J. OPERATOR THEORY
25(1991), 367–381

ON THE POLES OF THE SCATTERING MATRIX FOR ONE CONVEX AND ONE NONCONVEX BODIES

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1. INTRODUCTION

Let $Q \subset \mathbb{R}^3$ be an open bounded set with smooth boundary Γ. We set $\Omega = \mathbb{R}^3 \setminus Q$ and suppose that Ω is connected. Consider the following acoustic problem with Dirichlet boundary condition:

$$(1.1) \qquad \begin{cases} \Box u = 0 & \text{in} \quad \Omega \times (-\infty, +\infty) \\ u(x,t) = 0 & \text{on} \quad \Gamma \times (-\infty, +\infty) \\ u(x,0) = f_1(x) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = f_2(x) & \text{in} \quad \Omega \end{cases}$$

where $\Box = \partial^2/\partial t^2 - \sum\limits_{j=1}^{3} \partial^2/\partial x_j^2$.

Denote by $S(z)$ the scattering matrix for this problem. It is known [9] that $S(z)$ is an unitary operator from $L^2(\mathbb{S}^2)$ onto itself for all $z \in \mathbb{R}$ and extends to an operator valued function which is analytic in $\{\operatorname{Im} z \leqslant 0\}$ and meromorphic in the whole plane. The purpose of the present work is to find a relation between the geometry of Q and the location of the poles of the scattering matrix. When the obstacle Q is nontrapping (for definition of trapping and nontrapping obstacles see [12]) it was proved that there exist positive constants a and b such that the domain $\{z \in \mathbb{C}\ ; \operatorname{Im} z < a \cdot \log|z| + b\}$ is free from poles [11], [10], [14]. Lax and Phillips conjectured in [9] that for trapping obstacles $S(z)$ has a sequence of poles converging to the real axis. This hypothesis was rejected by Ikawa who constructed a counter example. Namely, he showed that for an obstacle which consists of two disjoint strictly convex bodies there exists $a > 0$ such that the domain $\{z \in \mathbb{C}\ ; 0 < \operatorname{Im} z < a\}$ is free from poles of $S(z)$ [4], [3].

In [8] Ikawa stated his modified hypothesis: when Q is trapping there exists $a > 0$ such that a slab domain $\{z \in \mathbb{C}\ ; 0 < \operatorname{Im} z < a\}$ contains an infinite number of poles. This was shown to be true by Ikawa and Gérard in [6], [4], [5] and [2] under the condition that the obstacle consists of two convex bodies. Moreover, these works show how the location of the poles is connected with the type of the Poincaré map for the periodic ray between the two bodies. In [8] Ikawa proved that his hypothesis is also true if Q consists of several strictly convex bodies and Neumann condition is posed on the boundary. Lately he proved the same thing for acoustic problem with Dirichlet boundary condition.

In this work we consider an obstacle which consists of two disjoint bodies. The first body is supposed to be strictly convex. For the second we assume that the principal curvatures of its boundary may be negative. So we have a new geometric situation which differs from that in the works of Ikawa and Gérard. However we pose conditions on the geometry of the second body in order to have only one hyperbolic periodic ray. Our main result is that the location of the poles for such obstacle is similar to the location found by Ikawa for the case of two strictly convex bodies [4], [6]. Namely, we get one string of poles (the closest to the real axis). As we follow the methods from [4], [7] and [6] we see that the main difficulties which we have to overcome are due to the negative principal curvature of the boundary near the periodic ray. Therefore, we have to investigate more carefully the dynamical system connected with (1.1).

The paper is organized as follows. In Section 2 we state precisely the main result and show how the study of the poles can be reduced to the analysis of the resolvent for the stationary problem, corresponding to (1.1), with boundary data concentrated on the convex boundary. In Section 3 we investigate the behaviour of the phase functions and broken rays. Section 4 is devoted to the construction of an asymptotic solution of (1.1) in such way that we avoid the consideration of solutions with caustics.

It is a pleasure to thank R. Denchev for his useful remarks and suggestions. I am also grateful to P. Stefanov who read the whole text.

optrm 2. MAIN RESULT AND REDUCTION OF THE PROBLEM

In what follows we suppose that $Q = Q_1 \cup Q_2$ and $Q_1 \cap Q_2 = \emptyset$, where Q_1 is strictly convex and Q_2 is nontrapping. We set $\Gamma_j = \partial Q_j$, $j = 1, 2$; $\Gamma = \Gamma_1 \cup \Gamma_2$ and $d = \text{dist}(Q_1; Q_2)$. We also pose:

$$K_i = \min_{x \in \Gamma_i, j=1,2} K_{ij}(x),$$

where $K_{ij}(x)$ are the principal curvatures of Γ_i at the point $x \in \Gamma_i$. Let h be a ray which starts from $x \in \Gamma_i$ in direction ξ_h and after one reflection from Γ_2 hits again Γ_1. Denote by H the set of all such rays, and by β_h the angle between ξ_h and the inner normal of Γ_2 at the reflection point y_h. Pose:

$$\beta = \max_{h \in H} \beta_h \quad \text{and} \quad l_0 = \max_{h \in H} |x_h - y_h|.$$

Suppose that the following conditions are satisfied:

(B1) $$\min_{l \leqslant l_0} \left(K_1 + \frac{K_2}{\cos \beta + 2lK_2} \right) > 0$$

(B2) *Consider a ray h which starts from $x \in \Gamma_1$ and tangents to Γ_2 at the point $y \in \Gamma_2$. We assume that $K_{2j}(y) > 0$, $j = 1, 2$ and $h \cap \Gamma = \{x\} \cup \{y\}$. We also assume $Q_1 \cap$ (convex hull of Q_2)$= \emptyset$.*

REMARK. Condition (B2) garantees that the rays starting from Γ_1 are not gliding to Γ_2.

