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A FAMILY OF DILATION CROSSED PRODUCT ALGEBRAS

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§0

ABSTRACT. We study a one-parameter family of crossed product C^*-algebras associated with automorphisms of certain solenoids. A canonical set of topological conjugacy invariants of the dynamical system (which are also known to be isomorphism invariants of the associated C^*-algebra) is shown to contain the entropy of the dynamical system.

We introduce a family of C^*-algebras B_a parametrized by strictly positive real numbers $a \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Each B_a is defined as the crossed product C^*-algebra $C(\hat{\Lambda}_a) \rtimes \mathbb{Z}$ arising from an abelian dynamical system, namely an action of $\mathbb{Z}$ on a compact space $\hat{\Lambda}_a$. As the notation suggests this dynamical system is obtained by dualizing a group automorphism of a discrete abelian group Λ_a.

We show that certain topological conjugacy invariants (including the entropy) involving the periodic points of the dynamical system are isomorphism invariants of the C^*-algebra.

A study of periodic points has always been an important part of the theory of dynamical systems. It seems well known that the number of points fixed by γ^n (γ a homeomorphism of a compact space X) is recoverable as an isomorphism invariant of $A = C(X) \rtimes \mathbb{Z}$. Only some knowledge of the structure of the $\hat{A}_n$ ($n \in \mathbb{N}$), the space of unitary equivalence classes of n-dimensional irreducibile representations is needed. As shown later an easy application of multiplicity theory provides the required information. Actually more than what is needed is true, namely $\hat{A}_n$ is homeomorphic with $O_n \times \mathbb{T}$ where O_n is the space of orbits in X consisting of n distinct points [5].

Of course this information may also exist in other algebraic objects associated with the dynamical system. In the special case of a zero dimensional compact metrizable space X (we will note that the dimension of $\hat{\Lambda}_a$ is the degree of the irreducible polynomial of a over $\mathbb{Q}$) it is shown in [10] that this information is recoverable from a certain ordered group associated with the dynamical system. As an unordered group it coincides with the (unordered) K_0 group of the crossed product algebra (again for X zero dimensional).

For the dynamical system under consideration, $(\hat{\Lambda}_a, \mathbb{Z})$, it is possible to compute the number of periodic points of period n, thus obtaining a sequence of (computable) isomorphism invariants of the algebra B_a.

For some dynamical systems it is known that the entropy coincides with the growth rate of the periodic points [11]. The entropy for a class of dynamical systems (including $(\hat{\Lambda}_a, \mathbb{Z})$) was computed by Yuzvinskii [8]. Comparing the entropy with growth rate of periodic points for $(\hat{\Lambda}_a, \mathbb{Z})$ we see that they coincide. This is far from true in general, even within the class of dynamical systems considered by Yuzvinskii.

There are many obvious questions and generalizations which are not addresed in what is to follow. For example, the invariants provide an indispensable tool for classifying the algebras B_a. These and other considerations will appear subsequently.

NOTATION. If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$ and $\mathrm{Id}_{\mathcal{H}}$ is the identity of $\mathcal{H}$. The unit circle in the complex plane $\mathbb{C}$ is denoted by $\mathbb{T}$. If $\mathcal{H} \cong \mathbb{C}^m$, $B(\mathcal{H}) = \mathcal{M}_m(\mathbb{C})$ and if $\mathcal{A}$ is an abelian algebra of operators, $\mathcal{M}_m(\mathcal{A}) \cong \mathcal{M}_m(\mathbb{C}) \otimes \mathcal{A}$. Where convenient $\langle , \rangle : \hat{G} \times G \to \mathbb{T}$ denotes the duality between the locally compact abelian groups G and $\hat{G}$.

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For $a \in \mathbb{R}^+$ let Λ_a denote the additive subgroup $\{\sum a^{n_i} m_i \mid n_i, m_i \in \mathbb{Z}, m_i$ non zero for finitely many $i\}$ of $\mathbb{R}$ and G_a the semidirect product of $\mathbb{Z}$ where $\mathbb{Z}$ acts on Λ_a by $n \cdot \lambda = a^n \lambda$ ($n \in \mathbb{Z}$, $\lambda \in \Lambda_a$). Unless $a \in \mathbb{Z}$, the group Λ_a is dense in $\mathbb{R}$ so the subspace topology of Λ_a is not one in which Λ_a is a locally compact group. Endow Λ_a and G_a with the discrete topology.

