \precapprox_n x$ denote that $f_\varepsilon(y) \precsim x$, respectively $f_\varepsilon(y) \precsim_n x$, for every $\varepsilon > 0$. Write $x \sim y$, respectively $x \approx y$, if $x \precsim y$ and $y \precsim x$, respectively $x \precapprox y$ and $y \precapprox x$. Let $C(y, x)$ denote the least integer n for which $y \precsim_n x$, and put $C(y, x) = \infty$ if no such n exists.

LEMMA 2.1. *Let $x, x_0 \in A^+$ with $x \leqslant x_0$ and put $r_\delta = x^{\frac{1}{2}} g_\delta(x_0), \delta > 0$. Then*

$$\lim_{\delta \to \infty} \|x - r_\delta x_0 r_\delta^*\| = 0.$$

Proof. Put $t_\delta = x^{\frac{1}{2}}(1 - g_\delta(x_0)x_0)^{\frac{1}{2}}$. Then

$$x - r_\delta x_0 r_\delta^* = t_\delta t_\delta^*,$$

and

$$t_\delta^* t_\delta = (1 - g_\delta(x_0)x_0)^{\frac{1}{2}} x (1 - g_\delta(x_0)x_0)^{\frac{1}{2}} \leqslant$$

$$\leqslant (1 - g_\delta(x_0)x_0)^{\frac{1}{2}} x_0 (1 - g_\delta(x_0)x_0)^{\frac{1}{2}}.$$

Hence

$$\lim_{\delta \to 0} \|x - r_\delta x_0 r_\delta^*\| = \lim_{\delta \to 0} \|t_\delta^* t_\delta\| = 0.$$

■

LEMMA 2.2. *Let $x, y \in A^+$ with $\|x - y\| \leqslant \varepsilon$. Then $f_{2\varepsilon}(y) \leqslant rxr^*$ for some $r \in A$ with $\|r\| \leqslant \varepsilon^{\frac{1}{2}}$.*

Proof. Since $y - \varepsilon \cdot 1 \leqslant x$, we get

$$\varepsilon f_{2\varepsilon}(y) \leqslant f_{2\varepsilon}(y)^{\frac{1}{2}}(y - \varepsilon 1) f_{2\varepsilon}(y)^{\frac{1}{2}} \leqslant f_{2\varepsilon}(y)^{\frac{1}{2}} x f_{2\varepsilon}(y)^{\frac{1}{2}},$$

and so we may take $r = \varepsilon^{-\frac{1}{2}} f_{2\varepsilon}(y)^{\frac{1}{2}}$. ■

LEMMA 2.3. *Let $x, y \in A^+$, let $\varepsilon > 0$ and $\delta > 0$, and assume that $y \precsim_n f_{2\delta}(x)$. Then*

$$f_\varepsilon(y) \leqslant \sum_{j=1}^{n} r_j f_\delta(x) r_j^*$$

for some $r_j \in A$ with $\|r_j\| \leqslant (\varepsilon^{-1}\|y\| + 1)^{\frac{1}{2}}$.

Proof. We have

$$y \leqslant y_0 = \sum_{j=1}^{n} s_j f_{2\delta}(x) s_j^*$$

for some $s_j \in A$. By Lemma 2.1, there is $t_1 \in A$ such that $\|y - t_1 y_0 t_1^*\| \leqslant \varepsilon/2$, and from Lemma 2.2 $f_\varepsilon(y) \leqslant t_2 t_1 y_0 t_1^* t_2^*$ for some $t_2 \in A$ with $\|t_2\| \leqslant \varepsilon^{-\frac{1}{2}}$. Put $r_j = t_2 t_1 s_j f_{2\delta}(x)^{\frac{1}{2}}$. Then

$$r_j f_\delta(x) r_j^* = t_2 t_1 s_j f_{2\delta}(x) s_j^* t_1^* t_2^*,$$

and so $f_\varepsilon(y) \leqslant \sum_{j=1}^{n} r_j f_\delta(x) r_j^*$. Also $r_j r_j^* \leqslant t_2 t_1 y_0 t_1^* t_2^*$, whence

$$\|r_j\|^2 \leqslant \|t_2\|^2 \cdot \|t_1 y_0 t_1^*\| \leqslant \varepsilon^{-1}(\|y\| + \varepsilon/2).$$

■

LEMMA 2.4. *Let $x, x_0 \in A^+$ with $x \leqslant x_0$, and let $\varepsilon > 0$. Then $f_{4\varepsilon}(x) \precsim f_\varepsilon(x_0)$.*

Proof. Put $r = x^{\frac{1}{2}}(1 - f_\varepsilon(x_0))^{\frac{1}{2}}$. Then

$$rr^* = x - x^{\frac{1}{2}} f_\varepsilon(x_0) x^{\frac{1}{2}},$$

and

$$r^*r = (1 - f_\varepsilon(x_0))^{\frac{1}{2}} x (1 - f_\varepsilon(x_0))^{\frac{1}{2}} \leqslant$$
$$\leqslant (1 - f_\varepsilon(x_0))^{\frac{1}{2}} x_0 (1 - f_\varepsilon(x_0))^{\frac{1}{2}} \leqslant 2\varepsilon \cdot 1.$$

Thus, from Lemma 2.2,

$$f_{4\varepsilon}(x) \precsim x^{\frac{1}{2}} f_\varepsilon(x_0) x^{\frac{1}{2}} \precsim f_\varepsilon(x_0).$$

■

Let $I_A(x)$ — or just $I(x)$ — denote the closed two-sided ideal in A generated by $x \in A$.