Now we state the main result in this work.

THEOREM 2.1. *Let the obstacle Q satisfy* (B1) *and* (B2). *Then there exist constants $C_0, C_1 > 0$ such that:*

(i) *for any $\varepsilon > 0$ the region*

$$\{z\ ;\ \operatorname{Im} z \leqslant C_0 + C_1 - \varepsilon\} \setminus \bigcup_{j=-\infty}^{+\infty} \{z\ ;\ |z - z_j| \leqslant C(|j| + 1)^{-\frac{1}{2}}\}$$

contains only a finite number of poles of $S(z)$, where

$$z_j = \mathrm{i}C_0 + (j\pi)/d$$

and C is independent of ε.

(ii) *there exist infinitely many poles of $S(z)$ in:*

$$\bigcup_{j=-\infty}^{-\infty} \{z\ ;\ |z - z_j| \leqslant C(|j| + 1)^{-\frac{1}{2}}\}$$

with the same C.

The results of [9] show that Theorem 2.1 follows immediately from:

THEOREM 2.2. *Suppose that Q satisfies* (B1) *and* (B2). *Denote by $U(p)g$ the solution in $\bigcap_{m>0} H^m(\Omega)$ of the problem:*

$$\begin{cases} (p^2 - \Delta)u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \Gamma \end{cases} \tag{2.1}$$

for $\operatorname{Re} p > 0$ *and $g \in C^\infty(\Gamma)$. Then $U(p)$ is analytic in* $\operatorname{Re} p > 0$ *as $\mathcal{L}(C^\infty(\Gamma), C^\infty(\overline{\Omega}))$ valued function and prolonged analytically into:*

$$D_\varepsilon = \{p\ ;\ \operatorname{Re} p \geqslant -C_0 - C_1 + \varepsilon\} \setminus \bigcup_{j=-\infty}^{+\infty} \{p\ ;\ |p - \mathrm{i}z_j| \leqslant C(|j| + 1)^{-\frac{1}{2}}\}$$

for any $\varepsilon > 0$, where C is independent of ε. And for $p \in D_\varepsilon$ it holds:

$$\sum_{|\beta| \leqslant m} \sup_{x \in \Omega \cap \{|x|<R\}} |D_x^\beta (U(p)g)(x)| \leqslant C_{R,m,\varepsilon} \sum_{j=0}^{m+7} |p|^j \|g\|_{H^s(\Gamma)}$$

for any $R > 0$ and $s = m + 7 - j$.

Moreover $U(p)$ has an infinite number of poles in the region

$$\bigcup_{j=-\infty}^{+\infty} \{p \; ; \; |p - iz_j| \leqslant C(|j| + 1)^{-\frac{1}{2}}\}.$$

REMARKS. 1. The formulations of the theorems are exactly the same as in [4], but here we have a new geometric situation which causes a lot of difficulties in the proofs

2. The constants C_0 and C_1 depend only on the geometry of Q.

3. Following the methods of [6] one can obtain more precise results on the location of the poles (see Theorems 1 and 2 from [6]).

Now we shall make an easy reduction of problem (2.1) which will be very useful in the course of the proof of Theorem 2.2.

Denote by $U_2(p)g$ the solution in $\bigcap_{m>0} H^m$ of the problem

$$\begin{cases} (p^2 - \Delta)u = 0 & \text{in} \quad \mathbb{R}^3 \setminus Q_2 \\ u = g & \text{on} \quad \Gamma_2 \;\; \text{for} \;\; g \in C^\infty(\Gamma_2). \end{cases}$$

Set $g = (g_1, g_2)$ for $g \in C^\infty(\Gamma)$, where $g_i \in C^\infty(\Gamma_i)$, $i = 1, 2$ and let $U(p)g$ be the solution in $\bigcap_{m>0} H^m(\Omega)$ of (2.1). Then we have for $\operatorname{Re} p \geqslant 0$:

$$\begin{aligned} U(p)g &= U(p)(g_1, g_2) = U(p)(g_1, 0) + U(p)(0, g_2) = \\ &= \widetilde{U}(p)(g_1 - U_2(p)(g_2)|_{\Gamma_1}) + U_2(p)(g_2), \end{aligned} \tag{2.3}$$

where $\widetilde{U}(p) : C^\infty(\Gamma_1) \longrightarrow C^\infty(\Omega)$ is such that:

$$\widetilde{U}(p)g = U(p)(g, 0).$$

We know [14] that $U_2(p)$ can be prolonged analytically into:

$$\{p \; ; \; \operatorname{Re} p \geqslant -b, |p| \geqslant C_b\}$$

for any $b > 0$, where C_b is a constant depending on b. Hence using (2.3) we see that it is sufficient to prove Theorem 2.2 only for $\widetilde{U}(p)$ instead of $U(p)$ and in what follows we shall consider only $\widetilde{U}(p)$.

The advantage of the above argument is that now we are able to avoid consideration of gliding rays (see the Remark after condition (B2)). This is important for the proof of Theorem 2.2 because we have to construct a parametrix of the problem:

$$\begin{cases} \Box u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u = f & \text{on} \quad \Gamma \times \mathbb{R} \\ \operatorname{supp} u \subset \overline{\Omega} \times [0, \infty) \end{cases}$$

and on Γ_2 we have gliding rays and inflection points.

