J. OPERATOR THEORY
25(1991), 347–366

INVARIANT SUBSPACES FOR MULTIVARIATE BISHOP-TYPE OPERATORS

GORDON W. MACDONALD

In [3], the author showed that many Bishop-type operators have nontrivial hyperinvariant subspaces. A Bishop-type operator is a weighted translation operator of the form $M_\varphi U_\alpha$ where $\varphi \in L^\infty[0,1)$ and α is an irrational number, defined on $f \in L^2[0,1)$ by

$$(M_\varphi U_\alpha f)(x) = \varphi(x) f(x+\alpha)$$

where addition is modulo 1.

In particular, it is shown in [3] that if φ is the restriction to $[0,1)$ of a function which is analytic in some open neighbourhood of $[0,1]$ then $M_\varphi U_\alpha$ has a nontrivial hyperinvariant subspace for almost all α. This entended a result of Davie [2], who showed $M_x U_\alpha$ has a nontrivial hyperinvariant subspace for almost all α.

In this paper, we are interested in the following multivariate generalization. The measure space is $([0,1)^n, \mathrm{d}\mu)$, where $n \in \mathbb{N}$ and $\mathrm{d}\mu$ is the product measure $\mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n$. Consider the point transformation $\tau_{\vec{\alpha}}: [0,1)^n \to [0,1)^n$ defined by

$$\tau_{\vec{\alpha}}(x_1, x_2, \ldots, x_n) = (\{x_1+\alpha_1\}, \{x_2+\alpha_2\}, \ldots, \{x_n+\alpha_n\})$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. This will be a measurable, measure-preserving transformation with measurable inverse. Let $U_{\vec{\alpha}}$ denote the unitary operator induced by $\tau_{\vec{\alpha}}$. Then for $\varphi \in L^\infty[0,1)^n$, $M_\varphi U_{\vec{\alpha}}$ defined by

$$(M_\varphi U_{\vec{\alpha}} f)(\vec{x}) = \varphi(\vec{x}) f(\tau_{\vec{\alpha}}(\vec{x})) \quad \text{for all } f \in L^2[0,1)^n$$

is a weighted translation operator which we shall refer to as a multivariate Bishop-type operator. (Here, $\vec{x}$ denotes a point in $[0,1)^n$.)

We shall try to emulate the methods of [3], to obtain nontrivial hyperinvariant subspaces for $M_\varphi U_{\vec{\alpha}} \in \mathcal{B}(L^2[0,1)^n)$. In the one variable case, $\tau(x) = \{x+\alpha\}$ was ergodic if and only of α was irrational. In the multivariate case, the analogous result is that the translation $\tau_{\vec{\alpha}}$ is ergodic if and only if $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Z}$.

In fact, $\tau_{\vec{\alpha}}$ will be uniquely ergodic when it is ergodic. We shall only consider the ergodic case. In the non-ergodic case there are many invariant subspaces.

The existence of nontrivial invariant or hyperinvariant subspaces for $M_\varphi U_{\vec{\alpha}}$ is easily established in some special cases. If φ is equal to zero on a set of positive measure, the $M_\varphi U_{\vec{\alpha}}$ has a nontrivial kernel which provides a hyperinvariant subspace. Thus, we shall assume that $\mu(\varphi^{-1}(0)) = 0$. We also obtain an invariant subspace in one other special case.

THEOREM 1. *If $\tau_{\vec{\alpha}}$ is ergodic and*

$$\varphi(x_1, x_2, \ldots, x_n) = \psi(x_{j_1}, x_{j_2}, \ldots, x_{j_k})\theta(x_{j_{k+1}}, x_{j_{k+2}}, \ldots, x_{j_n})$$

where $M_\psi U_{(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})} \in \mathcal{B}(L^2[0,1)^k)$ has a nontrivial invariant subspace, then $M_\varphi U_{\vec{\alpha}}$ has a nontrivial invariant subspace.

Proof. If $\mathcal{M}$ is a nontrivial invariant subspace for $M_\psi U_{(\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_k})}$, then the set of all $f \in L^2[0,1)^n$ such that f is independent of $(x_{j_{k+1}}, \ldots, x_{j_n})$ and f (considered as a function of $(x_{j_1}, \ldots, x_{j_k})$ is in $\mathcal{M}$ will be a nontrivial invariant subspace for $M_\varphi U_{\vec{\alpha}}$. ∎

Hence, if φ is independent of x_j for some j or more generally φ is of the form $\varphi(x_1, x_2, \ldots, x_n) = \psi(x_j)\theta(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and $M_\psi U_{\alpha_j}$ has a nontrivial invariant subspace then $M_\varphi U_{\vec{\alpha}}$ has a nontrivial invariant subspace.

As in [3], the following two theorems are the bases for showing the existence of nontrivial hyperinvariant subspaces.

THEOREM 2. (Wermer [5]). *If $T \in \mathcal{B}(H)$ is invertible and satisfies*

$$\sum_{n=-\infty}^{\infty} \frac{\log \|T^n\|}{1+n^2} < \infty,$$

and if $\sigma(T)$ is not a singleton, then T has a nontrivial hyperinvariant subspace.

In [4], Parrott shows that the spectrum of a weighted translation operator with ergodic translation is circularly symmetric about the origin and hence is never a singleton unless the weighted translation is quasinilpotent.

THEOREM 3. (Atzmon [1]). *Let E be a Banach space and $T \in \mathcal{B}(E)$. Suppose there exist sequences of vectors $\{x_n\}_{n\in\mathbb{Z}} \in E$ and $\{y_n\}_{n\in\mathbb{Z}} \in E^*$ with the following properties*

i) $Tx_n = x_{n+1}, T^*y_n = y_{n+1}$ *for all* $n \in \mathbb{Z}$ ($x_0 \neq 0$ *and* $y_0 \neq 0$);

ii) *There exist* $\{\rho_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}^+$ *such that* $\|x_n\| \leqslant k\rho_n$ *and* $\|y_n\| \leqslant k_1\rho_n$ *for some constants* k, $k_1 \in \mathbb{R}^+$ *and* $\rho_0 = 1$, $\rho_n \geqslant 1$, $\rho_{n+m} \leqslant \rho_n\rho_m$ *and*

$$\sum_{n=-\infty}^{\infty} \frac{\log \rho_n}{1+n^2} < \infty;$$

iii) *For*

$$G_x(z) = \begin{cases} \sum_{n=1}^{\infty} x_{-n} z^{n-1} & \text{if } |z| < 1 \\ -\sum_{n=-\infty}^{0} x_{-n} z^{n-1} & \text{if } |z| > 1 \end{cases}$$

$$G_y(z) = \begin{cases} \sum_{n=1}^{\infty} y_{-n} z^{n-1} & \text{if } |z| < 1 \\ -\sum_{n=-\infty}^{0} y_{-n} z^{n-1} & \text{if } |z| > 1 \end{cases}$$

two vector valued analytic functions defined for $\{z \in \mathbb{C} \mid |z| \neq 1\}$, *the union of the singularity sets of* G_x *and* G_y *is not a singleton.*

Then T *has a nontrivial hyperinvariant subspace.*

We shall show that for certain φ and $\vec{\alpha}$, $M_\varphi U_{\vec{\alpha}}$ satisfy the conditions of Wermer's theorem or Atzmon's theorem. Actually, to be more precise, a scalar multiple of $M_\varphi U_{\vec{\alpha}}$ does. By Proposition 1.3 of [3], when φ is "nice", the spectral radius of $M_\varphi U_{\vec{\alpha}}$ is $e^{\int \log|\varphi| d\mu}$. We normalize to an operator of spectral radius one by setting

$$T_\varphi = e^{-\int \log|\varphi| d\mu} M_\varphi U_{\vec{\alpha}}.$$

We first consider the case where $\log|\varphi|$ is the characteristic function of a very simple set.