DEFINITION. For $a \in \mathbb{R}^+$ denote by B_a the C^*-crossed product algebra $C^*(\Lambda_a) \rtimes \mathbb{Z}$.

Note that both $\mathbb{Z}$ and Λ_a sit naturally in B_a. Since $\mathbb{Z}$ and Λ_a are amenable the regular representation of B_a induced by the left regular representation of $C^*(\Lambda_a)$ on $\ell^2(\Lambda_a)$ is faithful and $B_a = C^*(\Lambda_a) \rtimes_{red} \mathbb{Z}$ (see [9]). Also since G_a is amenable the (right) regular representation of $C^*(G_a)$ is faithful and it easily follows that B_a is isomorphic to the group C^*-algebra $C^*(G_a)$. Note that the algebras B_a are nuclear. The group G_a ($a \neq 1$) is an infinite conjugacy class group so the von Neumann algebra generated by the left representation of B_a is a type II_1 factor, namely the matricial II_1 factor (since B_a is nuclear).

Although the connection may seem spurious to some we think it is worth noting, if only to expose our original motivation, that the algebras B_a may be contrasted with the irrational rotation algebras. The later arise as (an infinite dimensional and simple quotient of) the group C^*-algebra of a discrete Heisenberg group while the former are group C^*-algebras of subgroups of the $ax+b$ groups (with the discrete topology). Recall that the $ax+b$ group is the semidirect product $\mathbb{R}^+ \rtimes \mathbb{R}$ with multiplication defined by $(a,b)(a',b') = (aa', ab'+b)$. For $a \neq 1$, we see that G_a is isomorphic to the subgroup generated by the elements $(a,0)$ and $(1,1)$. The irrational rotation algebras are simple while the B_a are far from simple. The primary means used to classify the rotation algebras are K-theoretic while the entropy of the underlying dynamical system is zero in all cases. For the algebras B_a, our preliminary calculations seem to indicate that K-theoretic invariants are of limited use while the entropy (more precisely, the zeta function [1]) is the useful invariant.

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Several aspects of the representation theory of B_a are considered below.

The dual group $\hat{\Lambda}_a$ is compact, it is the Bohr compactification $b_{\Lambda_a}\mathbb{R}$ of $\mathbb{R}$ ([3]). Since Λ_a is torsion free $\hat{\Lambda}_a$ is connected. The map dual to the inclusion $i : \Lambda_a \to \mathbb{R}$ is a continuous group homomorphism $\varphi : \mathbb{R} \to \hat{\Lambda}_a$ with dense image such that $\varphi(r)(\lambda) = \exp(ir\lambda)$ for $\lambda \in \Lambda_a$, $r \in \mathbb{R}$. If $C^*(\Lambda_a)$ is identified with $C(\hat{\Lambda}_a)$ via the Gelfand isomorphism, the element $\lambda \in \Lambda_a$ is identified with the function in $C(\hat{\Lambda}_a)$ mapping x to $x(\lambda)$ ($x \in \hat{\Lambda}_a$). The action of $\mathbb{Z}$ on Λ_a by group isomorphisms yields an action, denoted $n \to \alpha^n$ ($n \in \mathbb{Z}$) of $\mathbb{Z}$ on $\hat{\Lambda}_a$ by group isomorphisms. Note that $\alpha^n(\varphi(r)) = \varphi(a^n r)$ for $n \in \mathbb{Z}$, $r \in \mathbb{R}$. The uniqueness of the Haar measure ν on $\hat{\Lambda}_a$ ensures that ν is invariant under α. The corresponding dual action $\hat{\alpha}$ of $\mathbb{Z}$ on $C(\hat{\Lambda}_a)$ agrees of course with the action of $\mathbb{Z}$ on $C(\hat{\Lambda}_a)$ defined by the Gelfand isomorphism. For $\lambda \in \Lambda_a$ viewed as an element of $C(\hat{\Lambda}_a)$ we have $\hat{\alpha}^n(\lambda) = n \cdot \lambda = a^n \lambda$.

The C^*-algebra $C(\hat{\Lambda}_a)$ may be identified (via the map $f \to f \circ \varphi$) with an algebra of almost periodic functions on $\mathbb{R}$, namely those functions with frequencies in Λ_a.