3. PROPERTIES OF PHASE FUNCTIONS AND BROKEN RAYS

Following Section 9 of [4] (also in [6]) one can see how the proof of Theorem 2.2 is reduced to the investigation of the asymptotic solution of the problem:

$$\begin{cases} \Box u = 0 & \text{in} \quad \Omega \times \mathbb{R} \\ u = e^{ik(\varphi(x,\beta',\eta)-t)}\omega(x,t) & \text{on} \quad \Gamma_1 \times \mathbb{R} \\ u = 0 & \text{on} \quad \Gamma_2 \times \mathbb{R} \\ \operatorname{supp} u \subset \overline{\Omega} \times [0, \infty) \end{cases} \tag{3.1}$$

where $\omega(x,t) \in C^\infty(\Gamma_1)$ and $\varphi(x, \beta', \eta)$ satisfies:

$$\begin{cases} |\nabla_x \varphi| = 1 \\ \varphi(y(\sigma), \beta', \eta) = \beta' \langle \sigma, \eta \rangle \\ \dfrac{\partial \varphi}{\partial n} > 0 \quad \text{on} \quad \Gamma_1 \end{cases}$$

Here $\eta \in \mathbb{S}^2$, $-2\beta'_0 \leqslant \beta' \leqslant 2\beta'_0$ for some small β'_0 and $y(\sigma)$ is a representation of Γ_1 near A_1, where $A_i \in \Gamma_i$, $i = 1, 2$ and $d = |A_1 - A_2|$.

REMARK. Lemma 3.8 from this section allows us to consider functions $\varphi(x, \beta', \eta)$ which are defined only in a small neighborhood of A_1 (see [3], Section 8).

As we build the asymptotic solution of (3.1) following the construction of geometric optics, the investigation of this solution is closely related with the behaviour of the broken rays and of the so called phase functions.

DEFINITION 3.1. A real valued smooth function $\varphi(x)$, defined in an open set $U \subset \mathbb{R}^3$ is called *phase function* if $|\nabla\varphi| = 1$ in U. A surface $B_\varphi(x) = \{y\, ;\, \varphi(x) = \varphi(y)\}$ is called the *wave front* of $\varphi(x)$ passing x.

Let $s_0 \in \Gamma$ and $\varphi^{(1)}$ and $\varphi^{(2)}$ be phase functions in $U \ni s_0$ such that:

$$\begin{cases} \varphi^{(1)} = \varphi^{(2)} & \text{on} \quad \Gamma \cap U \\ \dfrac{\partial \varphi^{(1)}}{\partial n} > 0, \quad \dfrac{\partial \varphi^{(2)}}{\partial n} < 0 & \text{on} \quad \Gamma \cap U, \end{cases}$$

where $n(x)$ is the unit outer normal at $x \in \Gamma$. In other words $\varphi^{(1)}$ and $\varphi^{(2)}$ are two different solutions of the equation $|\nabla\varphi| = 1$, $\varphi|_\Gamma = f(x)$ ($f(x) = \varphi^{(1)}(x) = \varphi^{(2)}(x)$ for $x \in \Gamma \cap U$).

Let $K_1^{(1)}$, $K_2^{(1)}$ and $K_1^{(2)}$, $K_2^{(2)}$ be the principal curvatures at s_0 of $B_{\varphi^{(1)}}(s_0)$ and $B_{\varphi^{(2)}}(s_0)$, respectively (we consider the principal curvatures with respect to $-\nabla\varphi(y)$, $y \in B_\varphi(s_0)$).

LEMMA 3.2 ([3]). *It holds that:*

$$\min_{j=1,2} K_j^{(2)} + 2\delta \min_{i=1,2} K_i(s_0) \leqslant K_{j'}^{(1)} \quad \text{for } j' = 1,2, \tag{3.2}$$

where $K_1(s_0)$, $K_2(s_0)$ are the principal curvatures of Γ at $s_0 \in \Gamma$, with respect to $n(s_0)$ and $\delta = 1$ for $s_0 \in \Gamma_1$ and $\delta = (\cos\beta)^{-1}$ for $s_0 \in \Gamma_2$. (β is defined before Condition (B1).*)*

Note that if $y = s_0 + l\nabla\varphi^{(1)}(s_0)$, then the principal curvatures of $B_{\varphi^{(1)}}(y)$ at the point y are given by the terms:

$$\frac{K_j^{(1)}}{1 + lK_j^{(1)}} \quad \text{for } j = 1,2 \tag{3.3}$$

(for example see [13]).

Let $\varphi_0(x)$ be a phase function defined in U ($U \cap \Gamma_1 \neq \emptyset$), such that $\dfrac{\partial\varphi_0}{\partial n} > 0$ for $x \in U \cap \Gamma_1$ and the principal curvatures of $B_{\varphi_0}(x)$ are non-negative at the points of $U \cap \Gamma_1$. Let $\varphi_1(x)$ satisfy:

$$\begin{cases} |\nabla\varphi_1| = 1 & \text{in } \omega_1 \\ \varphi_1 = \varphi_0 & \text{on } \omega_1 \cap \Gamma_2 \\ \dfrac{\partial\varphi_1}{\partial n} = -\dfrac{\partial\varphi_0}{\partial n} & \text{on } \omega_1 \cap \Gamma_2 \end{cases}$$

where ω_1 is the set of all rays reflected by Γ_2, which start from points $x \in U \cap \Gamma_1$ in direction $\xi_x = \nabla\varphi_0(x)$. Fix $\delta = (\cos\beta)^{-1}$.