LEMMA 2.5. *Let $x, y \in A^+$. Then $y \in I(x)$ if only it for each $\varepsilon > 0$ there are $\delta > 0$ and $n \in \mathbb{N}$ such that $f_\varepsilon(y) \precsim_n f_\delta(x)$.*

Proof. Clearly $f_\delta(x) \in I(x)$ for all $\delta > 0$, and so if $f_\varepsilon(y) \precsim_n f_\delta(x)$ for some $\delta > 0$ and some $n \in \mathbb{N}$, then $f_\varepsilon(y) \in I(x)$. Furthermore, if $f_\varepsilon(y) \in I(x)$ for all $\varepsilon > 0$, then $y \in I(x)$.

Suppose now that $y \in I(x)$, and let $\varepsilon > 0$. Approximate $y^{\frac{1}{2}}$ closely enough with an element z in

$$\bigcup_{\delta > 0} I^{\mathrm{alg}}(f_\delta(x))$$

where $I^{\mathrm{alg}}(f_\delta(x))$ denotes the algebraic ideal in A generated by $f_\delta(x)$, to obtain $\|y - z^* z\| \leqslant \varepsilon/2$. Now, $z^* z \in I^{\mathrm{alg}}(f_\delta(x))$ for some $\delta > 0$, and so

$$z^* z = \sum_{j=1}^{m} r_j f_\delta(x) s_j \quad \left(\leqslant \frac{1}{2} \sum_{j=1}^{m} r_j f_\delta(x) r_j^* + \frac{1}{2} \sum_{j=1}^{m} s_j^* f_\delta(x) s_j \right),$$

for some $m \in \mathbb{N}$ and $r_j, s_j \in A$. Hence, from Lemma 2.2,

$$f_\varepsilon(y) \precsim z^* z \precsim_{2m} f_\delta(x).$$

■

Let $\{f_n\}$ be an approximate unit of projections for K, the C^*-algebra of compact operators, where $f_0 = 0$ and $f_n - f_{n-1}$ is one-dimensional. Let A be a unital C^*-algebra, and put $e_n = 1_A \otimes f_n$, so that $\{e_n\}$ is an approximate unit for $A \otimes K$. Call an element $X \in \mathcal{M}(A \otimes K)$ diagonal (with respect to $\{e_n\}$) if there is a strictly increasing sequence $\{\alpha(n)\}$ of integers with $\alpha(0) = 0$ such that

$$[X, e_{\alpha(n)} - e_{\alpha(n-1)}] = 0, \quad n \in \mathbb{N}.$$

Write $X = \mathrm{Diag}(x_1, x_2, \ldots)$, where

$$x_n = X(e_{\alpha(n)} - e_{\alpha(n-1)}), \quad n \in \mathbb{N}.$$

Conversely, if $\{x_n\}$ is a bounded sequence with $x_n \in M_{k_n}(A)$, then upon identifying $M_{k_n}(A)$ with

$$(e_{\alpha(n)} - e_{\alpha(n-1)})\mathcal{M}(A \otimes K)(e_{\alpha(n)} - e_{\alpha(n-1)})$$

for appropriate $\alpha(n)$, we have $X = \mathrm{Diag}(x_1, x_2, \ldots)$ for some $X \in \mathcal{M}(A \otimes K)$.

A finite segment $\mathrm{Diag}(x_n, x_{n+1}, \ldots, x_m)$, $m \geqslant n$, of a diagonal element may be viewed as an element of $A \otimes K$, or as an element of $M_k(A)$ for suitable k.

Let $\{x_j\}$ and $\{y_j\}$ be sequences of positive elements in matrix algebras over A. Write $\{y_j\} \precsim_n \{x_j\}$ if $\forall\, m\ \exists\, k\ \forall\, k' \geqslant k\ \exists\, m' \geqslant m$:

$$\mathrm{Diag}(y_k, y_{k+1}, \ldots, y_{k'}) \precsim_n \mathrm{Diag}(x_m, x_{m+1}, \ldots, x_{m'}).$$

(The comparison of the diagonal segments is relative to $A \otimes K$ or to some large enough matrix over A.)

THEOREM 2.6. *Let A be a unital C^*-algebra, and let $X = \mathrm{Diag}(x_1, x_2, \ldots)$ and $Y = \mathrm{Diag}(y_1, y_2, \ldots)$ be positive diagonal elements of $\mathcal{M}(A \otimes K)$. Then $Y \in I(X) + A \otimes K$, if and only if for each $\varepsilon > 0$ there are $\delta > 0$ and $n \in \mathbb{N}$ such that $\{f_\varepsilon(y_j)\} \precsim_n \{f_\delta(x_j)\}$.*

Proof. Fix increasing sequences $\{\alpha(j)\}$ and $\{\beta(j)\}$ of integers such that

$$X = \sum_{j=1}^{\infty} x_j(e_{\alpha(j)} - e_{\alpha(j-1)}), \quad Y = \sum_{j=1}^{\infty} y_j(e_{\beta(j)} - e_{\beta(j-1)}).$$

Suppose that $Y = I(X) + A \otimes K$. Upon replacing X by $\mathrm{Diag}(1_A, x_1, x_2, \ldots)$, which does not alter the equivalence class of $\{x_j\}$, we way assume that, in fact, $Y \in I(X)$. Let $\varepsilon > 0$. From Lemma 2.5 there are $\delta > 0$ and $n \in \mathbb{N}$ such that

$$f_{\varepsilon/2}(Y) \leqslant Y_0 = \sum_{i=1}^{n} R_i f_\delta(X) R_i^*,$$

for some $R_i \in \mathcal{M}(A \otimes K)$. Put $\mu = \sum_{i=1}^{n} \|R_i\|$. For each $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that

$$\|(1 - e_{\beta(k-1)})R_i e_{\alpha(m-1)}\| \leqslant (64\mu)^{-1}, \quad i = 1, \ldots, n;$$

and for each $k' \geqslant k$ there is $m' \geqslant m$ such that

$$\|e_{\beta(k')} R_i (1 - e_{\alpha(m')})\| \leqslant (64\mu)^{-1} \quad i = 1, \ldots, n.$$