PROPOSITION 1. *For $a \neq 1$ the action α of $\mathbb{Z}$ on $\hat{\Lambda}_a$ is ergodic.*

Proof. By a result of Rohlin and Halmos, ergodicity follows by showing that the trivial character is the only character $\lambda \in \Lambda_a$ of $\hat{\Lambda}_a$ satisfying $n \cdot \lambda = \lambda$ for some $n \in \mathbb{N}$

([11]). However $n \cdot \lambda = \lambda$ if and only if $(a^n - 1)\lambda = 0$. Since $a \in \mathbb{R}^+$ and $a \neq 1$, a is not a root of unity and $\lambda = 0$. ■

Let M be the multiplication representation of $C(\hat{\Lambda}_a)$ on the Hilbert space $L^2(\hat{\Lambda}_a, \nu)$, $M_f\xi = f \cdot \xi$ (pointwise product for $f \in C(\hat{\Lambda}_a)$, $\xi \in L^2(\hat{\Lambda}_a, \nu)$). If U is the unitary operator acting on $L^2(\hat{\Lambda}_a, \nu)$ mapping ξ to $\xi \circ \alpha$ we obtain a covariant representation (M, U) of the dynamical system $(C(\hat{\Lambda}_a), \hat{\alpha}, \mathbb{Z})$ and thus a representation π of B_a on $L^2(\hat{\Lambda}_a, \nu)$. Note that the weak closure of the image of M is a maximal abelian $*$-subalgebra of $\mathcal{B}(L^2(\hat{\Lambda}_a, \nu))$ and thus, if the action of α is ergodic (ensured if $a \neq 1$), it easily follows that π is an irreducible representation.

THEOREM 1. *For $a \neq 1$ the irreducible representation π of B_a on $L^2(\hat{\Lambda}_a, \nu)$ defined above is faithful.*

Proof. Since any closed two sided ideal of $C(\hat{\Lambda}_a) \rtimes \mathbb{Z}$ contains elements of the form $f\rho(g)$ ($f \in C(\hat{\Lambda}_a)$), $g \in \ell^1(\mathbb{Z})$ with $\rho : \ell^1(\mathbb{Z}) \to B_a$ the integrated representation of the natural inclusion of $\mathbb{Z}$ in B_a it is sufficient to show that $\pi(f\rho(g)) = 0$ implies $f\rho(g) = 0$ ([6]). The functions $\lambda \in \Lambda_a$ form a complete orthonormal system in $L^2(\hat{\Lambda}_a, \nu)$ (recall $L^2(\hat{\Lambda}_a, \nu) \cong \ell^2(\Lambda_a)$). For each $\lambda \in \Lambda_a$, $U^n(\lambda) = \lambda \circ \alpha^n = a^n\lambda$ and since $a \neq 1$, $\{U^n(\lambda) \mid n \in \mathbb{Z}\}$ is an orthonormal set for nonzero λ. Let ξ_λ denote the element $\pi(\rho(g))(\lambda)$ of $L^2(\hat{\Lambda}_a, \nu)$. For $\lambda \neq 0$, $\|\xi_\lambda\|_2^2 = \|\sum g(n)a^n\lambda\|_2^2 = \sum |g(n)|^2$ is independent of λ.

If $\pi(f\rho(g)) = 0$ then $M_{f\circ\alpha^n}\pi(\rho(g)) = U^n\pi(f)\pi(\rho(g))U^{-n} = 0$ for all $n \in \mathbb{Z}$ and $(f \circ \alpha^n)\xi_\lambda = M_{f\circ\alpha^n}\pi(\rho(g))(\lambda) = 0$ (a.e. ν) for all $n \in \mathbb{Z}$ and $\lambda \in \Lambda_a$. It follows that if $X = \cup\{\alpha^n(\text{support}(f)) \mid n \in \mathbb{Z}\}$ then $\xi_\lambda = 0$ (a.e. ν) on X. However X is a measurable α-invariant subset of $\hat{\Lambda}_a$ so either $\nu(X) = 0$ or $\nu(\hat{\Lambda}_a \backslash X) = 0$. Thus either $f = 0$ or $\xi_\lambda = 0$ (a.e. ν). Since $\|\xi_\lambda\|_2 = 0$, for any nonzero λ, implies $g = 0$ we must have $f\rho(g) = 0$. ■