LEMMA 3.3. *The phase function $\varphi_1(x)$ is well defined on Γ_1 and for $\varphi_1(x)$ we can build a phase function $\varphi_2(x)$ following the above construction. Moreover, the principal curvatures of B_{φ_2} at the points of Γ_2 are positive.*

Proof. From (3.3) we see that the principal curvatures of $B_{\varphi_0}(x)$ at $x \in \Gamma_2$ are non-negative. Then inequality (3.2) shows that the principal curvatures of B_{φ_1} at $x \in \Gamma_2$ are not less than $2\delta K_2$. Hence the principal curvatures of B_{φ_1} at the points of Γ_1 are not less than

$$\frac{2\delta K_2}{1 + 2\delta l K_2},$$

where l is such that $y = x + l\nabla\varphi_1(x)$, $x \in \Gamma_2$, $y \in \Gamma_1$. Condition (B1) shows that the above fraction is well defined. Then $\varphi_1(x)$ is well defined on Γ_1 [13]. So we can construct $\varphi_2(x)$ near Γ_1.

Inequality (3.2) gives the minimal principal curvature of B_{φ_2} at the points of Γ_1:

$$2K_1 + \frac{2\delta K_2}{1 + 2l\delta K_2}$$

and this is positive by (B1). Thus we have the assertion.

REMARK. The phase function $\varphi_1(x)$ may be no well defined in some points $x \in \Omega \setminus \Gamma$ because of the existence of caustics.

CORLLARY 3.4. *Following the procedure described before Lemma 3.3 we can construct a sequence of phase functions* $\{\varphi_j\}_{j=0}^{\infty}$, *which are defined near* Γ.

Now we prove a lemma which is crucial in order to apply the methods of [4] in our case.

LEMMA 3.5. *There exist* $N > 0$, $K > 0$, *depending only on the geometry of* Q, *such that for* $j \geqslant N$ *the principal curvatures of* $B_{\varphi_j}(x)$ *at* $x \in \Gamma_{\varepsilon(j)}$ *are greater than* K, *where* $\varepsilon(j) = 1$ *if* j *is even and* $\varepsilon(j) = 2$ *if* j *is odd.*

Proof. Denote by a'_j the minimal value of the principal curvatures of B_{φ_j} at the points of $\Gamma_{\varepsilon(j)}$. Using (3.2) and (3.3) we get:

$$a'_{2n} \geqslant a_{2n} = 2K_1 + \frac{a_{2n-1}}{1 + l_0 a_{2n-1}}$$

$$a'_{2n+1} \geqslant a_{2n+1} = 2\delta K_2 + \frac{a_{2n}}{1 + l_0 a_{2n}}$$

$$a'_0 \geqslant a_0 = 0; \quad a'_1 \geqslant a_1 = 2\delta K_2.$$

Consider the sequence $\{a_{2j+1}\}_{j=0}^{\infty}$. We have:

$$a_{2n+1} = 2\delta K_2 + \frac{a_{2n}}{1 + l_0 a_{2n}} = 2\delta K_2 + \frac{2K_1 + a_{2n-1}(1 + l_0 a_{2n-1})^{-1}}{1 + 2l_0 K_1 + l_0 a_{2n-1}(1 + l_0 a_{2n-1})^{-1}} =$$

$$= 2\delta K_2 + \frac{2K_1(1 + l_0 a_{2n-1}) + a_{2n-1}}{(1 + 2l_0 K_1)(1 + l_0 a_{2n-1}) + l_0 a_{2n-1}} =$$

$$= \frac{2\delta K_2(1 + 2l_0 K_1)(1 + l_0 a_{2n-1}) + 2\delta K_2 l_0 a_{2n-1} + 2K_1 + 2l_0 K_1 a_{2n-1} + a_{2n-1}}{1 + 2l_0 K_1 + l_0 a_{2n-1} + 2l_0^2 K_1 a_{2n-1} + l_0 a_{2n-1}}.$$

Hence we have $a_{2n+1} - a_{2n-1} = A/B$, where:

$$B = 1 + 2l_0 K_1 + l_0 a_{2n-1} + 2l_0^2 K_1 a_{2n-1} + l_0 a_{2n-1}$$

$$A = 2\delta K_2 + 4l_0 K_1 K_2 \delta + 2K_1 + 2\delta l_0 K_2 a_{2n-1} + 4\delta K_1 K_2 l_0^2 a_{2n-1} + 2\delta K_2 l_0 a_{2n-1} +$$

$$+2l_0K_1a_{2n-1} + a_{2n-1} - (1+2l_0K_1)a_{2n-1} - 2l_0a_{2n-1}^2 - 2l_0^2K_1a_{2n-1} =$$

$$= -a_{2n-1}^2 2l_0(1+l_0K_1) + 4l_0\delta K_2 a_{2n-1}(1+l_0K_1) + 2(\delta K_2 + 2l_0K_1K_2\delta + K_1) =$$

$$= 2l_0(1+l_0K_1)\left[-a_{2n-1}^2 + 2\delta K_2 a_{2n-1} + \frac{(1+2\delta K_2 l_0)}{(1+l_0K_1)l_0}\left(K_1 + \frac{K_2\delta}{1+2\delta l_0 K_2}\right)\right] =$$

$$= 2l_0(1+l_0K_1)(-a_{2n-1}^2 + 2\delta K_2 a_{2n-1} + R) = 2l_0(1+l_0K_1)S.$$

Here we denote:

