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UNIT BALL DENSITY AND THE OPERATOR EQUATION $AX=YB$

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1. PRELIMINARIES AND NOTATION

Every injective operator A with dense range, acting on a complex separable Hilbert space H, has a finite rank "approximate inverse" in the sense that there exists a sequence (R_n) of finite rank operators such that the sequences (AR_n) and (R_nA) converge strongly to the identity, with AR_n and R_nA in the unit ball of $\mathcal{B}(H)$ for every n. This result is possibly well known at the "folk" level. Below, a short proof (using the spectral theorem) is given and the result is partially extended to closed densely defined linear transformations. The present paper is mainly concerned with applications of the above result and its extension.

The first application leads to a characterization of the solution pairs (X, Y) of the operator equation $AX = YB$ where A, or B, is an injective operator with dense range. This equation, in fact the slightly less general equation $AX = YA$, arises naturally in the study of operators leaving a pair of complementary subspaces invariant. The characterization presented is fairly close to the obvious solutions $X = CB$, $Y = AC$ (C any element of $\mathcal{B}(H)$), which unfortunately does not cover all solutions. However, in the case where both A and B are injective with dense range, a simple argument shows that every finite rank solution is of the obvious form. Another application gives a perhaps more transparent proof [1] of the density of the set of finite rank operators of the unit ball of $\mathrm{Alg}\,\mathcal{L}$ in the unit ball of $\mathrm{Alg}\,\mathcal{L}$ in the strong operator topology, where $\mathcal{L}$ is the subspace lattice $\{(0), L, M, H\}$ with $L \cap M = (0)$, $L \vee M = H$ and $L, M \neq (0), H$. The final application concerns a certain class of finite atomic Boolean subspace lattices (on possibly infinite-dimensional space). We show that, for each such subspace lattice, there is a constant $M \geqslant 1$ such that the set of finite rank operators in the ball of radius M of its Alg is dense in the unit ball of the Alg in the strong operator topology. (See "Added in proof": M can be taken to be unity for this class of examples).

In what follows H will denote a complex separable Hilbert space. By a *projection* we mean an orthogonal projection and by a *subspace* we mean a closed linear manifold. The norm closure of a linear manifold $L \subseteq H$ will be denoted by $\overline{L}$ and $L^{\perp}$ will denote its orthocomplement. A subspace M is called *non-trivial* if $M \neq (0), H$. By an *operator* on H we mean a bounded linear transformation acting on H. The set of all operators on H will be denoted, us usual, by $\mathcal{B}(H)$. The rank of an operator is the dimension of its range. The spectrum of an operator $T \in \mathcal{B}(H)$ is denoted by $\sigma(T)$ and if T is a trace class operator, its trace is denoted by $\mathrm{tr}(T)$. We shall often be concerned with not necessarily everywhere defined closed linear transformations. The domain of such a transformation A will be denoted by $\mathcal{D}(A)$, its range by $\mathcal{R}(A)$ and its graph, that is, the subspace $\{(x, Ax) : x \in \mathcal{D}(A)\}$ of $H \oplus H$, by $G(A)$. If A and B are closed linear transformations, we write $A \subseteq B$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ax = Bx$ for every $x \in \mathcal{D}(A)$. If A is an operator, or more generally a closed densely defined linear transformation, the polar decomposition of A is the familiar equation $A = U|A|$ where U is a partial isometry and $|A| = (A^*A)^{1/2}$. The Cayley transform of a self-adjoint closed densely defined linear tranformation A is the (everywhere defined) unitary operator $V = (A - \mathrm{i})(A + \mathrm{i})^{-1}$. Convergence in the strong operator topology of $\mathcal{B}(H)$ of a sequence (A_n) of operators to an operator A is denoted by $A_n \xrightarrow{s} A$ or by $A = \text{s-lim} A_n$. A collection $\mathcal{L}$ of subspaces of H is called a *subspace lattice* on H if $(0), H \in \mathcal{L}$ and, for every family $\{L_\gamma\}$ of elements of $\mathcal{L}$, $\vee L_\gamma \in \mathcal{L}$ and $\cap L_\gamma \in \mathcal{L}$ (where "$\vee$" denotes closed linear span). We denote by $\mathrm{Alg}\,\mathcal{L}$ the algebra of operators leaving every element of $\mathcal{L}$ invariant. Other standard notation follows Halmos [5, 6] and Radjavi and Rosenthal [9].