NOTATION. Let $\vec{a} = (a_1, a_2, \ldots, a_n)$, $\vec{b} = (b_1, b_2, \ldots, b_n)$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$ denote points in $[0,1)^n$. We shall say $\vec{a} \leqslant$ (resp. $<$) $\vec{x}$ if $a_i \leqslant$ (resp. $<$) x_i for all $i = 1, \ldots, n$. Similarly, $\vec{a} + \vec{b}$ will denote coordinatewise sum, and $\{\vec{a}\}$ will denote coordinatewise modulo one, and any other such operations will also be coordinatewise operations. Then $\vec{I} = [\vec{a}, \vec{b})$ will denote the "hyperrectangle" which is

$$\{\vec{x} \in [0,1)^n \mid \vec{a} \leqslant \vec{x} < \vec{b}\}.$$

Let $\varphi(\vec{x}) = e^{\chi_{[\vec{a},\vec{b})}(\vec{x})}$. We begin with bounds on the norms of certain powers of T_φ. To simplify notation, let $\prod q_i$ denote $\prod_{i=1}^{n} q_i$.

LEMMA 4. *If $\{1,\alpha_1,\alpha_2,\dots,\alpha_n\}$ is linearly independent over $\mathbb{Z}$ and there exist $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$, integers such that $\gcd(q_i,q_j)=1$ for all $i\neq j$ and such that for $i=1,\dots,n$, $q_i\geqslant 2$, $\gcd(q_i,p_i)=1$ and*

$$|\alpha_i q_i - p_i|\leqslant \prod_{k=1}^n \frac{1}{q_k}$$

then for all $\vec{x}\in[0,1)^n$

$$\prod_{i=1}^n (q_i(b_i-a_i)-2)\leqslant \sum_{j=0}^{(\prod q_i)-1} \chi_{[\vec{a},\vec{b})}\{\vec{x}+j\vec{\alpha}\}\leqslant \prod_{i=1}^n (q_i(b_i-a_i)+2).$$

Proof. Fix $\vec{x}\in[0,1)^n$ and for $j=0,\dots,\left(\prod q_i\right)-1$ set

$$S_j=\left\{\{\vec{t}\}\;\middle|\;\vec{x}+\frac{j\vec{p}}{\vec{q}}\leqslant\vec{t}<\vec{x}+\frac{j\vec{p}+\vec{1}}{\vec{q}}\right\}$$

where $\vec{1}=(1,1,\dots,1)$.

Then

1) The S_j are disjoint for $j=0,\dots,\left(\prod q_i\right)-1$.

2) $\bigcup_{j=0}^{(\prod q_i)-1} S_j=[0,1)^n$.

3) Each S_j contains exactly one $\{\vec{x}+j\vec{\alpha}\}$.

Proof of 1): If $\{t\}$ is in $S_{j_1}\cap S_{j_2}$ then

$$\vec{x}+\frac{j_k\vec{p}}{\vec{q}}\leqslant\vec{t}+\vec{n}_k<\vec{x}+\frac{j_k\vec{p}+\vec{1}}{\vec{q}}\quad\text{for } k=1,2$$

where $\vec{n}_k=(n_k,n_k,\dots,n_k)$.

This implies that $\vec{x}+j_1\frac{\vec{p}}{\vec{q}}+\vec{n}_1=\vec{x}+j_2\frac{\vec{p}}{\vec{q}}+\vec{n}_2$, so if $j_1\neq j_2$ then $\frac{p_i}{q_i}=\frac{n_2-n_2}{j_1-j_2}$. Now, $\gcd(p_i,q_i)=1$ so q_i must divide j_1-j_2, and this must be true for all q_i. The q_i are relatively prime, so $\prod q_i$ must divide j_1-j_2. But $j_1-j_2<\prod q_i$, so we have reached a contradiction. Thus $j_1=j_2$ and the $\{S_j\}_{j=1}^{(\prod q_i)-1}$ are disjoint.

Proof of 2): The measure of each S_j is $\prod_{i=1}^n\frac{1}{q_i}$, so the result follows from 1).

Proof of 3): If $\frac{\vec{p}}{\vec{q}}<\vec{\alpha}$ then $\vec{x}+j\frac{\vec{p}}{\vec{q}}<\vec{x}+j\vec{\alpha}$ and for $j=0,\dots,\left(\prod q_i\right)-1$

$$\left|x_i+j\frac{p_i}{q_i}-(x_i+j\alpha_i)\right|\leqslant j\left|\frac{p_i}{q_i}-\alpha_i\right|\leqslant\frac{1}{q_i}$$

so $\vec{x}+j\vec{\alpha} \leqslant \vec{x}+\frac{j\vec{p}+\vec{1}}{\vec{q}}$ and thus $\{\vec{x}+j\vec{\alpha}\} \in S_j$. The cases where $\frac{\vec{p}}{\vec{q}} \not< \vec{\alpha}$ follow similarly.

Now $\sum_{j=0}^{(\prod q_i)-1} \chi_{[\vec{a},\vec{b})}\{\vec{x}+j\vec{\alpha}\}$ counts the number of $\{\vec{x}+j\vec{\alpha}\}$ in $[\vec{a},\vec{b})$ and, since each $\{\vec{x}+j\vec{\alpha}\}$ is contained in a single S_j, this sum is greater than or equal to the number of "hyperrectangles" S_j which intersect $[\vec{a},\vec{b})$.

If $(k_1, k_2, \ldots, k_n)$ in $\mathbb{N}^n$ is such that for $i = 1, 2, \ldots, n$, $\frac{k_i - 1}{q_i} \leqslant b_i - a_i < \frac{k_i}{q_i}$, then the number of "hyperrectangles" completely contained in $[\vec{a},\vec{b})$ is at least $\prod_{i=1}^{n}(k_i - 2)$ and the number of "hyperrectangles" which intersect $[\vec{a},\vec{b})$ is at most $\prod_{i=1}^{n}(k_i + 1)$. Since $q_i(b_i - a_i) < k_i \leqslant q_i(b_i - a_i) + 1$, the lemma is established. ∎

We shall not show that $M_\varphi U_{\vec{\alpha}}$ has a hyperinvariant subspace for the largest class of φ and $\vec{\alpha}$ possible, since to do so would introduce many complications and we shall see that the set of $\vec{\alpha}$ for which the analysis is possible is already a set of measure zero.

The following corollary will be used to simplify the calculations as well as the statement of the results. However, it is easily seen that some generality is sacrificed by using this crude inequality.

COROLLARY 5. *For $\vec{\alpha}$, $\vec{a}$, $\vec{b}$ and $\{p_i\}_{i=1}^n$, $\{q_i\}_{i=1}^n$ as above*

$$\left\| \sum_{j=0}^{(\prod q_i)-1} \chi_{[\vec{a},\vec{b})} \circ \tau_{\vec{\alpha}}^{j} - \prod_{i=1}^{n} q_i(b_i - a_i) \right\|_\infty \leqslant 2^{2n} \frac{\prod_{i=1}^{n} q_i}{\min_{i=1,\ldots,n} q_i}.$$

Proof. From Lemma 4

$$\left| \sum_{j=0}^{(\prod q_i)-1} \chi_{[\vec{a},\vec{b})}\{\vec{x}+j\vec{\alpha}\} - \prod_{i=1}^{n} q_i(b_i - a_i) \right| \leqslant$$

$$\leqslant \max\left\{ \prod_{i=1}^{n}(q_i(b_i - a_i) + 2) - \prod_{i=1}^{n} q_i(b_i - a_i) - \right.$$

$$\left. - \prod_{i=1}^{n}(q_i(b_i - a_i) - 2) + \prod_{i=1}^{n} q_i(b_i - a_i) \right\}.$$