$$R = (1+2K_2\delta l_0)[K_1 + K_2\delta(1+2\delta l_0K_2)^{-1}][l_0(1+l_0K_1)]^{-1}$$

$$S = -a_{2n-1}^2 + 2\delta K_2 a_{2n-1} + R.$$

We are interested in the sign of S (note that by (B1) $R > 0$). Let us solve the equation $S = 0$ with respect to a_{2n-1}. We get:

$$a_{1,2}^{(0)} = \delta K_2 \pm (\delta^2 K_2^2 + R)^{\frac{1}{2}}.$$

We have

$$B = 1 + 2l_0K_1 + 2l_0a_{2n-1} + 2l_0^2 a_{2n-1}K_1 \geqslant$$

$$\geqslant 1 + 2l_0K_1 + 4\delta l_0 K_2 + 4\delta l_0^2 K_1K_2 =$$

$$= (1+2\delta l_0K_2) + 2l_0(K_1 + 2\delta l_0K_1K_2 + \delta K_2)$$

and by (B1) this is positive.

We have two possibilities:

(i) $a_{2n_0-1} > a_2^{(0)} > 0$ for some n_0;

(ii) $a_{2n-1} \leqslant a_2^{(0)}$ for all n.

Consider the first case. We have $a_{2n_0} > 2K_1$ and then

$$a_{2n_0+1} > 2\delta K_2 + 2K_1(1+2l_0K_1) > \varepsilon_1 > 0.$$

Denote $K = \min(\varepsilon_1, a_2^{(0)}, 2K_1)$ and for $j > N = \frac{1}{2}n_0$ it holds that $a_j \geqslant K$. Thus the lemma is proved in the first case.

Let $a_{2n-1} \leqslant a_2^{(0)}$ for all n. Then, since $a_1^{(0)} < 2\delta K_2 \leqslant a_{2j+1}$ for all j, we have: $a_1^{(0)} < a_{2j+1} \leqslant a_2^{(0)}$ for all j.

Hence the sequence $\{a_{2j+1}\}_{j=0}^{\infty}$ is increasing and tends to $a_2^{(0)}$. Thus the lemma is proved.

Next, we introduce a terminology connected with the broken rays which satisfy the law of geometric optics. We follow [3], [4] and [7].

For $x \in \Gamma$ denote by $n(x)$ the unit outer normal of Γ at the point x. Set $\Sigma_x^+ = \{\xi \in \mathbb{S}^2; |\xi| = 1 \text{ and } n(x)\cdot\xi \geqslant 0\}$. Denote by $X(x,\xi)$ the broken ray, according to the law of geometric optics, which starts from $x \in \Gamma_1$ in direction $\xi \in \Sigma_x^+$. Let

$X_1(x,\xi), X_2(x,\xi), \dots$ be the points of reflection of $X(x,\xi)$. By $\#X(x,\xi)$ denote the number of all such points.

Lemma 3.6. *We can choose a sequence of neighborhoods:*

$$\begin{cases} U_{10}^{(i)} \subset U_{11}^{(i)} \subset \Gamma_1; \quad U_{20}^{(i)} \subset U_{21}^{(i)} \subset \Gamma_2 \\ A_1 \in U_{10}^{(i)}, A_2 \in U_{20}^{(i)} \quad \text{and for } j = 1,2 \text{ and } i \longrightarrow \infty \\ \max_{x \in U_{j1}^{(i)}} |x - A_j| \longrightarrow 0 \end{cases}$$

such that there exist positive constants $K^{(i)}$ satisfying the properties:

(i) *If $x \in \Gamma_1$, $\xi \in \Sigma_x^+$ and $X(x,\xi) \cap (U_{10}^{(i)} \cup U_{20}^{(i)}) = \emptyset$, then $\#X(x,\xi) \leqslant K^{(i)}$;*

(ii) *If $x \in U_{11}^{(i)} \cup U_{21}^{(i)}$ and $\xi \in \Sigma_x^+$ is such that $X_1(x,\xi) \in (U_{11}^{(i)} \setminus U_{10}^{(i)}) \cup (U_{21}^{(i)} \setminus U_{20}^{(i)})$ then $\#X(x,\xi) \leqslant K^{(i)}$.*

Proof. Denote by α the line passing through A_1 and A_2. Choose points O_1 and O_2 such that:

$$\begin{cases} O_1 \in \alpha, \quad |O_1 - A_1| = 1/K_1 \quad \text{and} \quad A_1 \text{ is between } O_1 \text{ and } A_2 \\ O_2 \in \alpha, \quad |O_2 - A_2| = 1/K_2 \quad \text{and} \quad A_1 \text{ is between } O_2 \text{ and } A_2 \end{cases}$$

Then by (B1) $O_1 \neq O_2$ and O_1 is between O_2 and A_1. We also choose the points:

$$\begin{cases} O_1', \; O_2' \in \alpha, \; O_1' \neq O_2' \\ O_i' \text{ is between } O_1 \text{ and } O_2 \text{ for } i = 1,2 \\ O_2' \text{ is between } O_2 \text{ and } O_1' \end{cases}$$