The functions λ and $\lambda \circ \alpha = a\lambda \in \Lambda_a$ map $\varphi(r)$ in $\hat{\Lambda}_a$ to $\exp(ir\lambda)$ and $\exp(ira\lambda) = \exp(ir\lambda)^a$ respectively. Thus we may view the unitary operator $UM_\lambda U^{-1} = M_{a\lambda}$ as $(M_\lambda)^a$, an a-th power of M_λ. If V denotes M_1 then the previous theorem shows that the C^*-algebra generated by the unitary operators V and U on $L^2(\hat{\Lambda}_a, \nu)$ is isomorphic to B_a. Loosely speaking, we have $U^nVU^{-n} = V^{a^n}$ for $n \in \mathbb{Z}$ and $M_\lambda = V^\lambda$ ($\lambda \in \Lambda_a$). This relation seems to echo the universal property for irrational rotation algebras. It does not however, suffice to define such a property (for the algebras B_a) as can be easily seen by reflecting on the constraints (described below) imposed on representation of B_a by multiplicity theory.

A representation of $A = C(X) \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts by a homeomorphism γ of a compact Hausdorff space X is just a covariant representation of the dynamical system

$(X, \gamma, \mathbb{Z})$. Thus a representation ρ of the abelian C^*-algebra $C(X)$ can be extended to a representation of A if and only if the representations ρ and $\rho \circ \gamma$ are unitarily equivalent. This occurs if and only if ρ and $\rho \circ \gamma$ have the same null ideal sequence or equivalently, have the same multiplicity functions ([4]). It is straightforward to check that, if $\{N_j \mid j \in \mathbb{N}\}$ is the null ideal sequence for ρ, then $\{\gamma(N_j) \mid j \in \mathbb{N}\}$ is the null ideal sequence for $\rho \circ \gamma$; or equivalently if m is the multiplicity function for ρ then $m \circ \gamma^{-1}$ is the multiplicity function for $\rho \circ \gamma$. Thus ρ can be extended to A if and only if $m = m \circ \gamma^{-1}$.

If ρ is representation of uniform multiplicity n then ρ is unitarily equivalent to $\overset{n}{\oplus} M$ on $\mathcal{H} = \overset{n}{\oplus} L^2(X, \mu)$ for some uniquely determined regular Borel measure μ on X. In this case it is straightforward to conclude that ρ is unitarily equivalent to $\rho \circ \gamma$ if and only if μ is equivalent to $\mu \circ \gamma^{-1}$ (i.e., the measure μ is quasi-invariant). In fact any unitary in $\mathcal{B}(\mathcal{H})$ implementing the equivalence is of the form $b_1(\tilde{U} \otimes \mathrm{Id}_{\mathbb{C}^n})$ with $\tilde{U} g = (\mathrm{d}\mu_\gamma / \mathrm{d}\mu)^{1/2} g \circ \gamma^{-1}$ (for $g \in L^2(X, \mu)$) and b_1 a unitary in $\mathcal{M}_n(L^\infty(X, \mu))$. For fixed ρ, the equivalence classes of the possible representations of A extending ρ are in a one to one correspondence with equivalence classes of the unitary 1-cocycles of $\mathbb{Z}$ in $\mathcal{M}_n(L^\infty(X, \mu))$ determined by b_1 ([2]).

Let ρ be a representation of uniform multiplicity one on $L^2(X, \mu)$ for some quasi-invariant Borel measure μ on X. A representation of A extending ρ is irreducible if and only if μ is ergodic with respect to the automorphism γ (for the dynamical system $(\hat{A}_a, \hat{\alpha}, \mathbb{Z})$ the representation π introduced above (with $\mu = \nu =$Haar measure on $\hat{A}_a$) is an example). Also any quasi-invariant point measure supported on the orbit $\mathcal{O}_x = \{\gamma^n(x) \mid n \in \mathbb{Z}\}$ of a point x in X_a is ergodic. In this case the Hilbert space $L^2(\mathcal{O}_x, \mu)$ is isomorphic to $\ell^2(\mathbb{Z})$ or, if $\mathcal{O}_x$ is finite, $\mathbb{C}^p$ for some $p \in \mathbb{N}$. Restricting attention now to the finite orbits, it is easy to check that equivalence classes of unitary 1-cocycles in $L^\infty(\mathcal{O}_x, \mu)$ $(= \mathbb{C}^p)$ are parametrized by $\mathbb{T}$. Standard multiplicity theory shows that (up to unitary equivalence) all finite dimensional irreducible representations of A are accounted for in this manner (cf. [5]).