2. APPROXIMATE INVERSES OF FINITE RANK

Although the result stated below as Theorem 2.2 appears to be fairly widely known, we have been unable to find reference to it in the literature. We supply a short proof and partially extend it in Theorem 2.3 to closed densely defined linear transformations. Later sections concern applications of these two theorems. The proof of the following preliminary lemma is based on idea of Harrison (reported in [1]).

LEMMA 2.1. *Let A be a positive operator acting on a complex separable Hilbert space H and let $(\varepsilon_n), (\delta_n)$ be sequences of positive real numbers. There exists an* *increasing sequence (P_n) of non-zero finite rank projections, converging to the identity in the strong operator topology, such that*

$$\|(A+\varepsilon_n)^{-1}P_n(A+\varepsilon_n)\| \leqslant 1+\delta_n, \quad \textit{for every } n.$$

Proof. We may suppose that H is infinite-dimensional. Let E be the spectral measure of A. For each $n \geqslant 1$ let $\mathcal{P}_n$ be a partition of $[0, \|a\|]$ given by

$$\mathcal{P}_n : 0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{k_n}^{(n)} = \|A\|$$

fine enough so that $|t_i^{(n)} - t_{i-1}^{(n)}| \leqslant \delta_n/(2\|(A+\varepsilon_n)^{-1}\|)$ $(1 \leqslant i \leqslant k_n)$ and such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$. For each n put $\Delta_i^{(n)} = [t_{i-1}^{(n)}, t_i^{(n)}) \cap \sigma(A)$ $(1 \leqslant i \leqslant k_n)$ and put

$$\Delta_{k_n}^{(n)} = [t_{k_n-1}^{(n)}, t_{k_n}^{(n)}] \cap \sigma(A).$$

Then $\sum_{1 \leqslant i \leqslant k_n} E(\Delta_i^{(n)}) = E(\sigma(A)) = I.$

Let $\{e_j : j \geqslant 1\}$ be any orthonormal basis of H and, for each n, let P_n be the projection onto the (finite-dimensional) subspace spanned by

$$\{E(\Delta_i^{(n)})e_j : 1 \leqslant j \leqslant n, 1 \leqslant i \leqslant k_n\}.$$

Since $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, (P_n) is increasing. Since $P_n e_j = e_j$ whenever $1 \leqslant j \leqslant n$, $P_n \xrightarrow{s} I$. Note that, for each n, P_n commutes with $E(\Delta_i^{(n)})$ $(1 \leqslant i \leqslant k_n)$.

Choose $s_i^{(n)}$ in $\Delta_i^{(n)}$ (if $\Delta_i^{(n)}$ is empty, simply ignore this step) and put

$$A_n = \sum_1^{k_n} s_i^{(n)} E(\Delta_i^{(n)}).$$

By [2, p.264], $\|A - A_n\| \leqslant \max_{1 \leqslant i \leqslant k_n} |t_i^{(n)} - t_{i-1}^{(n)}|$, so $2\|(A+\varepsilon_n)^{-1}\|\|A - A_n\| \leqslant \delta_n$. Since P_n commutes with A_n we have

$$P_n(A+\varepsilon_n) = (A+\varepsilon_n)P_n + P_n(A - A_n) - (A - A_n)P_n,$$

so

$$\|(A+\varepsilon_n)^{-1}P_n(A+\varepsilon_n)\| \leqslant \|P_n\| + 2\|(A+\varepsilon_n)^{-1}\|\|A - A_n\|\|P_n\| \leqslant 1+\delta_n.$$

This completes the proof.