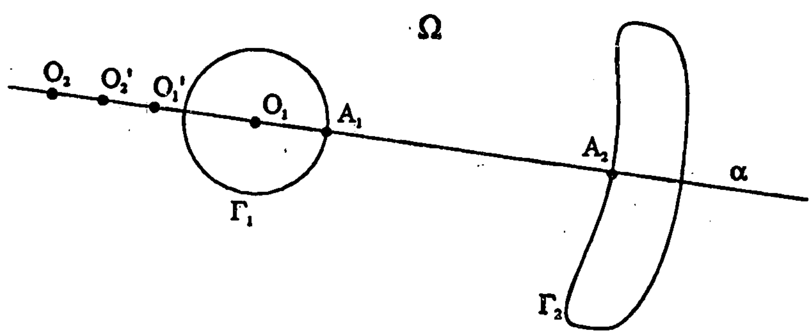

figure 1

(1) Let $x_1 \in \Gamma_1$ and consider the broken ray starting from x_1 and hitting Γ_2 at x_2. Let $V_1 = V_1(O_2', x_2)$ be the cone surface generated by rotation of $O_2'x_2$ (we mean the ray with origin O_2') around α. We say that $O_2'x_2$ generates V_1.

Assume $x_1 \in \overset{\circ}{V}_1$ ($\overset{\circ}{V}_1$ means the inside of V_1) and let

$$x_3 = X_3(x_1,\ (x_2 - x_1)/|x_2 - x_1|) \in \Gamma_1.$$

Consider the surface $V_2 = V_2(O_2, x_2)$. Then using (B2) we have:

$$\max_{x \in V_1 \cap \Gamma_1} |x'| \leqslant (1 - \delta_1) \min_{x \in V_2 \cap \Gamma_1} |x'|$$

for some $1 > \delta_1 > 0$, where $|x'| = \operatorname{dist}(x, \alpha)$. It is obvious that we can choose δ_1 =const if $|x_1''| > \varepsilon$ for some $\varepsilon > 0$ (δ_1 depends only on ε and on the geometry of Q). By taking the projection of x_1x_2, x_2x_3 and $n(x_2)$ onto the plane through α and x_2, and using (B1) and (B2) we get:

$$|x_1'| \leqslant (1 - \delta_1)|x_3'|.$$

Moreover, if $V_3 = V_3(O_1', x_3)$ then $x_2 \in \overset{\circ}{V}_3$.

The same argument shows that:

$$|x_2'| \leqslant (1 - \delta_2)|x_4'| \quad \text{for some } 1 > \delta_2 > 0.$$

Therefore, by induction we conclude that after s reflections we shall leave Γ, where s depends only on $|x_1'|$.

(2) Assume $x_1 \in \Gamma_1$ and $U \ni A_1$ is a neighborhood such that $x_1 \notin U$. Let $X(x_1, \xi) \cap U = \emptyset$ for some $\xi \in \Sigma_{x_1}^+$. Then we shall prove that $\# X(x_1, \xi) \leqslant M$ for some M depending only on $\max_{x \in U} |x'|$. Assume the contrary.

Consider the surface $V_1 = V_1(O_2', x_1)$. Then (1) shows that $x_2 \in \overset{\circ}{V}_1$. We also get $x_3 \in \overset{\circ}{V}_2$, where $V_2 = V_2(O_1', x_2)$. Following the above construction, we build $V_{2n-1} = V_{2n-1}(O_2', x_{2n-1})$ and we get $x_{2n} \in \overset{\circ}{V}_{2n-1}$. The choice of O_1' and O_2' leads to:

$$(V_{2n+1} \cup \overset{\circ}{V}_{2n+1}) \cap \Gamma_1 \longrightarrow A_1 \quad \text{for } n \to \infty.$$

Thus we get a contradiction and the statement of (2) holds.

(3) Finaly we describe the construction of the neighborhoods.

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of points on Γ_1 such that $x_j \to A_1$ and $|x_j'| > |x_{j+1}'|$. Denote by $\{V_j\}_{j=1}^{\infty}$ the sequence $V_j = V_j(O_1', x_j)$ for $j = 1, 2 \ldots$. We have for a suitable choice of x_1:

$$\begin{cases} \overset{\circ}{V}_{j+1} \subset \overset{\circ}{V}_j \\ \overset{\circ}{V}_j \cap \Gamma_1 \longrightarrow A_1 \quad \text{for } j \to \infty \\ \Gamma_2 \subset \overset{\circ}{V}_1 \quad \text{or} \quad \Gamma_1 \subset V_1 \end{cases}$$

Let $\mathbb{S}$ be the sphere with center O_1' and radius $|A_1 - O_1'|$. Denote $\gamma_j = V_j \cap \mathbb{S}$ and let V_j' be the cone surface with peak in O_2' and passing γ_j. Finaly we set for every j:

$$U_{10}^{(j)} = \overset{\circ}{V}_j \cap \Gamma_1; \quad U_{20}^{(j)} = \overset{\circ}{V}_j{}' \cap \Gamma_2$$

$$U_{11}^{(j)} = \overset{\circ}{V}_j{}' \cap \Gamma_1; \quad U_{21}^{(j)} = \overset{\circ}{V}_j \cap \Gamma_2$$

Using (1) and (2) we see that the above neighborhoods satisfy the requirements of the lemma.

REMARK. Note that using the proof of Lemma 3.6 one can prove the result of Section 2 of [6].