We shall see in the last section how many finite orbits there are in $\hat{A}_a$. In any case there are many finite dimensional (irreducible) representations of B_a, so there are many traces on B_a. Also any α-invariant Borel probability measure on $\hat{A}_a$ with support $\hat{A}_a$ gives rise to a faithful trace on B_a. The Haar measure on $\hat{A}_a$ is one such, and the representation associated to this trace is the left regular representation of G_a on $\ell^2(G_a)$ discussed above. We show below that if a is algebraic ($a \neq 1$) and $n \in \mathbb{N}$, the set of points of period n under α is finite and the set of all periodic points is dense in $\hat{A}_a$. This gives rise to another α-invariant probability measure with support $\hat{A}_a$ so a faithful trace on B_a.

§3

We describe in more detail the dynamical system $(\hat{\Lambda}_a, \alpha)$. The discrete space Λ_a is the ring of Laurent polynomials in a and a^{-1}, $\mathbb{Z}[a, a^{-1}]$. If a is transcedental then Λ_a is isomorphic as a group to $\bigoplus_{\mathbb{Z}} \mathbb{Z}$, so $\hat{\Lambda}_a$ is a countably infinite product of tori, $\mathbb{T}^{\mathbb{Z}}$. The action of α on $\hat{\Lambda}_a$ is the shift $(\alpha(t))(n) = t(n+1)$ for $t \in \hat{\Lambda}_a$, $n \in \mathbb{Z}$. If a is algebraic, denote by $f \in \mathbb{Q}[x]$ the monic irreducible polynomial of a over the field of rational numbers $\mathbb{Q}$. If c is the content of f then $f = c \cdot \ell$ with $\ell \in \mathbb{Z}[x]$. The coefficients of ℓ have greatest common divisor 1 and the ideal generated by ℓ in $\mathbb{Z}[x]$ is prime. If d is the degree of ℓ then the set of d elements (each of infinite order) $\{1, a, \dots, a^{d-1}\}$ is a maximal independent set in the group Λ_a. Thus Λ_a has torsion free rank d and the dimension of $\hat{\Lambda}_a$ is d, ([3] 3.11, 24.28). Note also that $\Lambda_a = \mathbb{Z}[x, x^{-1}]/(\ell)$.

For $\ell(x) = \sum_{j=0}^{d} a_j x^j$ let $F = [f_{jk}]$ be the $d \times d$ matrix with integer entries f_{jk} defined by $a_d a^j = \sum_{k=1}^{d} f_{jk} a^{k-1}$, $(a \le j \le d)$.

The matrix F has the form

$$\begin{pmatrix} 0 & a_d & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & a_d \\ -a_0 & -a_1 & \dots & \dots & a_{d-1} \end{pmatrix}.$$

Let σ be the shift of $(\mathbb{T}^d)^{\mathbb{Z}}$ and define K to be the connected component of the identity of the closed σ-invariant subgroup $\left\{ y = (y_n) \in (\mathbb{T}^d)^{\mathbb{Z}} \mid a_d y_{n+1} = F y_n \right\}$. In the terminology of [7], K is the generalized solenoid group of type (d, F, a_d). By Lemma 17 and Theorem 19 of [7], there is an isomorphism (of topological groups) $\psi : \hat{\Lambda}_a \to K$ such that $(\sigma|K) \circ \psi = \psi \circ \alpha$. It follows also that α is an expansive automorphism if and only if F has no eigenvalues of absolute value a_d, or equivalently $a_d^{-1} F$ has no eigenvalues of absolute value 1 ([7]).

PROPOSITION 2. *The characteristic polynomial of $a_d^{-1} F$ (viewed as a linear map of $\mathbb{R}^d$) is f.*

Proof. Let $c \in \mathbb{Q}[x]$ be the characteristic polynomial of $a_d^{-1} F$. By definition c is monic and has degree d. If g is the characteristic polynomial of F we have $g \in \mathbb{Z}[x]$ and $g(F) = 0$. Thus $g(a_d \sigma|K) = 0$ and since $g \in \mathbb{Z}[x]$, $g(a_d \alpha) = 0$ on $\hat{\Lambda}_a$. This occurs if and only if $g(a_d a) = 0$. Since $c(x) = a_d^{-d} g(a_d x)$, $c(a) = 0$ and f must divide c. Since f is monic and degree d it follows that $f = c$. ∎

Thus α is expansive if and only if f (or equivalently ℓ) has no roots of absolute value 1. Since the irreducible polynomial of a root of unity (a cyclotomic polynomial) has no positive real roots (except for $g(x) = x - 1$) it follows that if $a \neq 1$, ℓ has no root of unity as a zero.