Now

$$\prod_{i=1}^{n}(q_i(b_i - a_i) + 2) - \prod_{i=1}^{n} q_i(b_i - a_i) \leqslant$$

$$\leqslant \sum_{S\subset\{1,2,\ldots,n\},S\neq\emptyset} 2^{|S|}\prod_{i\notin S} q_i(b_i-a_i) \leqslant \sum_{S\subset\{1,2,\ldots,n\},S\neq\emptyset} 2^{|S|}\prod_{i\notin S} q_i \leqslant$$

$$\leqslant \left(\max_{S\subset\{1,2,\ldots,n\},S\neq\emptyset}\prod_{i\notin S} q_i\right)\sum_{S\subset\{1,2,\ldots,n\},S\neq\emptyset} 2^{|S|} \leqslant 2^{2n}\frac{\prod_{i=1}^{n} q_i}{\max_{S\subset\{1,2,\ldots,n\},S\neq\emptyset} q_i}.$$

We can obtain a similar bound for

$$-\prod_{i=1}^{n}(q_i(b_i-a_i)-2)+\prod_{i=1}^{n} q_i(b_i-a_i)$$

so the corollary is established. ∎

This generalizes quite easily to functions of the form $S=\sum_{k=1}^{l} r_k\lambda_{[\vec{a}_k,\vec{b}_k)}$. As in the single variable case, we shall refer to such functions as step functions. The following corollary is a direct consequence of Corollary 5 and the triangle inequality.

Corollary 6. *For $\vec{\alpha}$, $\{p_i\}_{i=1}^{n}$, and $\{q_i\}_{i=1}^{n}$ as above, and $S=\sum_{k=1}^{l} r_k\chi_{[\vec{a}_k,\vec{b}_k)}$ where $r_k\in\mathbb{R}$,*

$$\left\|\sum_{j=0}^{(\prod q_i)-1} S\circ\tau_{\vec{\alpha}}^{j}-\prod_{i=1}^{n} q_i\int S\,\mathrm{d}\mu\right\|_{\infty} \leqslant \left(\sum_{k=1}^{l}|r_k|\right)2^{2n}\frac{\prod_{i=1}^{n} q_i}{\min_{i=1,\ldots,n} q_i}.$$

So if S is a step function and $m=\left(\prod q_i\right)r+s$ where $s<\prod q_i$ then

$$\|T_{e^S}^{m}\| \leqslant \left\|T_{e^S}^{\prod q_i}\right\|^{r}\|T_{e^S}\|^{s} \leqslant e^{\left(\sum_{k=1}^{l}|r_k|\right)2^{2n}\frac{\prod q_i}{\min_{i=1,\ldots,n} q_i}r} e^{\|S\|_\infty s} \leqslant$$

$$\leqslant e^{\left(\sum_{k=1}^{l}|r_k|\right)2^{2n}\frac{m}{\min_{i=1,\ldots,n} q_i}} e^{\|S\|_\infty(\prod q_i)}.$$

Thus, we will need to be able to choose $\{q_i\}_{i=1}^{n}$ satisfying the above conditions and arbitrarily large in order to apply Wermer's theorem or Atzmon's theorem. This condition alone ensures that the set of $\vec{\alpha}$ for which our analysis can continue will be a set of measure zero when $n>1$.

However, we need an even more stringent condition on $\vec{\alpha}$.

DEFINITIONS. Consider the set of all $\vec{\alpha}$ such that there exist $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$ integers with $\{q_i\}_{i=1}^n$ arbitrarily large such that $\gcd(q_i, q_j) = 1$ for all $i \neq j$ and such that for all $i = 1, \dots, n$, $\gcd(q_i, p_i) = 1$ and

$$|\alpha_i q_i - p_i| \leqslant \prod_{k=1}^{n} \frac{1}{q_k}.$$

Then for $\kappa > 0$, $c < 1$ let $\mathcal{A}_{\kappa,c}$ be the set of all $\vec{\alpha}$ as above such that there exist positive constants K_1 and K_2 so that for all $m \in \mathbb{N}$, there exist $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$ as above with $K_1 m^\kappa \leqslant q_i$ for all $i = 1, \dots, n$ and $\prod q_i \leqslant K_2 m^c$.

Again, it is not at all obvious that $\mathcal{A}_{\kappa,c}$ is nonempty. After the statement and proofs of the main theorems, we shall construct some explicit $\vec{\alpha}$ in $\mathcal{A}_{\kappa,c}$.

We shall first look at the cases where $M_\varphi U_{\vec{\alpha}}$ is invertible. The following definitions are multivariate versions of similar definitions from [3].

DEFINITIONS. For $M \in \mathbb{R}$ define

$$\mathcal{S}_M^{(n)} = \left\{ S = \sum_{k=1}^{l} r_k \chi_{\vec{I}_k} \;\middle|\; r_k \in \mathbb{R},\ \vec{I}_k \text{ hyperrectangles and } \sum_{k=1}^{l} |r_k| \leqslant M \right\}.$$

Then set $\mathcal{L}^{(n)}$ to be all functions $f \in L^\infty[0,1)^n$ such that f is real valued and there exists $\gamma > 0$ and a constant K such that

$$\inf\left\{ \|f - S\|_\infty \mid S \in \mathcal{S}_M^{(n)} \right\} \leqslant K \frac{1}{M^\gamma}.$$

We shall show that $\mathcal{L}^{(n)}$ contains many well known functions, but first we shall proceed to one of the main theorems.

THEOREM 7. *If $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$ for some $\kappa > 0$ and $c < 1$ and $\log|\varphi| \in \mathcal{L}^{(n)}$, then $M_\varphi U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace.*

Proof. We shall apply Theorem 2. First, we fix a few constants. Since $\log|\varphi| \in \mathcal{L}^{(n)}$ there exists a constant K and $\gamma > 0$ such that $\inf\{\|\log|\varphi| - S\|_\infty | S \in \mathcal{S}_M^{(n)}\} \leqslant K \frac{1}{M^\gamma}$.

1) Fix such a K and γ.

2) Fix κ and c so that $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$.

3) Fix δ so that $0 < \delta < \kappa$.

4) Fix ρ so that

$$1 - \delta\gamma < \rho < 1$$

$$1 - \kappa + \delta < \rho < 1$$

$$c < \rho < 1.$$

For $m > 0$

$$\|T_\varphi^m\| = \left\| e^{-m\int \log|\varphi| d\mu} M_{(\varphi)(\varphi\circ\tau_{\vec\alpha})\cdots(\varphi\circ\tau_{\vec\alpha}^{m-1})} \right\| =$$

$$= \left\| e^{-m\int \log|\varphi| d\mu} (\varphi)(\varphi\circ\tau_{\vec\alpha})\cdots(\varphi\circ\tau_{\vec\alpha}^{m-1}) \right\|_\infty =$$

$$= \left\| e^{-m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec\alpha}^j|} \right\|_\infty \leqslant$$

$$\leqslant e^{\left\| -m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec\alpha}^j| \right\|_\infty}.$$

Similarly, for negative powers

$$\|T_\varphi^{-m}\| = \left\| e^{\int \log|\varphi| d\mu} M_{((\varphi\circ\tau_{\vec\alpha}^{-1})(\varphi\circ\tau_{\vec\alpha}^{-2})\ldots(\varphi\circ\tau_{\vec\alpha}^{-m}))^{-1}} \right\| =$$

$$= \left\| e^{m\int \log|\varphi| d\mu} \left((\varphi\circ\tau_{\vec\alpha}^{-1})(\varphi\circ\tau_{\vec\alpha}^{-2})\ldots(\varphi\circ\tau_{\vec\alpha}^{-m})\right)^{-1} \right\|_\infty =$$

$$= \left\| e^{\int \log|\varphi| d\mu - \sum_{j=1}^{m} \log|\varphi\circ\tau_{\vec\alpha}^{-j}|} \right\|_\infty \leqslant$$

$$\leqslant e^{\left\| m\int \log|\varphi| d\mu - \sum_{j=1}^{m} \log|\varphi\circ\tau_{\vec\alpha}^{-j}| \right\|_\infty} =$$

$$= e^{\left\| -m\int \log|\varphi| d\mu + \sum_{j=1}^{m} \log|\varphi\circ\tau_{\vec\alpha}^{-j}| \right\|_\infty}$$

and making the subsitution $\vec x \to \tau_{\vec\alpha}^m(\vec x)$ we obtain that

$$\|T_\varphi^{-m}\| \leqslant e^{\left\| -m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec\alpha}^j| \right\|_\infty}.$$