PROPOSITION 3.7. *Let $x, y \in (U_{11}^{(j)} \cup U_{21}^{(j)}) = F_1$ for some j and assume that $X_i(x, \nabla\varphi_N(x)) \in F_1$ and $X_i(y, \nabla\varphi_N(y)) \in F_1$ for $i = 1, \ldots, q$, where the positive integer N is the one of Lemma 3.5. Then there exists $j_0 > 0$ such that $|x - y| \leqslant C\alpha^{\frac{1}{2}q}$ for $j \geqslant j_0$ and $\alpha < 1$ (α is the one of Proposition 3.8 of [7]; for $\{\varphi_j\}_{j=0}^{\infty}$ see Corollary 3.4).*

Proof. Let $x, y \in U_{11}^{(j)}$, $x_1 = X_1(x, \nabla\varphi_N(x))$ and $y_1 = X_1(y, \nabla\varphi_N(y))$. Suppose that $\varphi_N(x_1) \leqslant \varphi_N(y_1)$ and denote by $x_1^{(r)}$ a point on the ray $h = \{x_1 + \tau\nabla\varphi_{N+1}(x_1);\ \tau \geqslant 0\}$ such that $\varphi_{N+1}(x_1^{(r)}) = \varphi_N(y_1)$.

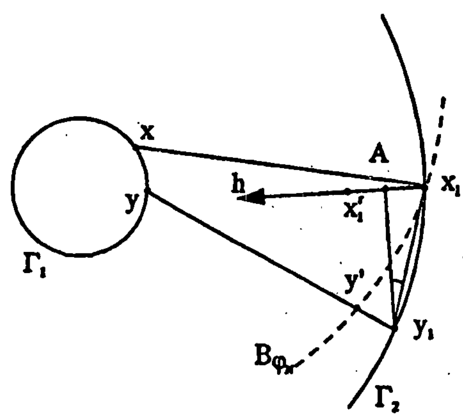

figure 2

We shall prove that $|x - y| \leqslant |y_1 - x_1^{(r)}|$.

Set $y' = B_{\varphi_N}(x_1) \cap yy_1$. Choose j_0 so large that if $j \geqslant j_0$ then the angle γ, between Ay_1 ($A \in h$ and $|A - y_1| = \text{dist}(h, y_1)$) and x_1y_1, is so small that $\alpha^{-1} = (\cos\gamma)(1 + dK_1) > 1$. (Recall that $U_{i1}^{(j)} \longrightarrow A_i$ for $j \to \infty$ (Lemma 3.6))

Using Lemma 3.6 of [7] we get:

$$|x_1 - y'| \geqslant (1 + dK_1)C_{K_1}|x - y|,$$

where C_{K_1} depends only on K_1. Hence:

$$C_{K_1}|x - y| \leqslant (1 + dK_1)^{-1}|x_1 - y'| \leqslant (1 + dK_1)^{-1}|y_1 - x_1| =$$

$$= [(\cos\gamma)(1 + dK_1)]^{-1}|A - y_1| = \alpha|A - y_1| \leqslant \alpha|x_1^{(r)} - y_1|.$$

Then following Proposition 3.8 of [7] we complete the proof.

COROLLARY 3.8. *Let $\varphi(x)$ be a phase function and suppose that the principal curvatures of $B_\varphi(x)$ at the points of Γ_1 in which φ is defined are greater than some $\varepsilon > 0$. Then if $x, y \in \Gamma_1$ and $X_q(x, \nabla\varphi(x))$, $X_q(y, \nabla\varphi(y)) \in \Gamma_1$ we have for large q:*

$$|x - y| \leqslant C\alpha^{\frac{1}{2}(q-N-2k-1)}, \tag{3.4}$$

where k and C are independent of q and α is fixed in Proposition 3.7.

Proof. We fix $j = j_0$ (see Propositin 3.7). Then using Lemma 3.6 we get $X_i(x, \nabla\varphi(x)), X_i(y, \nabla\varphi(y)) \in (U_{10}^{(j)} \cup U_{20}^{(j)})$ for large q and $i = K^{(j)}+1, \dots, q-K^{(j)}-1$. Hence (3.4) follows from Proposition 3.7 and from the inequality:

$$|x - y| \leqslant C|X_{N+K^{(j)}}(x, \nabla\varphi(x)) - X_{N+K^{(j)}}(y, \nabla\varphi(y))|.$$

Lemma 3.6 makes possible to prove (2.14) or [4]. Then following Section 3 of [4] and Section 3 of [7] we have:

PROPOSITION 3.9. *It holds that:*

$$|\nabla\varphi_q - \nabla\widetilde{\varphi}_q|_m(\Gamma_{\varepsilon(q)}) \leqslant C_m\alpha_m^{q-1}|\nabla\varphi - \nabla\widetilde{\varphi}|_m(\Gamma_1), \quad m = 1, 2, \dots; \alpha_m < 1,$$

where $\{\varphi_q\}_{q=1}^{\infty}$ and $\{\widetilde{\varphi}_q\}_{q=1}^{\infty}$ are built starting with phase functions $\varphi(x)$ and $\widetilde{\varphi}(x)$ (recall Corollary 3.4). The functions φ and $\widetilde{\varphi}$ are defined on Γ_1 and the principal curvatures of B_φ and $B_{\widetilde{\varphi}}$ are non-negative. (For definition of $|\cdot|_m(\Gamma_{\varepsilon(q)})$ see [4]).

Using Corollary 3.8, Proposition 3.9 and Lemma 3.6 we follow Section 5 of [7] in order to prove:

THEOREM 3.10 (convergence of phase functions). *Let $\varphi_0(x)$ be a phase function defined on Γ_1 such that the principal curvatures of B_{φ_0} are non-negative. Let $\{\varphi_j\}_{j=0}^{\infty}$ be the sequence of phase functions constructed in Corollary 3.4. There exist phase functions $\varphi_\infty(x)$ and $\widetilde{\varphi}_\infty(x)$ such that:*

(i) $|\varphi_{2p} - (\varphi_\infty + 2dp + d_0)|_m(\Gamma_1) \leqslant C_m\alpha^p$
$|\varphi_{2p+1} - (\widetilde{\varphi}_\infty + (2p+1)d + \widetilde{d}_0)|_m(\Gamma_2) \leqslant C_m\alpha^p$

where $d_0, \widetilde{d}_0$ depend only on φ_0.