Since $\Lambda_a \subseteq \mathbb{Q}^d$ the automorphism N_a of Λ_a defined by $\lambda \to a\lambda$ ($\lambda \in \Lambda_a$) extends to an element (also denoted N_a) of GL(d,$\mathbb{Q}$). Since $g(a) = 0$ if and only if $g(N_a) = = 0$ ($g \in \mathbb{Z}[x]$) it follows that f is the characteristic polynomial of N_a (or one can check that the transpose of the matrix of N_a is $a_d^{-1}F$). A result of Yuzvinskii [8] shows that the (topological) entropy $h(\alpha)$ of the automorphism α of $\hat{\Lambda}_a$ is equal to $\log s + \sum_{|r_i|>1} \log|r_i|$ where $\{r_i \mid i = 1, \ldots, d\}$ are the roots of f (eigenvalues of N_a) and s is the least common multiple of the denominators of the coefficients of f. Since $f = a_d^{-1}\ell$ and the coefficients of ℓ have 1 as a common factor it follows that $s = |a_d|$. Thus $h(\alpha) = \log|a_d| + \sum_{|r_i|>1} \log|r_i|$ where $\{r_i \mid i = 1, \ldots, d\}$ are the roots of ℓ. In the case that a is transcedental, α is the shift on $\times_{\mathbb{Z}} \mathbb{T}$ and it is well known that $h(\alpha) = \infty$ ([11]). Note that if $a = 1$ then $h(\alpha) = 0$. Conversely suppose $h(\alpha) = 0$, so a must be (real) algebraic. If $\ell \in \mathbb{Z}[X]$ is irreducible in $\mathbb{Q}[X]$ with $\ell(a) = 0$ it follows that ℓ is monic and all roots of ℓ lie in the closed unit disk. However $g(x) = x^d \ell(x^{-1}) \in \mathbb{Z}[X]$ (with $d =$ degree of ℓ) is irreducible in $\mathbb{Q}[X]$ with $g(a^{-1}) = 0$. Since $h(\alpha^{-1}) = 0$ where α^{-1} is the dual of multiplication by a^{-1} on $\Lambda_{a^{-1}} = \Lambda_a$ it follows that g is monic and all roots of g lie in the closed unit disk. The roots of ℓ therefore lie in $\mathbb{T}$. Since $a \in \mathbb{R}^+$ we have $a = 1$.

§4

For $A = C(X) \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts by a homeomorphism γ of a compact Hausdorff space X we have noted that the number of points in X having orbits (under γ) of length n is recoverable from $\hat{A}_n$ and is thus an isomorphism invariant of A. In particular $i(A)_n$, the cardinality of the set of points in X fixed by γ^n is information contained in $\{\hat{A}_d \mid d \in \mathbb{N}, d|n\}$. The sequence $i(A)$ is an isomorphism invariant of A.

The first named author is thankful to D. Lind for a comment which led to a substantial simplification of our original proof of following result.

PROPOSITION 3. *Let $a \in \mathbb{R}^+$, $a \neq 1$ be algebraic. Then*

$$i(B_a)_n = \left| \prod_{j=1}^{n} \ell\left(\exp(2\pi i j n^{-1})\right) \right|$$

where $\ell \in \mathbb{Z}[x]$ is irreducible and $\ell(a) = 0$.