THEOREM 2.2. *Let B be an injective operator with dense range acting on a complex separable Hilbert space H. There exists a sequence (R_n) of finite rank operators on H such that* s-lim BR_n =s-lim $R_nB = I$ *and* $\|BR_n\| \leqslant 1$, $\|R_nB\| \leqslant 1$ *for every n.*

Proof. Suppose first that B is also positive. We may suppose that $\|B\| = 1$.

Let (ε_n) be a sequence of positive numbers converging to 0, take $\delta_n = \varepsilon_n$ and $A = B$ in the preceding lemma and put $R_n = P_n(B+\varepsilon_n)^{-1}$.

Then $\|R_nB\| \leqslant \|B(B+\varepsilon_n)^{-1}\| = (1+\varepsilon_n)^{-1} \leqslant 1$. Also, $\|(B+\varepsilon_n)R_n\| \leqslant 1+\varepsilon_n$ (by the lemma) so $BR_n = B(B+\varepsilon_n)^{-1}(B+\varepsilon_n)R_n$ gives $\|BR_n\| \leqslant \|B(B+\varepsilon_n)^{-1}\|\|(B+ +\varepsilon_n)R_n\| \leqslant 1$.

Next we show that $R_nB \xrightarrow{s} I$. Since $\|R_nB\| \leqslant 1$ for every n and B has dense range, it is enough to show that $R_nB^2x \to Bx$ for every $x \in H$. Now $B^2-(B+\varepsilon_n)B = = -\varepsilon_nB$ so $(B+\varepsilon_n)^{-1}B^2 - B = -\varepsilon_nB(B+\varepsilon_n)^{-1}$. Hence $\|(B+\varepsilon_n)^{-1}B^2 - B\| \leqslant \varepsilon_n$. For $x \in H$ we have

$$\|R_nB^2x - Bx\| = \|P_n(B+\varepsilon_n)^{-1}B^2x - Bx\| =$$

$$= \|P_n[(B+\varepsilon_n)^{-1}B^2 - B]x + P_nBx - Bx\| \leqslant$$

$$\leqslant \varepsilon_n\|x\| + \|P_nBx - Bx\|,$$

so $R_nB^2s \to Bx$.

Next, $BR_n \xrightarrow{s} I$. For $R_nB \xrightarrow{s} I$ gives $(BR_n)B \xrightarrow{s} B$. Since $\|BR_n\| \leqslant 1$ for every n and B has dense range, it follows that $BR_n \xrightarrow{s} I$.

Without the positivity assumption we argue follows. By the polar decomposition theorem $B = U|B|$ where $|B| = (B^*B)^{1/2}$ and U is a unitary operator. By the above, there exists a sequence (S_n) of finite rank operators such that $S_n|B| \xrightarrow{s} I$, $|B|S_n \xrightarrow{s} I$ and $\|S_n|B|\| \leqslant 1$, $\||B|S_n\| \leqslant 1$ for every n. It is readily verified that the conclusion of the theorem holds by taking $R_n = S_nU^*$. This completes the proof.

The next theorem partially extends the preceding one to the case where B is a closed densely defined linear transformation.

THEOREM 2.3. *Let B be a closed densely defined injective linear transformation with dense range acting on a complex separable Hilbert space H. There exists a sequence (R_n) of finite rank operators on H such that $\mathcal{R}(R_n) \subseteq \mathcal{D}(B)$ for every n, $BR_nx \to x$ for every $x \in H$ and $R_nBy \to y$ for every $y \in \mathcal{D}(B)$.*

Proof. By the polar decomposition theorem for closed linear transformations [10, p.297] we have $B = U|B|$ with $|B|$ a positive closed injective linear transformation with dense range and with domain $\mathcal{D}(|B|) = \mathcal{D}(B)$. Here, as in Theorem 2.2, U is a unitary operator and as the domains of $|B|$ and B are identical, we need only consider the case where B is positive.