So we just need to bound

$$\left\| \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec\alpha}^j| - m\int \log|\varphi| d\mu \right\|_\infty .$$

For S a step function,

$$\left\| \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec\alpha}^j| - m\int \log|\varphi| d\mu \right\|_\infty = \left\| \sum_{j=0}^{m-1} \left(\log|\varphi\circ\tau_{\vec\alpha}^j| - S\circ\tau_{\vec\alpha}^j \right) + \right.$$

$$+m\int(S-\log|\varphi|)\mathrm{d}\mu+\sum_{j=0}^{m-1}S\circ\tau_{\vec\alpha}^{j}-m\int S\mathrm{d}\mu\Bigg\|_{\infty}\leqslant\sum_{j=0}^{m-1}\left\|\log|\varphi\circ\tau_{\vec\alpha}^{j}|-S\circ\tau_{\vec\alpha}^{j}\right\|_{\infty}+$$

$$+m\left|\int(S-\log|\varphi|)\mathrm{d}\mu\right|+\left\|\sum_{j=0}^{m-1}S\circ\tau_{\vec\alpha}^{j}-m\int S\mathrm{d}\mu\right\|_{\infty}\leqslant$$

$$\leqslant 2m\|\log|\varphi|-S\|_{\infty}+\left\|\sum_{j=0}^{m-1}S\circ\tau_{\vec\alpha}^{j}-m\int S\mathrm{d}\mu\right\|_{\infty}.$$

Now $\log|\varphi|$ is in $\mathcal{L}^{(n)}$ so we can choose $S=\sum_{k=1}^{l}r_k\chi_{\vec I_k}$ such that $\sum_{k=1}^{l}|r_k|\leqslant m^{\delta}$ and $\|\log|\varphi|-S\|_{\infty}\leqslant K\frac{1}{m^{\delta\gamma}}$. Hence,

$$2m\|\log|\varphi|-S\|_{\infty}\leqslant 2mK\frac{1}{m^{\delta\gamma}}\leqslant Km^{1-\delta\gamma}\leqslant$$

$$\leqslant Km^{\rho}\qquad\text{by the choice of }\rho.$$

If we choose $\{p_i\}_{i=1}^{n}$ and $\{q_i\}_{i=1}^{n}$ as in the definition of $\mathcal{A}_{\kappa,c}$, and write $m=\left(\prod q_i\right)r+s$ where $s<\prod q_i$ then $m\geqslant\left(\prod q_i\right)r$ so

$$\left\|\sum_{j=0}^{m-1}S\circ\tau_{\vec\alpha}^{j}-m\int S\mathrm{d}\mu\right\|_{\infty}\leqslant r\left\|\sum_{j=0}^{(\prod q_i)-1}S\circ\tau_{\vec\alpha}^{j}-\prod q_i\int S\mathrm{d}\mu\right\|_{\infty}+2s\|S\|_{\infty}\leqslant$$

$$\leqslant 2^{2n}\left(\sum_{k=1}^{l}|r_k|\right)\frac{\prod q_i}{\min\limits_{i=1,\ldots,n}q_i}r+2s\|S\|_{\infty}\leqslant\qquad\text{by Corollary 6}$$

$$\leqslant 2^{2n}m^{\delta}\frac{m}{\min\limits_{i=1,\ldots,n}q_i}+4\prod q_i\left\|\log|\varphi|\right\|_{\infty}\leqslant 2^{2n}K_1m^{\delta}m^{1-k}+4K_2\left\|\log|\varphi|\right\|_{\infty}m^{c}\leqslant$$

$$\leqslant 2^{2n}K_1m^{\rho}+4K_2\left\|\log|\varphi|\right\|_{\infty}m^{\rho}\leqslant\qquad\text{by the choice of }\rho$$

$$\leqslant K_3m^{\rho}\qquad\text{for some constant }K_3.$$

Thus, there exists a constant $K_4>0$ such that

$$\left\|\sum_{j=0}^{m-1}\log|\varphi\circ\tau_{\vec\alpha}^{j}|-m\int\log|\varphi|\mathrm{d}\mu\right\|_{\infty}\leqslant K_4m^{\rho}\quad\text{for all }m>0.$$

Therefore,

$$\|T_\varphi^m\| \leqslant e^{K_4 m^\rho} \quad \text{for all } m \in \mathbb{Z}$$

so the result follows by Theorem 2. ∎

We now consider the case where $M_\varphi U_{\vec{\alpha}}$ is not invertible. We shall construct functions f and g such that $T_\varphi^m f$ and $T_\varphi^{*m} g$ satisfy the proper bounds.

The following technical lemmas are needed. The proofs are identical to the one variable case and can be found in [3].

LEMMA 8. *If $g \in L^p([0,1)^n, \mathbb{R})$ for $p > 1$ and $\delta > 0$ such that $p \cdot \delta > 1$, there exists a constant K independent of t such that*

i) $\mu\{\vec{x} \in [0,1)^n \mid g(\tau_{\vec{\alpha}}^m) < -tm^\delta \text{ for some } m \in \mathbb{N}\} \leqslant \frac{K}{t^p}$

and

ii) $\mu\{\vec{x} \in [0,1)^n \mid g(\tau_{\vec{\alpha}}^{-m}) > tm^\delta \text{ for some } m \in \mathbb{N}\} \leqslant \frac{K}{t^p}$

hold for all $t > 0$.

LEMMA 9. *If $g \in L^p([0,1)^n, \mathbb{R})$, $p > 1$ and $\delta > 0$ then*

$$\int_{\{\vec{x} \in [0,1)^n \,\mid\, g(\vec{x}) < -tm^\delta\}} |g| \mathrm{d}\mu \leqslant \left(\frac{\|g\|_p^p}{t^{p-1}}\right) \frac{1}{m^{\delta(p-1)}}$$

and

$$\int_{\{\vec{x} \in [0,1)^n \,\mid\, g(\vec{x}) > tm^\delta\}} |g| \mathrm{d}\mu \leqslant \left(\frac{\|g\|_p^p}{t^{p-1}}\right) \frac{1}{m^{\delta(p-1)}}.$$

DEFINITION. For $\theta > 0$, set $\mathcal{M}_\theta^{(n)}$ to be the set of all measurable real valued functions f on $[0,1)^n$ such that there exists positive constants γ, c_f and K_f depending only on f such that

$$\inf\left\{\|(-N) \vee f \wedge N - S\|_\infty \mid S \in \mathcal{S}_M^{(n)}\right\} \leqslant K_f \frac{1}{M^\gamma}$$

whenever $M > c_f N^\theta$.

We are now ready to state our theorem for the noninvertible case.

THEOREM 10. *If $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$ and $\log|\varphi| \in L^p \cap \mathcal{M}_\theta^{(n)}$ for some p, κ, c, θ, satisfying*

$$p > \max\left\{\frac{1}{1-c}, \frac{\theta}{\kappa}\right\}$$

then $M_\varphi U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace.

Note that since $c < 1$, the conditions of Theorem 10 imply that $p > 1$. This will be needed in the proof.

Proof. To apply Theorem 3, we must show three things.