Put

$$z_i = (e_{\beta(k')} - e_{\beta(k-1)})R_i(e_{\alpha(m')} - e_{\alpha(m-1)}),$$

$$y' = (e_{\beta(k')} - e_{\beta(k-1)})Y_0(e_{\beta(k')} - e_{\beta(k-1)}),$$

$$y'' = \sum_{i=1}^{n} z_i f_\delta(X) z_i^*$$

$$\left(= \sum_{i=1}^{n} z_i \operatorname{Diag}(f_\delta(x_m), f_\delta(x_{m+1}), \dots, f_\delta(x_{m'}))z_i^* \right).$$

Then $\operatorname{Diag}(f_{\varepsilon/2}(y_k), f_{\varepsilon/2}(y_{k+1}), \dots, f_{\varepsilon/2}(y_{k'})) \leqslant y'$, and a brief computation shows that

$$\|y' - y''\| \leqslant \sum_{i=1}^{n} 2\|R_i\| \left[\|(1 - e_{\beta(k-1)})R_i e_{\alpha(m-1)}\| + \|e_{\beta(k')}R_i(1 - e_{\alpha(m')})\|\right] \leqslant$$

$$\leqslant 1/16.$$

Hence, from Lemma 2.2, $f_{1/8}(y') \precsim y''$. Using Lemma 2.4 and the fact that $f_\varepsilon \leqslant f_{1/2} \circ f_{\varepsilon/2}$, we get

$$\operatorname{Diag}(f_\varepsilon(y_k), f_\varepsilon(y_{k+1}), \dots, f_\varepsilon(y_{k'})) \leqslant f_{1/2}(\operatorname{Diag}(f_{\varepsilon/2}(y_k), f_{\varepsilon/2}(y_{k+1}), \dots, f_{\varepsilon/2}(y_{k'}))) \precsim$$

$$\precsim f_{1/8}(y') \precsim y'' \precsim_n \operatorname{Diag}(f_\delta(x_m), f_\delta(x_{m+1}), \dots, f_\delta(x_{m'})),$$

and so $\{f_\varepsilon(y_j)\} \precsim_n \{f_\delta(x_j)\}$.

Suppose conversely that $\{f_\varepsilon(y_j)\} \precsim_n \{f_\delta(x_j)\}$ for some $\varepsilon, \delta > 0$ and $n \in \mathbb{N}$. Find integers $m_1, k_1, l_1, k_2, m_2, k_3, l_2, \dots$ (in that order), so that

$$1 = m_1 < m_2 < \dots,$$

$$1 = l_1 < l_2 < \dots,$$

$$k_1 < k_2 < \dots,$$

and

$$\operatorname{Diag}(f_\varepsilon(y_{k_1}), \dots, f_\varepsilon(y_{k_2-1})) \precsim_n \operatorname{Diag}(f_\delta(x_1), \dots, f_\delta(x_{m_2-1})),$$

$$\operatorname{Diag}(f_\varepsilon(y_{k_2}), \dots, f_\varepsilon(y_{k_3-1})) \precsim_n \operatorname{Diag}(f_\delta(x_1), \dots, f_\delta(x_{l_2} - 1)),$$

$$\operatorname{Diag}(f_\varepsilon(y_{k_3}), \dots, f_\varepsilon(y_{k_4-1})) \precsim_n \operatorname{Diag}(f_\delta(x_{m_2}), \dots, f_\delta(x_{m_3-1})),$$

$$\dots .$$

By Lemma 2.3 and because $f_{2\varepsilon} \leqslant f_{1/2} \circ f_\varepsilon$, there are elements $r_{i,j}$ and $s_{i,j}$, $i = 1, \ldots, n$, and $j \in \mathbb{N}$, in $A \otimes K$ with $\|r_{i,j}\|, \|s_{i,j}\| \leqslant (2\|Y\| + 1)^{\frac{1}{2}}$ such that

$$\mathrm{Diag}(f_{2\varepsilon}(y_{k_1}), \ldots, f_{2\varepsilon}(y_{k_2-1})) \leqslant \sum_{i=1}^{n} r_{i,1} \mathrm{Diag}(f_{\delta/2}(x_1), \ldots, f_{\delta/2}(x_{m_2-1})) r_{i,1}^*,$$

$$\mathrm{Diag}(f_{2\varepsilon}(y_{k_2}), \ldots, f_{2\varepsilon}(y_{k_3-1})) \leqslant \sum_{i=1}^{n} s_{i,1} \mathrm{Diag}(f_{\delta/2}(x_1), \ldots, f_{\delta/2}(x_{l_2-1})) s_{i,1}^*,$$

$$\ldots,$$

and such that the sums

$$R_i = \sum_{j=1}^{\infty} r_{i,j}, \quad S_i = \sum_{j=1}^{\infty} s_{i,j}$$

are strictly convergent in $\mathcal{M}(A \otimes K)$. Hence

$$(1 - e_{\beta(k_1-1)}) f_{2\varepsilon}(Y) \leqslant \sum_{i=1}^{n} R_i f_{\delta/2}(X) R_i^* + \sum_{i=1}^{n} S_i f_{\delta/2}(X) S_i^*,$$

which proves that $f_{2\varepsilon}(Y) \in I(f_{\delta/2}(X)) + A \otimes K$ $(\subseteq I(X) + A \otimes K)$. ■

COROLLARY 2.7. *Let A be a unital C^*-algebra, and let $X = \mathrm{Diag}(x_1, x_2, \ldots)$ be a diagonal element in $\mathcal{M}(A \otimes K)^+$. Then $I(X) = \mathcal{M}(A \otimes K)$ if and only if there are $\delta > 0$ and $n \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ there is $m' \geqslant m$ with*

$$1_A \precsim_n \mathrm{Diag}(f_\delta(x_m), \ldots, f_\delta(x_{m'})).$$

Proof. Set $y_j = 1_A$ for all j. Then $\{y_j\} \precsim_n \{f_\delta(x_j)\}$ if and only if for each $m \in \mathbb{N}$ there is $m' \leqslant m$ with $1_A \precsim_n \mathrm{Diag}(f_\delta(x_m), \ldots, f_\delta(x_{m'}))$. Since $\mathrm{Diag}(y_1, y_2, \ldots) = 1$; $f_\delta(y_j) = y_j, 0 < \delta < 1$; and since $\{y_j\} \precsim_n \{f_\delta(x_j)\}$ implies $I(f_\delta(X)) = \mathcal{M}(A \otimes K)$, the corollary follows from Theorem 2.6. ■

We end this section with a result, rephrasing G. Elliott [6], stating that $\mathcal{M}(A \otimes K)$ has a large supply of diagonal elements.