Let B be positive. As B is self-adjoint it has a Cayley transform [11, p.320] $V = (B-\mathrm{i})(B+\mathrm{i})^{-1}$ which is unitary and which also satisfies $B = \mathrm{i}(I+V)(I-V)^{-1}$

(implicit in this is that $\mathcal{D}(B) = \mathcal{D}((I-V)^{-1})$). The assumption that B is injective and has dense range implies that $I+V$ also has these properties.

By Theorem 2.2 applied to $\mathrm{i}(I+V)$, there exists a sequence (S_n) fo finite rank operators such that $\mathrm{i}S_n(I+V)\xrightarrow{s}I$ and $\mathrm{i}(I+V)S_n\xrightarrow{s}I$. Put $R_n = (I-V)S_n$. We show that the sequence (R_n) has the desired properties. Clearly, for every n, $\mathcal{R}(R_n) \subseteq \mathcal{R}(I-V) = \mathcal{D}(B)$. Also, for any $x \in H$, $BR_nx = \mathrm{i}(I+V)(I-V)^{-1}(I-V)S_nx = \mathrm{i}(I+V)S_nx \to x$. If $y \in \mathcal{D}(B) = \mathcal{D}((I-V)^{-1})$ we have $R_nBy = (I-V)(\mathrm{i}S_n(I+V))(I-V)^{-1}y \longrightarrow (I-V)(I-V)^{-1}y = y$. This completes the proof.

REMARKS. (i) With notation as in the statement of Theorem 2.2, if B is also positive we may assume the sequence (R_n) also satisfies $\|BR_n - R_nB\| \to 0$. To show this it is enough to consider the case where $\|B\| = 1$.

With notation as in the proof of Lemma 2.1, note that since $A_nP_n = P_nA_n$ for every n,

$$\begin{aligned}\|AP_n - P_nA\| &= \|(A-A_n)P_n - P_n(A-A_n)\| \leqslant \\ &\leqslant 2\|A-A_n\|.\end{aligned}$$

Hence

$$\begin{aligned}\|AP_n(A+\varepsilon_n)^{-1} - P_n(A+\varepsilon_n)^{-1}A\| &= \|(AP_n - P_nA)(A+\varepsilon_n)^{-1}\| \leqslant \\ &\leqslant \|AP_n - P_nA\|\|(A+\varepsilon_n)^{-1}\| \leqslant \\ &\leqslant 2\|(A+\varepsilon_n)^{-1}\|\|A-A_n\|,\end{aligned}$$

and, as we observed, the latter is at most δ_n.

Thus, in the proof of Theorem 2.2, when we take $A = B$ and $\delta_n = \varepsilon_n$ in Lemma 2.1 we obtain $\|BR_n - R_nB\| \leqslant \varepsilon_n$, so $\|BR_n - R_nB\| \to 0$.

(ii) There is a version of Theorem 2.2 (and a corresponding one for Theorem 2.3) for not necessarily injective dense range operators. For any operator B on H there exists a sequence (R_n) of finite rank operators $R_n\colon \overline{\mathcal{R}(B)} \to (\ker B)^{\perp}$ such that

(a) $BR_nx \to x$, for every $x \in \overline{\mathcal{R}(B)}$,

(b) $R_nBy \to y$, for every $y \in (\ker B)^{\perp}$,

(c) $\|BR_nx\| \leqslant \|x\|$, for every $x \in \overline{\mathcal{R}(B)}$ and every n,

(d) $\|R_nBy\| \leqslant \|y\|$, for every $y \in (\ker B)^{\perp}$ and every n.