1) That there exists $f \neq 0$ such that $\|T_\varphi^n\|$ satisfies the bounds in Theorem 3 (and of course $T^n f \in L^2$ for all $n \in \mathbb{Z}$).

2) That there exists $g \neq 0$ such that $\|T_\varphi^{*n} f\|$ satisfies the bounds in Theorem 3 (and $T^{*n} g \in L^2$ for all $n \in \mathbb{Z}$).

3) That the union of the two singularity sets is not a singleton.

We shall take

$$f = \prod_{n=1}^{\infty} \chi_{\{x \mid |\varphi(\tau_{\bar\alpha}^{-n} x)| < a_n\}} \prod_{m=1}^{\infty} \chi_{\{x \mid |\varphi(\tau_{\bar\alpha}^{m} x)| > \frac{1}{a_{m+1}}\}}$$

and

$$g = \prod_{n=1}^{\infty} \chi_{\{x \mid |\varphi(\tau_{\bar\alpha}^{n-1} x)| < a_n\}} \prod_{m=1}^{\infty} \chi_{\{x \mid |\varphi(\tau_{\bar\alpha}^{-m-1} x)| > \frac{1}{a_{m+1}}\}}$$

where $a_n = e^{tn^\delta}$, t and δ chosen as follows.

We shall only show the bound for $\|T_\varphi^m f\|$, and only positive powers of m. The bound for negative powers and for $\|T_\varphi^{*m} g\|$ follow similarly.

First we must fix a few constants.

1) Fix $\delta > 0$ such that $\frac{1}{p} < \delta < \min\{1 - c, \frac{\kappa}{\theta}\}$.

2) For δ as in 1), by Lemma 8 we can fix $t > 0$ large enough that $f \neq 0$.

3) Fix δ_1 such that $\delta\theta < \delta_1 < \kappa$.

4) Fix γ as in the definition of $\mathcal{M}_\theta^{(n)}$.

5) Finally fix ρ such that

$$1 - k + \delta_1 < \rho < 1$$

$$1 - \delta_1 \gamma < \rho < 1$$

$$1 - \delta(p-1) < \rho < 1$$

$$\delta + c < \rho < 1.$$

Now we must bound $\|T^m f\|$.

Fix $m > m_0$ (yet to be specified); then

$$\|T_\varphi^m f\| = \left\| e^{-m \int \log|\varphi| d\mu} e^{\sum_{i=0}^{m-1} \log|\varphi \circ \tau_{\bar\alpha}^{j}|} f \circ \tau_{\bar\alpha}^{m} \right\|_2 \leqslant L_m \|f\|_2,$$

where

$$L_m = \text{ess sup}\left\{ e^{-m\int \log|\varphi| d\mu} e^{\sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec{\alpha}}^j|} \Big|_{\text{support } f\circ\tau_{\vec{\alpha}}^m} \right\}.$$

Now $x \in \text{support}\, f \circ \tau_{\vec{\alpha}}^m$ if and only if $\tau_{\vec{\alpha}}^m(x) \in \text{support}\, f$. By the definition of f, if $\tau_{\vec{\alpha}}^m(x) \in \text{support}\, f$ then $|\varphi(\tau_{\vec{\alpha}}^{-k}(\tau_{\vec{\alpha}}^m(x)))| < a_k$ for $k = 1, \ldots, m$. Our sequence $\{a_m\}_{m\in\mathbb{N}}$ is increasing, so setting $m - k = j$ we get that

$$\left|\varphi(\tau_{\vec{\alpha}}^j(x))\right| < a_j < a_m \quad \text{for } j = 0, 1, \ldots, m-1.$$

Since $a_m = e^{tm^\delta}$ (δ, t as chosen above), taking logs yields

$$\log|\varphi \circ \tau_{\vec{\alpha}}^j| \leqslant tm^\delta \text{ on support } f \circ \tau_{\vec{\alpha}}^m \quad \text{for } j = 0, 1, \ldots, m-1.$$

Hence

$$L_m \leqslant \text{ess sup}\left\{ e^{-m\int \log|\varphi| d\mu} e^{\sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec{\alpha}}^j|\wedge tm^\delta} \Big|_{\text{support } f\circ\tau_{\vec{\alpha}}^m} \right\} \leqslant$$

$$\leqslant \text{ess sup}\left\{ e^{-m\int \log|\varphi| d\mu} e^{\sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec{\alpha}}^j|\wedge tm^\delta} \right\} \leqslant$$

$$\leqslant e^{\text{ess sup}\left\{-m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} \log|\varphi\circ\tau_{\vec{\alpha}}^j|\wedge tm^\delta\right\}} \leqslant$$

$$\leqslant e^{\text{ess sup}\left\{-m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} -tm^\delta \vee \log|\varphi\circ\tau_{\vec{\alpha}}^j|\wedge tm^\delta\right\}} \leqslant$$

$$\leqslant e^{\left\|-m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} -tm^\delta \vee \log|\varphi\circ\tau_{\vec{\alpha}}^j|\wedge tm^\delta\right\|_\infty}.$$

So, we just need to bound $\left\|-m\int \log|\varphi| d\mu + \sum_{j=0}^{m-1} -tm^\delta \vee \log|\varphi\circ\tau_{\vec{\alpha}}^j| \wedge tm^\delta\right\|_\infty$.

We have $\log|\varphi| \in \mathcal{M}_\theta^{(n)}$, so there exists a step function $S = \sum_{k=1}^{l} r_k \chi_{\vec{I}_k}$ such that

$$\| -tm^\delta \vee \log|\varphi| \wedge tm^\delta - S\|_\infty \leqslant K_\varphi m^{-\delta_1\gamma}$$

and $\sum_{k=1}^{l} |r_k| \leqslant m^{\delta_1}$. (Here, $M = m^{\delta_1}$, and $N = (tm)^\delta$, so our conditions ensure that $M > c_{\log|\varphi|} N^\theta$, for $m > m_0$ large enough.)

Now,

$$\left\| -m\int \log|\varphi|\mathrm{d}\mu + \sum_{j=0}^{m-1} -tm^{\delta} \vee \log|\varphi\circ\tau_{\vec{\alpha}}^{j}| \wedge tm^{\delta} \right\|_{\infty} =$$

$$= \left\| \left(-m\int S\mathrm{d}\mu + \sum_{j=0}^{m-1} S\circ\tau_{\vec{\alpha}}^{j} \right) + \left(-m\int (\log|\varphi| - S)\mathrm{d}\mu \right) + \right.$$

$$\left. + \sum_{j=0}^{m-1} (-tm^{\delta} \vee \log|\varphi\circ\tau_{\vec{\alpha}}^{j}| \wedge tm^{\delta} - S\circ\tau_{\vec{\alpha}}^{j}) \right\|_{\infty} \leqslant$$

$$\leqslant \left\| -m\int S\mathrm{d}\mu + \sum_{j=0}^{m-1} S\circ\tau_{\vec{\alpha}}^{j} \right\|_{\infty} + m\left| \int (\log|\varphi| - S)\mathrm{d}\mu \right| +$$

$$+ \sum_{j=0}^{m-1} \| -tm^{\delta} \vee \log|\varphi\circ\tau_{\vec{\alpha}}^{j}| \wedge tm^{\delta} - S\circ\tau_{\vec{\alpha}}^{j} \|_{\infty}.$$

We shall bound each term separately.