PROPOSITION 2.8. *Let A be a unital C^*-algebra.*

(i) *For every $Y \in \mathcal{M}(A \otimes K)$, there is a diagonal element $X \in \mathcal{M}(A \otimes K)^+$ such that $I(X) + A \otimes K = I(Y) + A \otimes K$.*

(ii) *Assume I and J are ideals of $\mathcal{M}(A \otimes K)$ both containing $A \otimes K$ and such that $I \not\subseteq J$. Then there is a positive diagonal element in $I \setminus J$.*

Proof. (i) We assume that $Y \notin A \otimes K$ (otherwise, take $X = 0$). As in Elliott's proof of [6, Theorem 3.1], we have

$$Y = y_0 + Y_1 + Y_2 + Y_3,$$

with $y_0 \in A \otimes K$; $Y_j \in I(Y)$ and

$$Y_j = \mathrm{Diag}(y_{1j}, y_{2j}, \ldots), \quad j = 1, 2, 3,$$

diagonal with respect to some fixed approximate unit $\{e_n\}$ for $A \otimes K$. Put

$$X = \mathrm{Diag}(y_{1,1}, y_{1,2}, y_{1,3}, y_{2,1}, \ldots).$$

Then

$$I(X) = I(Y_1, Y_2, Y_3) \subseteq I(Y) \subseteq I(y_0, Y_1, Y_2, Y_3) \subseteq I(X) + A \otimes K,$$

which proves $I(X) + A \otimes K = I(Y) + A \otimes K$. (Replace X by X^*X to get X positive.)

(ii) Choose $Y \in I \backslash J$ and use (i) to find a diagonal $X \in \mathcal{M}(A \otimes K)^+$ with $I(X) + A \otimes K = I(Y) + A \otimes K$. Then $X \in I \backslash J$. ■

3. SIMPLICITY OF $\mathcal{M}(A \otimes K)/A \otimes K$

It is decided in this section for which unital C^*-algebras A the corona algebra $\mathcal{M}(A \otimes K)/A \otimes K$ is simple (Theorem 3.2). Recall that A is finite if $u^*u = 1$ implies $uu^* = 1$ for $u \in A$, and that A is stably finite if $M_n(A)$ is finite for all $n \in \mathbb{N}$. A simple unital C^*-algebra A is purely infinite if $A \neq \mathbb{C}$ and for every $x, y \in A^+$ with $x \neq 0$ there is $r \in A$ with $y = r^*xr$. If A is purely infinite and simple, then so is $M_n(A)$ for all n.

LEMMA 3.1. *Let A be a unital simple C^*-algebra which is neither purely infinite nor finite dimensional. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ in A^+ satisfying*

(i) $x_n x_m = 0, \quad n \neq m,$

(ii) $\|x_n\| = 1,$

(iii) $C(1_A, x_n) \geqslant 2^n,$

(iv) $\tau(f_\varepsilon(x_n)) \leqslant 2^{-n}$ *for each normalized positive trace τ on A and each $\varepsilon > 0$,*

and

(v) $\sum_{j=n+1}^{m} x_j \precsim x_n$ *for all n and all $m \geqslant n+1$.*

Proof. From the hypothesis that A is not purely infinite and $A \neq \mathbb{C}$, there is $y_0 \in A^+$ with $\|y_0\| = 1$ and $C(1_A, y_0) \geqslant 2$. We construct for each $n \in \mathbb{N}$, x_n, y_n and z_n in A^+ and $u_n \in U(A)$ such that

(a) $x_n, y_n \in \overline{f_{1/3}(y_{n-1}) A f_{1/3}(y_{n-1})},$

(b) $x_n = x_n z_n$ and $y_n z_n = 0,$

(c) $x_n = u_n y_n u_n^*$

(d) $\|x_n\|\ (= \|y_n\|) = 1.$

Indeed, let $y_{n-1} \in A^+$ with $\|y_{n-1}\| = 1$ be given and note that $f_{1/3}(y_{n-1}) \neq 0$. Since A is simple and not finite dimensional we can find

$$x', y' \in \overline{f_{1/3}(y_{n-1})Af_{1/3}(y_{n-1})}^{+}$$

with $\|x'\| = \|y'\| = 1$ and $x' \perp y'$. Put $z_n = 1 - f_{1/6}(y')$ and $y'' = f_{1/3}(y')$ $(\neq 0)$. From [9, Lemma 3.4] there is $u_n \in U(A)$ such that

$$(B =)\ \overline{x'Ax'} \cap u_n \overline{y''Ay''} u_n^* \neq \{0\}.$$

Choose $x_n \in B^+$ with $\|x_n\| = 1$ and set $y_n = u_n^* x_n u_n$. It is now easy to check that x_n, y_n, z_n and u_n satisfy (a) to (d).