Indeed, let $B \in \mathcal{B}(H)$ and let S be any invertibile operator from $\ker B \oplus H_1$ to $\overline{\mathcal{R}(B)} \oplus H_1$ where H_1 is any infinite dimensional complex separable Hilbert space. Consider the operator

$$T = \begin{pmatrix} B & 0 \\ 0 & S \end{pmatrix} : (\ker B)^{\perp} \oplus (\ker B \oplus H_1) \longrightarrow \overline{\mathcal{R}(B)} \oplus (\mathcal{R}(B)^{\perp} \oplus H_1)$$

acting on $H \oplus H_1$, with respect to the splittings indicated (here we regard B as a map from $(\ker B)^{\perp}$ to $\overline{\mathcal{R}(B)}$). Clearly this operator is injective and has dense range, so, by Theorem 2.2, there exists a sequence (F_n) of finite rank operators on $H \oplus H_1$ such that $TF_n \xrightarrow{s} I$, $F_nT \xrightarrow{s} I$ and $\|TF_n\| \leqslant 1$, $\|F_nT\| \leqslant 1$ for every n. Now

$$F_n = \begin{pmatrix} R_n & S_n \\ T_n & U_n \end{pmatrix} : \overline{\mathcal{R}(B)} \oplus (\mathcal{R}(B)^{\perp} \oplus H_1) \to (\ker B^{\perp}) \oplus (\ker B \oplus H_1)$$

for some finite rank operators R_n, S_n, T_n, U_n with $R_n \colon \overline{\mathcal{R}(B)} \to (\ker B)^{\perp}$ etc, and

$$TF_n = \begin{pmatrix} BR_n & BS_n \\ ST_n & SU_n \end{pmatrix}, \quad F_nT = \begin{pmatrix} R_nB & S_nS \\ T_nB & U_nS \end{pmatrix}.$$

Since $TF_n \xrightarrow{s} I$ and $F_nT \xrightarrow{s} I$, (a) and (b) hold. Since $\|TF_n\| \leqslant 1$ and $\|F_nT\| \leqslant 1$ for every n, (c) and (d) hold.

(iii) If $B \in \mathcal{B}(H)$ is injective, our second remark shows that there exists a sequence (R_n) of finite rank operators on H such that $R_nB \xrightarrow{s} I$, $BR_n \xrightarrow{s} P_R$ and $\|R_nB\| \leqslant 1$, $\|BR_nz\| \leqslant \|P_Rz\|$ for every $z \in H$ and every n, where P_R denotes the projection onto $\overline{\mathcal{R}(B)}$. (In (ii), extend the definition of R_n to all of H by defining it to be zero on $\mathcal{R}(B)^{\perp}$.) That the full conclusions of Theorem 2.2 need not hold for such injective operators B is easily seen by examples. For instance, if $B = S$ the unilateral shift on ℓ^2, then B is injective and, since $B(R_ne_1) \perp e_1$ for any choice of R_n, it is impossible to have $BR_n \xrightarrow{s} I$.

(iv) If $B \in \mathcal{B}(H)$ has dense range, (ii) shows that there exists a sequence (R_n) of finite rank operators on H such that $BR_n \xrightarrow{s} I$, $R_nB \xrightarrow{s} I - P_K$ and $\|BR_n\| \leqslant 1$, $\|R_nBz\| \leqslant \|(I - P_K)z\|$ for every $z \in H$ and every n, whereP_K denotes the projection onto $\ker B$. Again the full conclusions of Theorem 2.2 need not hold for such operators. For example, if $B = S^*$ the adjoint of the unilateral shift, then since $R_nBe_1 = 0$ for any choice of R_n, it is impossible to have $R_nB \xrightarrow{s} I$.

3. APPLICATION TO THE EQUATION $AX=YB$

The equation $AX = YA$ arises quite naturally in the study of operators leaving a pair of complementary subspaces invariant. Here A is some injective operator with dense range. In this section we will describe the solution set of this equation. In fact we will solve the slightly more general equation $AX = YB$ with either A or B an injective operator with dense range (or still more generally, a closed densely defined linear transformation).