(ii) *the principal curvatures of B_{φ_∞} ($B_{\widetilde{\varphi}_\infty}$) at $x \in \Gamma_1$ (Γ_2) are greater than $K > 0$ (K is defined in Lemma 3.5).*

THEOREM 3.11 (convergence of broken rays). *It holds that:*

$$|X_{-j}(\cdot, \nabla\varphi_{2q}) - X_{-j}(\cdot, \nabla\varphi_\infty)|_m(\Gamma_1) \leqslant C_m\alpha^{\frac{1}{2}q}$$

$$|X_{-2q+j}(\cdot,\nabla\varphi_{2q}) - X(A',\nabla\varphi_0)|_m(\Gamma_1) \leqslant C_m\alpha^{\frac{1}{2}q}$$

where $\alpha < 1$ *and* $A' \in \Gamma_1$ *depends only on* φ_0. *Analogous inequalities also hold for* $\widetilde{\varphi}_\infty(x)$.

($X_{-j}(x,\nabla\varphi_q)$ *is the point* y *from which a broken ray must start in direction* $\nabla\varphi_{q-j}(y)$ *in order to reach* x *after* j *reflections.)*

REMARK. Some of the above results and an outline of the proofs were announced in [1].

4. ASYMPTOTIC SOLUTION OF (3.1) AND PROOF OF THEOREM 2.2

We should like to use the methods of [4] in the proof of Theorem 2.2. The direct application leads to the fact that the asymptotic solution of (3.1) is defined only in a small neighborhood of Γ because the first N phase functions may have negative principal curvatures at the points of their wave front set. In other words the Lagrangian manifolds Λ_j which consist of the bicharacteristics generated by the equations $|\nabla\varphi_{2j-1}| = 1$, $j = 1,2,\ldots,\frac{1}{2}N$, have not proper projection on $\mathbb{R}^3$. It is possible to look for an asymptotic solution in terms of Fourier Integral Operators. But the fact that the region in which the phase functions are well defined includes Γ_1 (see Lemma 3.3) leads to a simpler solution.

The construction of an asymptotic solution of (3.1) consists of several steps [4], [6]:

(i) construction of the sequence of phase functions $\{\varphi_j\}_{j=0}^{\infty}$

(ii) introduction of the sums:

$$u_q(x,t,k) = e^{ik(\varphi_{2q}(x)-t)} \sum_{r=0}^{M} v_{r,q}(x,t)k^{-r}$$

$$\widetilde{u}_q(x,t,k) = e^{ik(\varphi_{2q+1}(x)-t)} \sum_{r=0}^{M} \widetilde{v}_{r,q}(x,t)k^{-r}$$

where $v_{r,q}$ and $\widetilde{v}_{r,q}$ are solutions of the corresponding transport equations and $M > 0$ is a fixed integer.

(iii) the asymptotic solution is given by:

$$u(x,t,k) = \sum_{q=0}^{\infty}(u_q(x,t,k) - \widetilde{u}_q(x,t,k)).$$

In our case the functions $\{u_q(x,t,k)\}_{q=0}^{\frac{1}{2}N}$ are well defined in a neighborhood U of Γ_2 and $\Gamma_1 \subset U$. Each of the functions $u_q,\widetilde{u}_q, q = 0,1,\ldots,\frac{1}{2}N$ is an approximation of a certain boundary value problem outside a nontrapping obstacle. We replace this functions with the exact solutions $u'_q(x,t)$, $\widetilde{u}'_q(x,t)$, $q = 0, 1, \ldots$. Then the asymptotic solution is given by:

$$u'(x,t,k) = \sum_{q=0}^{\frac{1}{2}N}(u'_q(x,t) - \widetilde{u}'_q(x,t)) + \sum_{q=\frac{1}{2}N+1}^{\infty}(u_q(x,t,k) - \widetilde{u}_q(x,t,k)).$$

The second sum we treat in the same way as in [4]. The last step is to observe that if we take the Laplace transform of the first sum we obtain $\sum_{q=0}^{\frac{1}{2}N}(u'_q(x,p) - -\widetilde{u}'_q(x,p))$, where

$$u'_q(x,p) = \int_0^{\infty} e^{-pt} u'_q(x,t)\mathrm{d}t$$

$$\widetilde{u}'_q(x,p) = \int_0^{\infty} e^{-pt} \widetilde{u}'_q(x,t)\mathrm{d}t.$$

We know that u'_q and $\widetilde{u}'_q$ extend to meromorphic functions in the whole plane, which are analytic in some regions:

$$\{p\,;\mathrm{Re}\,p > -b, |p| \geqslant C_b\} \quad \text{for all } \; b > 0.$$

Here C_b depends only on b.

Hence only the second sum is important for the poles of the resolvent $U(p)$ and using the results of Section 3 we follow [4] in order to prove Theorem 2.2.

REMARK. After proving Proposition 5.6 of [4] we can also follow [6] and prove Theorem 2 of [6] for the obstacle of this work.

Acknowledgements. This work is supported by the Bulgarian Commitee of Sciences and High Education, Contract No 39.

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Received November 14, 1989.