Proof. By definition $i(B_a)_n$ is the cardinality of the group $\{x \in \hat{\Lambda}_a \mid \alpha^n x = x\} = \{x \in \hat{\Lambda}_a \mid x((a^n - 1)(\lambda)) = 1,\ (\lambda \in \Lambda_a)\}$. This is the cardinality of the dual group $(\Lambda_a/(a^n - 1)\Lambda_a)\hat{}$ ([3]). It will turn out that $\Lambda_a/(a^n - 1)\Lambda_a$ is a finite group, so it has $i(B_a)_n$ elements. If $g(x) = x^n - 1$ then x is invertible in the ring $\mathbb{Z}[x]/g\mathbb{Z}[x] = R$ so $\Lambda_a/\overline{g}\Lambda_a = \mathbb{Z}[a, a^{-1}]/\overline{g}\mathbb{Z}[a, a^{-1}] = R/\overline{\ell}R$ where $\overline{g}$, $\overline{\ell}$ is the class of g, ℓ in Λ_a, R respectively. As an additive group R is just $\mathbb{Z}^n$. Since ℓ and g have no common factors, $\overline{\ell}R$ is a rank n free submodule of R. The cardinality of $R/\overline{\ell}R$ is $|\det M_\ell|$ where M_ℓ is the linear map of R defined by $r \to \overline{\ell}r$. If M_x denotes the map of R defined by $r \to \overline{x}r$ then $M_\ell = \ell(M_x)$. Since M_x is the cyclic shift on $\mathbb{Z}^n$ it has spectrum $\{\sigma_j \mid 1 \leqslant j \leqslant n\}$ with $\sigma_j = \exp(2\pi i j n^{-1})$. The polynomial spectral mapping theorem implies that M_ℓ has spectrum $\{\ell(\sigma_j) \mid j \in \mathbb{N}\}$ and the theorem follows. ■

Under the hypothesis of the theorem it follows that $0 < i(B_a)_n < \infty$.

An alternante expression for $i(B_a)_n$ is available. Let $\{r_k \mid k = 1, \ldots, d\}$ be the d roots of ℓ in $\mathbb{C}$. Then $\ell(x) = a_d \prod_{k=1}^{d} (x - r_k)$ and

$$i(B_a)_n = \prod_{j=1}^{n} |\ell(\sigma_j)| = \prod_{j=1}^{n} \left| a_d \left(\prod_{k=1}^{d} (\sigma_j - r_k) \right) \right| = |a_d|^n \prod_{k=1}^{d} \prod_{j=1}^{n} |\sigma_j - r_k| =$$

$$= |a_d|^n \prod_{k=1}^{d} \left| \prod_{j=1}^{n} (1 - r_k \sigma_j) \right| = |a_d|^n \prod_{k=1}^{d} |(1 - r_k^n)|$$

since $x^n - b^n = \prod_{j=1}^{n} (x - b\sigma_j)$.

COROLLARY 1. *With the hypothesis of the previous theorem,*

$$i(B_a)_n = |a_d|^n \prod_{k=1}^{d} |1 - r_k^n|$$

where $\ell(x) = \sum_{j=0}^{d} a_j x^j$ *has roots* $\{r_k \mid k = 1, \ldots, d\}$.

THEOREM 2. *Let $a \in \mathbb{R}^+$. The entropy $h(\alpha)$ of the dynamical system $(\hat{\Lambda}_a, \alpha)$ is an isomorphism invariant of the C^*-algebra $B_a = C(\hat{\Lambda}_a) \rtimes_\alpha \mathbb{Z}$.*

Proof. For a algebraic in $\mathbb{R}^+$, $a \neq 1$, $\lim_{n \to \infty} n^{-1} \log i(B_a)_n$ is an isomorphism invariant of C^*-algebra B_a. This is equal to

$$\lim_{n \to \infty} n^{-1} \log \left(|a_d|^n \prod_{k=1}^{d} |1 - r_k^n| \right) = \lim_{n \to \infty} \left[\log |a_d| + \sum_{k=1}^{d} n^{-1} \log |1 - r_k^n| \right] =$$

$$= \log |a_d| + \sum_{|r_k|>1} \log |r_k|$$

which by Yuzvinskii's result $=h(\alpha)$.

For a transcendental all the algebras B_a are isomorphic. The dynamical system $(\hat{\Lambda}_a, \alpha)$ with a transcedental are exactly those with infinite entropy. Also for each n $(\hat{B}_a)_n$ is nonempty and is not a finite union of tori for these a.

The algebra $B_1 \cong C(\mathbb{T} \times \mathbb{T})$ is the only abelian algebra in $\{B_a \mid a \in \mathbb{R}^+\}$. Also the dynamical system $(\hat{\Lambda}_a, \alpha)$ has zero entropy if and only if $a = 1$ and $(\hat{B}_1)_n = \emptyset$ for $n > 1$. ■

The proof of the theorem yields a little more, namely each of the three assertions $h(\alpha) = \infty$, a is transcendental, $(\hat{B}_a)_n$ is not a finite union of tori for each n, are equivalent. Also each of the three statements $h(\alpha) = 0$, $a = 1$, $(\hat{B}_a)_n = \emptyset$ for $n > 1$, are equivalent.