An obvious family of solutions of the equation $AX = YB$ is given parametrically by $X = CB$, $Y = AC$ as C runs through $\mathcal{B}(H)$. Choosing C to be an operator of finite rank, it is clear that the equation has an abundance of solutions (X, Y) with X and Y of finite rank. The argument that follows shows that if A *and* B are injective with dense ranges, then all solutions with X or Y (and hence both) of finite rank arise in this way.

For instance, suppose that Y is of finite rank and $AX = YB$ for some X. As $\overline{\mathcal{R}(B)} = H$ we have $\mathcal{R}(Y) = Y\overline{\mathcal{R}(B)} \subseteq \overline{Y\mathcal{R}(B)}$. As $Y\mathcal{R}(B)$ is a finite-dimensional linear manifold it is closed. Thus $\mathcal{R}(Y) \subseteq Y\mathcal{R}(B) = \mathcal{R}(YB) = \mathcal{R}(AX) \subseteq \mathcal{R}(A)$. By the range inclusion theorem of Halmos and Douglas [3, 6], there exists an operator C with $Y = AC$. Since A is injective and Y has finite rank so has C. We have $AX = YB = ACB$, so $X = CB$ by the injectivity of A.

If, instead, X is if finite rank, a similar argument starting from the equivalent equation $B^*Y^* = X^*A^*$ shows the existance of a finite rank operator C parametrizing the solution by $X^* = B^*C^*$, $Y^* = C^*A^*$. Hence $X = CB$ and $Y = AC$ once again.

If X and Y are not of finite rank, the parametrization $X = CB$, $Y = AC$ does not cover all solutions of $AX = YB$, where each of A, B is injective with dense range. For exemple, if $B = A$ and A is not surjective, the solution $X = Y = I$ of $AX = YA$ is not of the form $X = CA$, $Y = AC$ because of the non-invertibility of A. Thus a more general parametrization to describe all the solution pairs (X, Y) is unavoidable. For instance, suppose that (C_n) is a sequence of operators such that (C_nB) and (AC_n) converge in the strong operator topology to operators X and Y respectively. Then $AX = A(\text{s-lim}\, C_nB) = \text{s-lim}(AC_nB) = \text{s-lim}(AC_n)B = YB$. The main result of this section is that the converse is true. Moreover, it is true if only one of A, B is injective with dense range and (even if X or Y is not finite rank) we can choose (C_n) to be a sequence of finite rank operators.

THEOREM 3.1. *Let A and B be operators acting on a complex separable Hilbert space H such that A or B is injective with dense range. The following are equivalent for a pair (X, Y) of operators of $\mathcal{B}(H)$.*

(i) $AX = YB$,

(ii) *there exists a sequence (C_n) of finite rank operators on H such that $C_nB \xrightarrow{s} X$ and $AC_n \xrightarrow{s} Y$,*

(iii) *there exists a sequence (C_n) of operators on H such that $C_nB \xrightarrow{s} X$ and $AC_n \xrightarrow{s} Y$.*

Proof. The implications (ii)⇒(iii)⇒(i) are obvious. Assume that (i) holds. If A is injective with dense range, Theorem 2.2 shows that there exists a sequence (R_n) of finite rank operators on H such that $R_nA \xrightarrow{s} I$ and $AR_n \xrightarrow{s} I$. Thus $AC_n \xrightarrow{s} Y$ where $C_n = R_nY$. For this C_n we also have $C_nB = R_nYB = R_nAX \xrightarrow{s} X$ and the proof is complete in this case.

If, instead, B is injective with dense range, then there exists a sequence (S_n) of finite rank operators such that $S_nB\xrightarrow{s}I$ and $BS_n\xrightarrow{s}I$. It is easily verified that, with $C_n = XS_n$, we have $C_nB\xrightarrow{s}X$ and $AC_n\xrightarrow{s}Y$. This completes the proof.