We prove next that $\{x_n\}$ satisfy (i) to (v). Note that $x_m, y_m \in \overline{f_{1/3}(y_n)Af_{1/3}(y_n)}$ if $m \geqslant n+1$, and so (using(d)),

$$x_m, y_m \leqslant f_{1/6}(y_n), \quad m \geqslant n+1.$$

In particular, $x_m \perp x_n$ for $m \geqslant n+1$, wich proves (i). Also, for $m \geqslant n+1$,

$$\sum_{j=n+1}^{m} x_j \leqslant f_{1/6}(y_n) = u_n^* f_{1/6}(x_n) u_n \precsim x_n,$$

and this proves (v).

We also have $f_\varepsilon(x_m) \perp f_\varepsilon(y_m)$ and $f_\varepsilon(x_m), f_\varepsilon(y_m) \leqslant f_{1/6}(b_n)$ for $m \geqslant n+1$ and for all $\varepsilon > 0$. Let τ be a positive normalized trace on A, and put $\alpha_n = \lim_{\varepsilon \to 0} \tau(f_\varepsilon(x_n))$. Then

$$2\alpha_n \leqslant \tau(f_{1/6}(y_{n-1})) \leqslant \alpha_{n-1}, \quad (\alpha_0 = 1),$$

which proves $\alpha \leqslant 2^{-n}$ and (iv) follows.

Put $k_n = C(1_A, x_n)$ and $k_0 = C(1_A, y_0)$ $(\geqslant 2)$. Let l_n be the least integer $\geqslant k_n/2$. Then there are elements $r_1, \ldots, r_{l_n}$ and $s_1, \ldots, s_{l_n}$ in A such that

$$1_A \leqslant \sum_{j=1}^{l_n} r_j x_n r_j^* + \sum_{j=1}^{l_n} s_j x_n s_j^*.$$

Put $t_j = r_j z_n + s_j z_n u_n^*$. Then

$$t_j(x_n + y_n)t_j^* = r_j x_n r_j^* + s_j x_n s_j^*,$$

and so $C(1_A, x_n + y_n) \leqslant l_n$. Since

$$x_n + y_n \leqslant f_{1/6}(y_{n-1}) = u_n^* f_{1/6}(x_{n-1}) u_n \precsim x_{n-1},$$

we find that

$$2k_{n-1} = 2C(1_A, x_{n-1}) \leqslant 2C(1_A, x_n + y_n) \leqslant 2l_n \leqslant k_n + 1.$$

This proves that $k_n \geqslant 2^n$, and hence (iii). ∎

THEOREM 3.2. *Let A be a unital C^*-algebra. Then $\mathcal{M}(A \otimes K)/A \otimes K$ is simple if and only if either $A \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ or A is a purely infinite simple C^*-algebra.*

Proof. If $A \cong M_n(\mathbb{C})$, then $A \otimes K \cong K$ and $\mathcal{M}(K)/K \cong B(H)/K$ is simple. Assume that A is simple and purely infinite, and let I be an ideal of $\mathcal{M}(A \otimes K)$ properly containing $A \otimes K$. Then from the Proposition 2.8, there is a positive diagonal element $X = \mathrm{Diag}(x_1, x_2, \ldots) \in I \backslash A \otimes K$. Now since $\lim_{j \to \infty} \|x_j\| \neq 0$, there is $\delta > 0$ such that $f_\delta(x_j) \neq 0$ for infinitely many $j \in \mathbb{N}$. Hence, for each $m \in \mathbb{N}$ there is $m' \geqslant m$ such that $f_\delta(x_j) \neq 0$ for some j with $m \leqslant j \leqslant m'$, and since A is purely infinite,

$$1_A \precsim \mathrm{Diag}(f_\delta(x_m), \ldots, f_\delta(x_{m'})).$$

From Corollary 2.7 it follows that $I(X) = \mathcal{M}(A \otimes K)$, and so $I = \mathcal{M}(A \otimes K)$, and this proves that $\mathcal{M}(A \otimes K)/A \otimes K$ is simple.

Suppose now that A is not simple, and let J be a non-trivial ideal in A. Then $J \otimes K$ is a non-trivial ideal in $A \otimes K$, and

$$I_0 = \{X \in \mathcal{M}(A \otimes K) \mid X(A \otimes K) \subseteq J \otimes K\}$$

is a closed two-sided ideal in $\mathcal{M}(A \otimes K)$. Put $I = I_0 + A \otimes K$. Choose $x \in J^+, x \neq 0$, and put $X = \mathrm{Diag}(x, x, \ldots)$. Then $X \in I_0 \subseteq I$ but $X \notin A \otimes K$, and so $I \neq A \otimes K$.

Assume — to reach a contradiction — that $I = \mathcal{M}(A \otimes K)$. Then

$$1_{\mathcal{M}(A \otimes K)} = X + y$$

for some $X \in I_0$ and $y \in A \otimes K$. Let $\{f_n\}$ be an approximate unit of projections for K, and put $e_n = 1_A \otimes f_n$. Choose n large enough such that $\|(1 - e_n)y(1 - e_n)\| < 1$. Then

$$\|(1 - e_n) - (1 - e_n)X(1 - e_n)\| < 1,$$

and since $1 - e_n$ is a projection, there is $Z \in \mathcal{M}(A \otimes K)$ such that $1 - e_n = Z(1 - e_n)X(1 - e_n)$. Thus $1 - e_n \in I_0$, contradicting the fact that

$$e_m - e_n = e_m(1 - e_n) \notin J \otimes K$$

for $m > n$. Hence $I \neq \mathcal{M}(A \otimes K)$ and $\mathcal{M}(A \otimes K)/A \otimes K$ is not simple.