First we bound

$$\left\| -m\int S\mathrm{d}\mu + \sum_{j=0}^{m-1} S\circ\tau_{\vec{\alpha}}^{j} \right\|_{\infty}.$$

Since $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$ we can find $\{p_i\}_{i=1}^{n}$ and $\{q_i\}_{i=1}^{n}$ such that $K_1 m^{\kappa} \leqslant q_i$, for all $i = 1,\dots,n$, $\prod q_i \leqslant K_2 m^c$ and $|q_i\alpha_i - p_i| \leqslant \frac{1}{\prod q_i}$. Hence, by Corollary 6,

$$\left\| \sum_{j=0}^{(\prod q_i)-1} S\circ\tau_{\vec{\alpha}}^{j} - \prod_{i=1}^{n} q_i \int S\mathrm{d}\mu \right\|_{\infty} \leqslant \left(\sum_{k=1}^{l} |r_k| \right) 2^{2n} \frac{\prod_{i=1}^{n} q_i}{\min_{i=1,\dots,n} q_i}.$$

So if $m = \prod q_i r + s$ with $s < \prod q_i$ then $m \geqslant \prod q_i r$,

$$\left\| -m\int S\mathrm{d}\mu + \sum_{j=0}^{m-1} S\circ\tau_{\vec{\alpha}}^{j} \right\|_{\infty} \leqslant r\left\| -\prod q_i \int S\mathrm{d}\mu + \sum_{j=0}^{\prod q_i - 1} S\circ\tau_{\vec{\alpha}}^{j} \right\|_{\infty} + 2s\|S\|_{\infty} \leqslant$$

$$\leqslant \left(\sum_{k=1}^{l} |r_k| \right) 2^{2n} \frac{\prod q_i r}{\min_{i=1,\dots,n} q_i} + 2s\|S\|_{\infty} \leqslant \left(\sum_{k=1}^{l} |r_k| \right) 2^{2n} \frac{m}{\min_{i=1,\dots,n} q_i} + 2\prod q_i \|S\|_{\infty} \leqslant$$

$$\leqslant \frac{2^{2n}}{K_1} m^{\delta_1} m m^{-\kappa} + 2K_2 m^c 2tm^{\delta} \leqslant \frac{2^{2n}}{K_1} m^{1+\delta_1-\kappa} + 4tK_2 m^{c+\delta} \leqslant$$

$$\leqslant \frac{2^{2n}}{K_1} m^\rho + 4tK_2 m^\rho \leqslant \qquad \text{by the choice of } \rho$$

$$\leqslant K_3 m^\rho \qquad \text{for some constant } K_4.$$

The second item is

$$m\left|\int(\log|\varphi| - S)\mathrm{d}\mu\right| \leqslant m\left|\int(\log|\varphi| - (-tm^\delta \vee \log|\varphi| \wedge tm^\delta))\mathrm{d}\mu\right| +$$

$$+m\left|\int(-tm^\delta \vee \log|\varphi| \wedge tm^\delta - S)\mathrm{d}\mu\right| \leqslant$$

$$\leqslant m \int\limits_{\{x \mid \log|\varphi(x)| > tm^\delta\} \cup \{x \mid \log|\varphi| < -tm^\delta\}} \left|\log|\varphi|\right| \mathrm{d}\mu +$$

$$+m\|-tm^\delta \vee \log|\varphi| \wedge tm^\delta - S\|_\infty.$$

By Lemma 9, the first part is bounded by $mK_4/m^{\delta(p-1)}$ and the second part is bounded by $mK_5m^{-\delta_1\gamma}$ by the choice of S above. (K_4, K_5 are two positive constants depending only on φ and t.)

Thus

$$m\left|\int(\log|\varphi| - S)\mathrm{d}\mu\right| \leqslant K_4 m^{1-\delta(p-1)} + K_5 m^{1-\delta_1\gamma} \leqslant$$

$$\leqslant K_4 m^\rho + K_5 m^\rho \leqslant \qquad \text{by the choice of } \rho$$

$$\leqslant K_6 m^\rho \qquad \text{for some constant } K_6.$$

The third term is

$$\sum_{j=0}^{m-1} \|-tm^\delta \vee \log|\varphi \circ \tau_{\bar\alpha}^j| \wedge tm^\delta - S \circ \tau_{\bar\alpha}^j\|_\infty \leqslant$$

$$\leqslant m\|-tm^\delta \vee \log|\varphi| \wedge tm^\delta - S\|_\infty \leqslant$$

$$\leqslant mK_5 m^{-\delta_1\gamma} \leqslant K_5 m^\rho,$$

by the choices of S and ρ made above.

Hence, there exists a constant $K > 0$ independent of m such that,

$$\|T_\varphi^m f\|_2 \leqslant \mathrm{e}^{Km^\rho}\|f\|_2 \quad \text{for all } m > m_0.$$

So there exists another constant $C > 0$ such that,

$$\|T_\varphi^m f\|_2 \leqslant C\mathrm{e}^{Km^\rho}\|f\|_2 \quad \text{for all } m > 0.$$

As mentioned previously, similar bounds for negative powers and the adjoint follow from an almost identical argument. Thus we have shown i) and ii) of Theorem 3 are satisfied with $\rho_m = e^{K|m|^{\rho}}$.

To show iii) of Theorem 3 is satisfied, first note that T_φ is unitarily equivalent to $e^{2\pi i\alpha_k}T_\varphi$ via the operator $M_{e^{2\pi i x_k}}$, and note that if we replace f by $e^{2\pi i x_k}f$ in the above argument we change nothing. If $\mathrm{Sing}(G_f)$ denotes the singularity set of $G_{\vec{x}}$ in Theorem 3 with $\vec{x}_m = T^m f$, then, as in proof of Theorem 2.6 of [3]

$$\mathrm{Sing}(G_{e^{2\pi i x_k} f}) = e^{2\pi i\alpha_k}\mathrm{Sing}(G_f).$$

Thus if $\mathrm{Sing}(G_f)\cup\mathrm{Sing}(G_g)$ is a singleton, then $\mathrm{Sing}(G_{fe^{2\pi i x_k}})\cup\ \mathrm{Sing}(G_g)$ is not a singleton. The proof of Theorem 10 is now complete. ■

Now we shall give a few concrete examples of $\vec{\alpha}$ and φ for which Theorems 7 and 10 are valid.

Let us first consider the question: "Which $\vec{\alpha}$ are $\mathcal{A}_{\kappa,c}$?"

Let $r_1,\dots,r_n$ be any relatively prime set of natural numbers, each r_i for $i = 1,\dots,n$ strictly greater than 2. For $i = 1,\dots,n$, let $\{a_{i,j}\}_{j=0}^{\infty}$ be a sequence such that each $a_{i,j}$ is either 1 or 2. Let t be a natural number yet to be determined. Then set

$$\alpha_i = \sum_{j=0}^{\infty} a_{i,j}\left(\frac{1}{r_i}\right)^{t^j}.$$

CLAIM. *If $\kappa > 0$ is small enough, and $c < 1$ is large enough, then $\vec{\alpha} = (\alpha_1,\dots,\alpha_n)$ is in $\mathcal{A}_{\kappa,c}$.*

Proof of Claim. For our approximants to α_i we shall take the k^{th} partial sum

$$\sum_{j=0}^{k} a_{i,j}\left(\frac{1}{r_i}\right)^{t^j} = \frac{p_{i,k}}{r_i^{t^k}} = \frac{p_{i,k}}{q_{i,k}}.$$

So $q_{i,k} = r_i^{t^k}$ and it is easily seen that each of the fractions above is in lowest from and that $\gcd(q_{i_1,k}, q_{i_2,k}) = 1$ for $i_1 \neq i_2$.

Also, for $i = 1,\dots,n$,

$$\left|\alpha_i - \frac{p_{i,k}}{q_{i,k}}\right| = \left|\alpha_i - \sum_{j=0}^{k} a_{i,j}\left(\frac{1}{r_i}\right)^{t^j}\right| = \sum_{j=k+1}^{\infty} a_{i,j}\left(\frac{1}{r_i}\right)^{t^j} \leqslant$$

$$\leqslant 2\left(\frac{1}{r_i}\right)^{t^{k+1}}\left\{\sum_{j=k+1}^{\infty}\left(\frac{1}{r_i}\right)^{t^j - t^{k+1}}\right\} \leqslant 4\left(\frac{1}{r_i}\right)^{t^k t} \leqslant 4\left(\frac{1}{q_{i,k}}\right)^{t}.$$

(We have that $\sum_{j=k+1}^{\infty}\left(\frac{1}{r_i}\right)^{t^j-t^{k+1}} \leqslant 2$ since this sum converges even faster than a geometric series.)