PROPOSITION 4. *The subgroup* $F = \{x \in \hat{\Lambda}_a \mid \alpha^n(x) = x$ *for some* $n \in \mathbb{N}\}$ *of periodic points is dense in* $\hat{\Lambda}_a$.

Proof. This is clear for $a = 1$ or a transcendental. Assume a is algebraic and let $\ell \in \mathbb{Z}[x]$ be irreducible with $\ell(a) = 0$. The group F is dense in $\hat{\Lambda}_a$ if and only if the subgroup $G = \{\lambda \in \Lambda_a \mid \langle F, \lambda\rangle = 1\}$ of Λ_a is zero. Since $\alpha^n(F) = F$ it follows that $a^n G = G$ $(n \in \mathbb{Z})$ and G is an ideal in the ring Λ_a. If we assume G is nonzero there is $g \in \mathbb{Z}[x]$ with $g(a) = \lambda$ nonzero in G. Since ℓ is also irreducible in $\mathbb{Q}[x]$, g and ℓ have no common factor in the principal ring $\mathbb{Q}[x]$ and there are $p, q \in \mathbb{Z}[x]$ and a nonzero $r \in \mathbb{N}$ with $p\ell + qg = r$. Thus $r\Lambda_a \subseteq G$ and F may be identified with a subgroup of the dual group $(\Lambda_a/r\Lambda_a)^\wedge$. In particular each element of F is a torsion element and has order dividing r.

If $m \in \mathbb{N}$ is prime to the leading and constant coefficient of ℓ then $\Lambda_a/m\Lambda_a$ is a finite group isomorphic to $(\mathbb{Z}/m\mathbb{Z})^d$. The dual group $(\Lambda_a/m\Lambda_a)^\wedge$ may be identified with the subgroup $F_m = \{x \in \hat{\Lambda}_a \mid \langle x, m\Lambda_a\rangle = 1\}$ of $\hat{\Lambda}_a$. This is a finite α-invariant group so $F_m \subseteq F$. Since there are elements of F_m with torsion order m this contradicts the conclusion that $m|r$. Thus G must be zero. ■

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Addendum. Proposition 3 in a more general setting of n commuting automorphisms but with an additional assumption on expansiveness is essentially contained in a preprint of D. Lind, K. Schmidt and T. Ward entitled *Mahler measure and entropy for commuting automorphisms of compact groups.* We received this after our work was completed.

REFERENCES

1. ARTIN M.; MAZUR, B., On periodic points, *Ann. Math.*, **81**(1965), 82–99.
2. BRENKEN, B., Representations and automorphisms of the irrational rotation algebra, *Pacific J. Math.*, **111**(1984), 257–282.
3. HEWITT, E.; ROSS, K. A., *Abstract harmonic analysis*, Volume 1, Springer-Verlag, Berlin, 1963.
4. KADISON, R. V.; RINGROSE, J., *Fundamentals of the theory of operator algebras*, Volume II, Academic Press, 1986.
5. KAWAMURA, S.; TOMIYAMA, J.; WATATANI, Y, Finite dimensional irreducible representations of C^*-algebras associated with topological dynamical systems, *Math. Scand.*, **56**(1985), 241–248.
6. KISHIMOTO, A., *Ideals of C^*-crossed products by locally compact abelian groups*, Proceedings of Symposia in Pure Mathematics, Volume **38**(1982), Part I, AMS, Providence, Rhode Island.
7. LAWTON, W., The structure of compact connected groups which admit an expansive automorphism, *Springer Lecture Notes in Math.*, **318**(1973), pp. 182–196.
8. LIND, D.; WARD, T., Automorphisms of solenoids and p-adic entropy, *Ergodic Th. Dynamical System*, **8**(1988), 411–419.
9. PEDERSEN, G., *C^*-algebras and their automorphism groups*, Academic Press, 1979.
10. POON, Y. T., A K-theoretic invariant for dynamical systems, *Trans. Amer. Math. Soc.*, **311**(1989), 515–533.
11. WALTERS, P., *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

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Note added in proof. Professor J. Tomiyama has informed us (1992) that he has a general framework which includes the crossed product from our Theorem 1, and also yields the faithfulness; — this is in his recent book (World Scientific Publ. 1987), and in a set of his lecture notes from the Seoul Nath. University (1992).