Suppose finally that A is simple, but not finite dimensional and not purely infinite. Let $\{x_n\}$ be the sequence in A^+ constructed in Lemma 3.1, and put $X = \mathrm{Diag}(x_1, x_2, \ldots)$. Then for every $\delta > 0, m \in \mathbb{N}$ and $m' \geqslant m$ we have

$$C(1_A, \mathrm{Diag}(f_\delta(x_m), \ldots, f_\delta(x_{m'})) = C\left(1_A, \sum_{j=m}^{m'} f_\delta(x_j)\right) \geqslant$$

$$\geqslant C\left(1_A, \sum_{j=m}^{m'} x_j\right) \geqslant C(1_A, x_{m-1}) \geqslant 2^{m-1}.$$

From Corollary 2.7, this implies $I(X) \neq \mathcal{M}(A \otimes K)$. Since $\|x_j\| = 1$ for all $j \in \mathbb{N}$, $X \notin A \otimes K$, and it follows that $A \otimes K \subsetneq I(X) \subsetneq \mathcal{M}(A \otimes K)$ and $I(X) \neq A \otimes K, \mathcal{M}(A \otimes K)$. ■

REMARK 3.3. The class of C^*-algebras B with $\mathcal{M}(B)/B$ simple, when B is not assumed to be isomorphic to $A \otimes K$ for some unital C^*-algebra A (in particular, when B is non-stable), is more complicated to describe:

If B is a simple AF-algebra and all traces on B extend to bounded traces on $\mathcal{M}(B)$, then $\mathcal{M}(B)/B$ is simple (see [6] and [8]).

If B is such that $\mathcal{M}(B)/B$ is simple and C is unital, then $\mathcal{M}(B \oplus C)/B \oplus C \cong \mathcal{M}(B)/B$ is simple.

Let A be a simple unital C^*-algebra which is not finite dimensional or purely infinite, so that $\mathcal{M}(A \otimes K)/A \otimes K$ is not simple. Let I be a maximal proper ideal of $\mathcal{M}(A \otimes K)$. Then $A \otimes K \subseteq I, A \otimes K \neq I$ and $\mathcal{M}(I) = \mathcal{M}(A \otimes K)$. Hence $\mathcal{M}(I)/I$ is simple and I has non-trivial essential ideal.

REMARK 3.4. In [2] it is proved that $A \otimes K$ is algebraically simple if and only if A is simple and not stably finite. It is not clear whether this relates to simplicity of $\mathcal{M}(A \otimes K)/A \otimes K$ (this certainly is not the case for $A \otimes K \cong K$). Still, it appears to add the mystery of the (maybe vacuous) class of simple C^*-algebras which are neither stably finite nor purely infinite (see also Remark 4.6).

4. IDEALS IN $\mathcal{M}(A \otimes K)$ AND TRACES ON A

Let A be a unital C^*-algebra, and let τ be a (positive, normalized) trace (or quasi-trace) on A. Extend τ to a semi-finite trace on $A \otimes K$, and to a trace function on $\mathcal{M}(A \otimes K)^+$ by

$$\tau(X) = \sup \tau(e_n X e_n)$$

where $\{e_n\}$ is any approximate unit for $A \otimes K$. As in [6,8], define I_τ to be the closure of the set of all elements X in $\mathcal{M}(A \otimes K)$ with $\tau(X^*X) < \infty$. It is easily checked that I_τ is a closed two-sided ideal in $\mathcal{M}(A\otimes K)$. Note that if $X \in \mathcal{M}(A\otimes K)^+$, then $X \in I_\tau$ if and only if $\tau(f_\varepsilon(X)) < \infty$ for all $\varepsilon > 0$ (use Lemma 2.2 for this).

Let $T(A)$ denote the set of all positive normalized traces on A. If $\tau_1, \tau_2 \in T(A)$, then write $\tau_1 \precsim \tau_2$ if there is $c > 0$ so that $c\tau_1 \leqslant \tau_2$. Write $\tau_1 \sim \tau_2$ if $\tau_1 \precsim \tau_2$ and $\tau_2 \precsim \tau_1$.

PROPOSITION 4.1. *Let A be a unital C^*-algebra, let τ and τ' be in $T(A)$, and let I_τ and $I_{\tau'}$ denote the corresponding ideals in $\mathcal{M}(A\otimes K)$. Then*

(i) $A\otimes K \subseteq I_\tau$ and $I_\tau \neq \mathcal{M}(A\otimes K)$,

(ii) $I_\tau \subseteq I_{\tau'}$ *if and only if* $\tau' \precsim \tau$,

(iii) $I_\tau = I_{\tau'}$ *if and only if* $\tau \sim \tau'$,

(iv) *if I_τ is a maximal ideal in $\mathcal{M}(A\otimes K)$, then τ is an extreme point in $T(A)$.*

Proof. (i) obvious.

(ii) If $c\tau' \leqslant \tau$ for some $c > 0$ then

$$\tau'(f_\varepsilon(X)) \leqslant c^{-1}\tau(f_\varepsilon(X)) < \infty$$

for each $X \in I_\tau^+$ and each $\varepsilon > 0$, and so $X \in I_{\tau'}$. Thus $I_\tau \subseteq I_{\tau'}$.

Suppose now that $\tau' \precsim \tau$. Let $M = (\varphi_\tau \oplus \varphi_{\tau'})(A)''$, where φ_τ is the GNS-representation of A with respect to τ, and note that τ and τ' extend to traces $\tilde{\tau}$ and $\tilde{\tau'}$ on M with $\tilde{\tau'} \precsim \tilde{\tau}$. For each $n \in \mathbb{N}$ find a projection p_n in M with $2^{-(n+1)}\tilde{\tau'}(p_n) > \tilde{\tau}(p_n)$. Put $\delta_n = \tilde{\tau'}(p_n)$ and choose $m_n \in \mathbb{N}$ such that

$$(*) \qquad \frac{\frac{2}{3}\delta_n - \left(\frac{1}{2}\right)^{m_n}}{1 - \left(\frac{1}{2}\right)^{m_n}} \geqslant \frac{1}{2}\delta_n.$$