Thus

$$\left|\alpha_i - \frac{p_{i,k}}{q_{i,k}}\right| \leqslant \frac{4}{(\min_{j=1,\dots,n} q_{j,k})^t} \quad \text{for } i = 1, \dots, n.$$

So, if we choose t large enough that $(\min_{i=1,\dots,n} q_{j,k})^{t-1} \geqslant 4\prod q_{i,k}$, then

$$|\alpha_i q_{i,k} - p_{i,k}| \leqslant \frac{1}{\prod q_{i,k}} \quad \text{for } i = 1, \dots, n.$$

To show that $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$ for some κ, c it only remains to show that for any $m > 0$ we can take a partial sum up to k so that the $\{q_{i,k}\}_{i=1}^n$ chosen as above satisfy the appropriate bounds.

It is easy to see that given $\varepsilon > 0$, we can choose $\kappa_1, \dots, \kappa_n$ in the interval $(0, \varepsilon)$ such that

$$r_1^{\frac{1}{\kappa_1}} = r_2^{\frac{1}{\kappa_2}} = \cdots = r_n^{\frac{1}{\kappa_n}}.$$

Now consider a generic r and κ. Partition $[0, \infty)$ into intervals $J_0 = [0, r^{\frac{1}{\kappa}})$ and for $k = 1, 2, \dots$, $J_k = [r^{\frac{1}{\kappa}t^{k-1}}, r^{\frac{1}{\kappa}t^k})$. Then $[0, \infty) = \bigcup_{k=0}^{\infty} J_k$, so $m \in J_k$ for some k. This implies that $m < r^{\frac{1}{\kappa}t^k}$, so $m^\kappa < r^{t^k} = q_k$. Also, if m is greater that some constant $(r^{\frac{1}{\kappa}})$ then $r^{\frac{1}{\kappa}t^{k-1}} \leqslant m$, so $m^\kappa \geqslant r^{t^{k-1}} = r^{t^k t^{-1}} = \left(r_i^{t^k}\right)^{t^{-1}} = q_k^{t^{-1}}$. Hence, $q_k \leqslant m^{\kappa t}$.

Applying this to r_i and κ_i chosen above we obtain that

$$m^{\kappa_i} \leqslant q_{i,k} \leqslant m^{\kappa_i t} \quad \text{for } i = 1, \dots, n$$

for m sufficiently large. Hence,

$$m^{\min \kappa_i} \leqslant q_{i,k} \quad \text{and} \quad \prod q_{i,k} \leqslant m^{nt \max \kappa_i}.$$

If we choose ε sufficiently small (so that $\varepsilon nt < 1$) then there exist constants K_1 and K_2 and $\kappa > 0, c < 1$ such that

$$K_1 m^\kappa \leqslant q_{i,k} \quad \text{and} \quad \prod q_{i,k} \leqslant K_2 m^c$$

for some k and all $i = 1, \dots, n$. Thus $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$ and the claim is established. ∎

In fact, given $\vec{\alpha}$ as constructed above in $\mathcal{A}_{\kappa,c}$, if $\frac{\vec{p}}{\vec{r}} = \left(\frac{p_1}{r_1}, \frac{p_2}{r_2}, \dots, \frac{p_n}{r_n}\right)$ then $\vec{\alpha} + \frac{\vec{p}}{\vec{r}}$ is also in $\mathcal{A}_{\kappa,c}$ for all $\vec{p} \in \mathbb{Z}^n$. Hence, for κ small enough, and c large enough, $\mathcal{A}_{\kappa,c}$ is dense in $[0,1)^n$.

The above $\vec{\alpha}$ actually provide an uncountable subset of $\mathcal{A}_{\kappa,c}$. Obviously there are uncountably many sequences $\{a_{i,j}\}_{j=0}^{\infty}$ so to conclude that $\mathcal{A}_{\kappa,c}$ is uncountable, it only remains to show that for t large enough all the $\vec{\alpha}$ constructed above are unique.

We will use the elementary fact that given $r \in \mathbb{N}$

$$r^{t^k} \sum_{j=k+1}^{\infty} \left(\frac{1}{r}\right)^{t^j} \leqslant \frac{1}{\ln t \ln r}$$

for all $t \in \mathbb{N}$ and $k = 0, 1, \ldots$.

This allows us to fix t large enough ($t > e^{\frac{1}{\ln r}}$) that

$$r^{t^k} \sum_{j=k+1}^{\infty} \left(\frac{1}{r}\right)^{t^j} < 1$$

for all $k = 0, 1, \ldots$. Then if

$$\sum_{j=0}^{\infty} a_j \left(\frac{1}{r}\right)^{t^j} = \sum_{j=0}^{\infty} b_j \left(\frac{1}{r}\right)^{t^j}$$

we obtain that

$$a_0 - b_0 = r \sum_{j=1}^{\infty} (a_j - b_j) \left(\frac{1}{r}\right)^{t^j} \leqslant r \sum_{j=1}^{\infty} \left(\frac{1}{r}\right)^{t^j} < 1$$

so $a_0 = b_0$.

Suppose that $a_j = b_j$ for $j = 0, 1, \ldots, l-1$. Then

$$a_k - b_k = r^{t^k} \sum_{j=k+1}^{\infty} (a_j - b_j) \left(\frac{1}{r}\right)^{t^j} \leqslant r^{t^k} \sum_{j=k+1}^{\infty} \left(\frac{1}{r}\right)^{t^j} < 1$$

so $a_k = b_k$.

Hence, by induction, $a_j = b_j$ for all $j = 0, 1, \ldots$ and therefore for a given $r_1, r_2, \ldots, r_n$, each different set of n sequences generates a different number $\vec{\alpha}$ in $\mathcal{A}_{\kappa,c}$. Thus $\mathcal{A}_{\kappa,c}$ is an uncountable set.

Now we consider the question: "For which φ do Theorem 7 and 10 apply?", or more specifically: "Which φ are such that $\log|\varphi|$ is in $\mathcal{L}^{(n)}$?"

LEMMA 11. *If φ is the restriction to $[0,1)^n$ of a function which is analytic is some open neighbourhood of $\mathbb{C}^n$ containing $[0,1]^n$ and $\varphi(\vec{z}) \neq 0$ for all $\vec{z} \in [0,1]^n$, then* $\log|\varphi| \in \mathcal{L}^{(n)}$.

Proof. The conditions of the theorem imply that $\log|\varphi|$ is differentiable and continuous on $[0,1]^n$, so, by the Mean Value Theorem,

$$\log|\varphi(\vec{x}_1)| - \log|\varphi(\vec{x}_2)| = (\nabla \log|\varphi|)(\vec{x}) \cdot (\vec{x}_1 - \vec{x}_2)$$

where $\vec{x}$ is a point on the line segment joining $\vec{x}_1$ and $\vec{x}_2$ and $\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$. Thus

$$\left|\log|\varphi(\vec{x}_1)| - \log|\varphi(\vec{x}_2)|\right| \leqslant \max_{\vec{x}\in[0,1]^n} \|\nabla \log|\varphi|(\vec{x})\|_1 \max_{i=1,\ldots,n} |x_{1i} - x_{2i}| =$$

$$= K \max_{i=1,\ldots,n} |x_{1i} - x_{2i}|$$

for some constant $K > 0$.