From Kaplansky's density theorem, $(\varphi_\tau \oplus \varphi_{\tau'})(A_1 \cap A^+)$ is strong operator dense in $M_1 \cap M^+$; and since the maps $x \to \tilde{\tau}(x)$ and $x \to \tilde{\tau'}(x^{m_n})$ are strong operator continuous, we can find $x_n \in A_1 \cap A^+$ with

$$|\tilde{\tau}(p_n - (\varphi_\tau \oplus \varphi_{\tau'})(x_n))| \leqslant 2^{-(n+1)}\delta_n,$$

$$|\tilde{\tau'}(p_n - (\varphi_\tau \oplus \varphi_{\tau'})(x_n)^{m_n})| \leqslant \frac{1}{3}\delta_n.$$

Then $\tau(x_n) \leqslant 2^{-n}\delta_n$ and $\tau'(x_n^{m_n}) \geqslant \frac{2}{3}\delta_n$. There is a probability measure μ on [0,1] such that

$$\tau'(f(x_n)) = \int_0^1 f(t)d\mu(t), \qquad f \in C([0,1]).$$

It follows from (*) that $\mu([\frac{1}{2},1]) \geqslant (1/2)\delta_n$, and so $\tau'(f_{1/4}(x_n)) \geqslant (1/2)\delta_n$. Let k_n be an integer with $1 \leqslant k_n\delta_n \leqslant 2$, and put

$$X = \mathrm{Diag}(\underbrace{x_1,\ldots,x_1}_{k_1 \text{ times}},\underbrace{x_2,\ldots,x_2}_{k_2 \text{ times}},\ldots).$$

Then

$$\tau(X) = \sum_{n=1}^{\infty} k_n\tau(x_n) \leqslant \sum_{n=1}^{\infty} 2^{-n}k_n\delta_n \leqslant 2,$$

and so $X \in I_\tau$. On the other hand,

$$\tau'(f_{1/4}(X)) = \sum_{n=1}^{\infty} k_n\tau'(f_{1/4}(x_n)) \geqslant \sum_{n=1}^{\infty}\frac{1}{2}k_n\delta_n = \infty,$$

whence $X \notin I_{\tau'}$ and so $I_\tau \not\subseteq I_{\tau'}$.

(iii) follows from (ii).

(iv) Assume that τ is not extreme in $T(A)$. Then $M = \varphi_\tau(A)''$ is not a factor. Choose a non-trivial central projection p in the center of M and put

$$\tau_1(x) = \tau(p)^{-1}\tau(px), \quad \tau_2(x) = \tau(1-p)^{-1}\tau((1-p)x).$$

Then $\tau_1, \tau_2 \in T(A), \tau_1 \not\sim \tau_2$ and $\tau_j \precsim \tau$. Hence, by (ii) and (iii), $I_\tau \subseteq I_{\tau_j}$ and $I_{\tau_1} \neq I_{\tau_2}$, and so I_τ is not maximal. ∎

REMARK 4.2. If A is a unital simple C^*-algebra which is not finite dimensional, and $\tau \in T(A)$, then one can use Lemma 3.1 (iv) to prove that $A \otimes K \neq I_\tau$. One has $I_\tau \neq A \otimes K$ under more general assumptions on A.

REMARK 4.3. If τ and τ' are distinct extreme points of $T(A)$, then $\tau \not\sim \tau'$ and so $I_\tau \neq I_{\tau'}$. If $T(A)$ has precisely n extreme points, then $T(A)$ contains exactly $2^n - 1$ equivalence classes, and there are exactly $2^n - 1$ distinct ideals of $\mathcal{M}(A \otimes K)$ of the form I_τ for some $\tau \in \tau(A)$.

For each $\tau \in T(A)$ associate a dimension function d_τ (cf. Cuntz [4]) on $(A \otimes \mathcal{F})^+$, where $\mathcal{F}$ denotes the algebra of finite rank operators on a Hilbert space, by

$$d_\tau(x) = \lim_{\varepsilon\to\infty} \tau(f_\varepsilon(x)).$$

We say that A has comparison given by its traces if for $x, y \in (A \otimes \mathcal{F})^+, d_\tau(x) < < d_\tau(y)$ for all $\tau \in T(A)$ implies $x \precsim y$ (see also B. Blackadar [1]). It is known that von Neumann algebras, AF-algebras and purely infinite simple C^*-algebras have this property.

The following theorem which extends Theorem 2 of H. Lin [8], states that the lattice $T(A)/\sim$ is isomorphic to the lattice of ideals in $\mathcal{M}(A \otimes K)$ under certain (rather strong) assumption on A.

THEOREM 4.4. *Let A be a simple unital infinite dimensional C^*-algebra which has comparison given by its traces, and assume further that $T(A)$ has only finitely many extreme points. Then*

(i) *I is a maximal ideal in $\mathcal{M}(A \otimes K)$ if and only if $I = I_\tau$ for some extremal trace τ,*

(ii) *if I is a proper ideal in $\mathcal{M}(A \otimes K)$ properly containing $A \otimes K$, then $I = I_\tau$ for some $\tau \in T(A)$,*

(iii) *there are exactly $2^n - 1$ proper ideals in $\mathcal{M}(A \otimes K)$ properly containing $A \otimes K$.*

Proof. Let $\{\tau_1, \tau_2, \ldots, \tau_n\}$ denote the set of extreme points $T(A)$.

(i). From (iii) of Proposition 4.1, $I_{\tau_k} \not\subseteq I_{\tau_l}$ if $k \neq l$. It suffices therefore to prove that if I is an ideal in $\mathcal{M}(A \otimes K)$ not contained in any I_{τ_j}, then $I = \mathcal{M}(A \otimes K)$.

Let I be such an ideal, and find $Y_j \in I \setminus I_{\tau_j}, Y_j \geqslant 0$. Put $Y = Y_1 + Y_2 + \ldots + {}+Y_n$, and find a diagonal $X = \mathrm{Diag}(x_1, x_2, \ldots)$ in $\mathcal{M}(A \otimes K)^+$ with $I(X) = I(Y)$ (cf. Proposition 2.8). Then $X \in I \setminus I_{\tau_j}, j = 1, 2, \ldots, n$, and so, for some $\delta > 0$, $\tau_j(f_\delta(X)) = \infty$ for $j = 1, \ldots, n$. Hence

$$\sum_{i=1}^{\infty} d_{\tau_j}(f_\delta(x_i)) = \infty \quad j = 1, \ldots, n.$$

Using the comparison property of A and Corollary 2.7, we find that $I \supseteq I(X) = {} = \mathcal{M}(A \otimes K)$.

(ii) and (iii). For each non-empty subset σ of $\{1, 2, \ldots, n\}$ $(= \Sigma)$, set

$$I_\sigma = \bigcap_{j \in \sigma} I_{\tau_j}, \quad \tau_\sigma = |\sigma|^{-1} \sum_{j \in \sigma} \tau_j,$$

and note that $I_\sigma = I_{\tau_\sigma}$. From (i) and the Chinese Remainder Theorem,

$$\mathcal{M}(A \otimes K)/I_\Sigma \cong \bigoplus_{j=1}^{n} \mathcal{M}(A \otimes K)/I_{\tau_j}.$$

Hence, if I is an ideal in $\mathcal{M}(A \otimes K)$ and $I_\Sigma \subseteq I$, then $I = I_\sigma (= I_{\tau_\sigma})$ for some $\sigma \subseteq \Sigma$; and $I_{\sigma_1} \neq I_{\sigma_2}$ if $\sigma_1 \neq \sigma_2$. It therefore suffices to prove that $I_\Sigma / A \otimes K$ is simple. Assume that I is an ideal properly containing $A \otimes K$. By Proposition 2.8 we can find a positive diagonal $X = \mathrm{Diag}(x_1, x_2, \ldots)$ in $I \setminus A \otimes K$. Assume — to reach a contradiction — that $I_\Sigma \not\subseteq I$. Then again by Proposition 2.8, there is a positive diagonal $Y = \mathrm{Diag}(y_1, y_2, \ldots)$ in $I_\Sigma \setminus I$. Since $X \notin A \otimes K$, there is $\delta > 0$ so that $f_\delta(x_j) \neq 0$ for infinitely many $j \in \mathbb{N}$, and $\tau_j(f_\varepsilon(Y)) < \infty$ for $j = 1, 2, \ldots, n$ and for all $\varepsilon > 0$ because $Y \in I_\Sigma$. Hence

$$\sum_{i=m}^{\infty} d_{\tau_j}(f_\delta(x_i)) > 0 \quad j = 1, \ldots, n,$$

for all $m \in \mathbb{N}$, and

$$\lim_{k \to \infty} \sum_{i=k}^{\infty} d_{\tau_j}(f_\varepsilon(y_i)) = 0 \quad j = 1, \ldots, n.$$

From the comparison property of A we conclude that $\{f_\varepsilon(y_i)\} \precsim \{f_\delta(x_i)\}$. This holds for all $\varepsilon > 0$, and so from Theorem 2.6, $Y \in I(X) \subseteq I$, a contradiction. ■

REMARK 4.5. It is not true in general that maximal ideals of $\mathcal{M}(A \otimes K)$ are of the form I_τ for some extreme trace τ. Indeed, if $T(A)$ is not the norm closure of conv E, where E is the set of extreme points of $T(A)$, then there is $\tau \in T(A)$ for which $\tau \precsim \tau'$ for all $\tau' \in E$. From Proposition 4.1, $I_\tau \not\subseteq I_{\tau'}$ for all $\tau' \in E$ and so , if I is a maximal ideal in $\mathcal{M}(A \otimes K)$ containing I_τ, then $I \neq I_{\tau'}$ for $\tau' \in E$ (or for $\tau' \in T(A)$).

REMARK 4.6. If A is a simple unital C^*-algebra which is neither stably finite nor purely infinite, then by Theorem 3.2, $\mathcal{M}(A \otimes K)$ has a proper ideal properly containing $A \otimes K$. However, A has no traces (and even no quasi-traces).

REFERENCES

1. BLACKADAR, B., Comparison theory for simple C^*-algebras, *Operator Algebras and Applications* (editors, D. E. Evans and M. Takesaki), Cambridge University Press, 1988, Volume 1, pp. 21-54.
2. BLACKADAR, B.; CUNTZ J., The structure of algebraically simple C^*-algebras, *Math. Scand.*, **104**(1982), 813-822.
3. CUNTZ J., The structure of addition and multiplication in simple C^*-algebras, *Math. Scand.*, **40**(1977), 215-233.
4. CUNTZ, J., Dimension functions on simple C^*-algebras, *Math. Ann.*, **233**(1978), 145-153.
5. CUNTZ J., The internal structure of simple C^*-algebras, operator algebras and applications, *Proc. Sympos. Pure Math.*, **38**, Amer. Math. Soc., Providence, (1981), part 1, 85-115.
6. ELLIOTT, G., Derivations of matroid C^*-algebras. II, *Ann. of Math.*, **100**(1974), 407-422.
7. ELLIOTT, G., The ideal structure of the multiplier algebra of an AF-algebra, *C. R. Math. Rep. Acad. Sci. Canada*, **9**(1987), 225-230.
8. LIN H., Ideals of multiplier algebras of simple AF C^*-algebras, *Proc. Amer. Math. Soc.*, **104**(1988), 239-244.

9. RØRDAM, M., On the structure of simple C^*-algebras tensored with a UHF-algebra, *J. Func. Anal.* **100**(1991), 1-17.
10. ZHANG, S., A Riesz descomposition property and ideal structure of multiplier algebras, *J. Operator Theory*, **24**(1990), 209-226.

MIKAEL RØRDAM
Department of Mathematics,
Odense University,
DK-5230 Odense M,
Denmark.

Received October 4, 1989; revised December 21, 1989.