Now, given $\varepsilon > 0$ partition $[0,1)^n$ into "hypersquares" of size $\frac{\varepsilon}{K}$. So we get $\left(\frac{K}{\varepsilon}\right)^n$ squares $\vec{I_j}$. Pick $\vec{x}_j \in \vec{I_j}$ and set $r_j = \log|\varphi(\vec{x}_j)|$. Then, if $S = \sum_j r_j \chi_{\vec{I}_j}$, from the above inequality we see that

$$\left\|S - \log|\varphi|\right\|_\infty \leqslant \varepsilon$$

and

$$\sum_j |r_j| \leqslant \left\|\log|\varphi|\right\|_\infty \{\text{the number of hypersquares}\} \leqslant$$

$$\leqslant \left\|\log|\varphi|\right\|_\infty \left(\frac{K}{\varepsilon}\right)^n .$$

Thus $\log|\varphi| \in \mathcal{L}^{(n)}$. ∎

COROLLARY 12. *For φ as in Lemma 11, there exist an uncountable dense set of $\vec{\alpha}$ such that $\tau_{\vec{\alpha}}$ is ergodic and $M_\varphi U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace.*

Proof. Theorem 7 and Lemma 11 imply that $M_\varphi U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace for all $\vec{\alpha} \in \mathcal{A}_{\kappa,c}$. The above construction shows that for $\kappa > 0$ small enough and $c < 1$ large enough, $\mathcal{A}_{\kappa,c}$ is an uncountable dense subset of $[0,1)^n$. ∎

Unfortunately, I can not say whether every function on $[0,1)^n$ which is the restriction of some function analytic in some open neighbourhood in $\mathbb{C}^n$ of $[0,1]^n$ is in $\mathcal{M}_\theta^{(n)}$ for some θ. However, in many special cases, this is the case.

Note that $M_{x_1x_2\cdots x_n}U_{\vec{\alpha}}$ has a nontrivial invariant subspace for almost all $\vec{\alpha}$ by the remarks following Theorem 1. Theorem 10 easily implies that $M_{x_1x_2\cdots x_n}U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace for a dense set of $\vec{\alpha}$. Rather than showing this, consider a similar class of operators, which are another possible multivariate generalization of Bishop operators. We shall show that operators in this class also have nontrivial hyperinvariant subspaces, at least for some $\vec{\alpha}$.

THEOREM 13. *The operator $M_{x_1+x_2+\cdots+x_n}U_{\vec{\alpha}}$ has a nontrivial hyperinvariant subspace for an uncountable dense set of $\vec{\alpha}$.*

Proof. To avoid technical difficulties, we shall only prove this for the case $n = 2$. The proof for arbitrary number of variables, although more intricate, has no significant variations from the following.

Note that $\log(x_1 + x_2) \in L^p[0,1)^2$ for all $p < \infty$. So to show this result, by Theorem 10 and the comments on the properties of $\mathcal{A}_{\kappa,c}$ given above, it is enough to show that $\log(x_1 + x_2) \in \mathcal{M}_\theta^{(2)}$ for some θ.

Fix $N \in \mathbb{N}$. Then the range of $(-N) \vee \log(x_1 + x_2)$ is $[-N, \log 2)$. Fix $\varepsilon \in (0,1)$ and divide this interval into subintervals J_i of length ε. So $J_i = [i\varepsilon, (i+1)\varepsilon)$ where $i = \left[\frac{-N}{\varepsilon}\right], \dots, \left[\frac{\log 2}{\varepsilon}\right]$. Then $(\log(x_1 + x_2))^{-1}(J_i)$ are strips at an angle of 45 degrees which get narrower as i gets smaller. We can approximate these strips by rectangles as follows.

Set

$$I_{i,j} = \left\{(x_1, x_2) | e^{i\varepsilon} \leqslant x_1 < e^{(i+1)\varepsilon}, \text{ and } e^{j\varepsilon} \leqslant x_2 < e^{(j+1)\varepsilon}\right\}$$

where $i, j = \left[\frac{-N}{\varepsilon}\right], \dots, \left[\frac{\log 2}{\varepsilon}\right]$. This divides $[0,1)^2$ into rectangles of various sizes.

CLAIM. *Each $I_{i,j}$ intersects at most two $(\log(x_1 + x_2))^{-1}(J_k)$.*

Proof of Claim. From the geometric picture, it is enough to check the southwest northeast corners of the rectangles $I_{i,j}$. These are $(e^{i\varepsilon}, e^{j\varepsilon})$ and $(e^{(i+1)\varepsilon}, e^{(j+1)\varepsilon})$ respectively.

If $(e^{i\varepsilon}, e^{j\varepsilon}) \in (\log(x_1 + x_2))^{-1}(J_k)$ then

$$e^{k\varepsilon} \leqslant e^{i\varepsilon} + e^{j\varepsilon} < e^{(k+1)\varepsilon}$$

so, multiplying by e^{ε}

$$e^{(k+1)\varepsilon} \leqslant e^{(i+1)\varepsilon} + e^{(j+1)\varepsilon} < e^{(k+2)\varepsilon}$$

and we obtain that

$$(e^{(i+1)\varepsilon}, e^{(j+1)\varepsilon}) \in (\log(x_1 + x_2))^{-1}(J_{k+1}).$$

Hence $I_{i,j} \subset (\log(x_1 + x_2))^{-1}(J_k) \cap (\log(x_1 + x_2))^{-1}(J_{k+1})$ and the claim is established.

Therefore, (and this shall be a rough approximation), if we assign $r_{i,j}$ the value of $\log(x_1 + x_2)$ for some point $(x_1, x_2) \in I_{i,j}$ and set

$$S = \sum_{i,j=[\frac{-N}{\varepsilon}]}^{[\frac{\log 2}{\varepsilon}]} r_{i,j}\chi_{I_{i,j}}$$

then $\|S - (-N) \vee \log(x_1 + x_2)\|_\infty \leqslant 2\varepsilon$ and

$$\sum_{i,j=[\frac{-N}{\varepsilon}]}^{[\frac{\log 2}{\varepsilon}]} |r_{i,j}| \leqslant (\max |r_{i,j}|) K \left(\frac{N}{\varepsilon}\right)^2 \leqslant K \frac{N^3}{\varepsilon^2}$$

for some constant $K > 0$. Hence, setting $M = \dfrac{KN^3}{\varepsilon}$, we obtain

$$\inf \left\{ \|(-N) \vee \log(x_1 + x_2) - S\|_\infty \;\middle|\; S \in \mathcal{S}_M^{(n)} \right\} \leqslant K \frac{1}{M}$$

whenever $M > c_{\log(x_1+x_2)} N^3$. Thus $\log(x_1 + x_2) \in \mathcal{M}_3^{(2)}$ and the result follows from Theorem 10. ■

These results are much less general than those for the single variable case for two main reason. First, the conditions required of $\vec{\alpha}$ are much more stringent, and restrict us to a set of measure zero. Second, in the multivariate case, the basic sets (the "hyperrectangles") are not as general as intervals in the single variable case. For example, for an analityc real valued function on some open interval containing $[0, 1]$, the inverse image of an interval is a finite union of intervals. In the multivariate case, no such result is true.

REFERENCES

1. ATZMON, A., On the existence of hyperinvariant subspaces, *J. Operator Theory*, **11**(1984), 3–40.
2. DAVIE, A. M., Invariant subspaces for Bishop's operators, *Bull. London Math. Soc.*, **6**(1974), 343–348.
3. MACDONALD, G. W., Invariant subspaces for Bishop-type operators, *J. Funct. Anal.*, to appear.
4. PARROTT, S. K., Weighted Translation Operators, Ph. D. Dissertation, University of Michigan, 1965.
5. WERMER, J., The existence of invariant subspaces, *Duke Math. J.*, **19**(1952), 615–622.

GORDON W. MACDONALD
Department of Mathematics, Evans Hall,
University of California at Berkeley,
Berkeley, CA 94720, U. S. A.

Received November 9, 1989.