In the previous theorem we may actually assume less. Basing the proof on Theorem 2.3 rather than on Theorem 2.2 we conclude the following.

THEOREM 3.2. *Let A and B be closed densely defined linear transformations acting on a complex separable Hilbert space H such that A or B is injective with dense range. The following are equivalent for a pair (X,Y) of operators of $\mathcal{B}(H)$.*

(i) *$AX \supseteq YB$ in the sense that $X\mathcal{D}(B) \subseteq \mathcal{D}(A)$ and $AXx = YBx$ for every $x \in \mathcal{D}(B)$,*

(ii) *there exists a sequence (C_n) of finite rank operators on H such that $\mathcal{R}(C_n) \subseteq \subseteq \mathcal{D}(A)$ for every n, $C_nBx \to Xx$ for every $x \in \mathcal{D}(B)$ and $AC_ny \to Yy$ for every $y \in H$,*

(iii) *the same as* (ii) *but without the finite rank assumption on the C_n.*

The proof is similar to that of Theorem 3.1 and is omitted.

4. UNIT BALL DENSITY

Given the subspace lattice $\mathcal{L} = \{(0), L, M, H\}$ where L and M are non-trivial subspaces of the complex separable Hilbert space H satisfying $L \cap M = (0)$ and $L \vee \vee M = H$, in this section we use the above results to study certain density properties of the set of finite rank operators of $\operatorname{Alg}\mathcal{L}$. As is shown in [1] some questions concerning the properties of such an algebra can be reduced to the case where L and M are in generic position. Following Halmos [4] we say that two subspaces L, M are in *generic position* if $L \cap M = L^\perp \cap M = L \cap M^\perp = L^\perp \cap M^\perp = (0)$. In [4] Halmos gives two elegant characterizations of generic positioning. The first says that, up to unitary equivalence, the subspaces are of the form $G(B)$ and $G(-B)$ with respect to a direct sum $K \oplus K$ of some Hilbert space K with itself, with B an injective positive contraction on K for which $I - B$ is also injective. The second characterization states that, again up to unitary equivalence, the subspaces are of the form $K \oplus (0)$ and $G(A)$ with A an injective closed densely defined linear transformation on K with dense range. With these characterizations a description of $\operatorname{Alg}\mathcal{L}$ (given that we also have $L^\perp \cap M = L \cap M^\perp = (0)$, that is, generic positioning) becomes easy to obtain. For instance, suppose that $L = G(B)$ and $M = G(-B)$ with B as above. Then $\begin{pmatrix} X & Z \\ W & Y \end{pmatrix}$ belongs to $\operatorname{Alg}\mathcal{L}$ if and only if $BX + BZB = W + YB$ and $-BX + BZB = W - YB$ or, equivalently, $BX = YB$ and $W = BZB$. Thus

$$\mathrm{Alg}\,\mathcal{L} = \left\{ \begin{pmatrix} X & Z \\ BZB & Y \end{pmatrix} : BX = YB \right\}.$$

If $\begin{pmatrix} X & Z \\ BZB & Y \end{pmatrix}$ is of finite rank so must X, Y and Z be. Moreover, by the arguments at the beginning of Section 3, if also $BX = YB$ then $X = CB$ and $Y = BC$ for some finite rank operator C. It follows that the finite rank operators of $\mathrm{Alg}\,\mathcal{L}$ are those operators of the form $\begin{pmatrix} CB & Z \\ BZB & BC \end{pmatrix}$ with C and Z of finite rank.

In [8] Papadakis has shown that if $\mathcal{L}$ is as above (not necessarily in generic position) then the finite rank operators of $\mathrm{Alg}\,\mathcal{L}$ are dense in $\mathrm{Alg}\,\mathcal{L}$ in the ultraweak topology, thereby improving the result in [1] that the same set of operators is dense in $\mathrm{Alg}\,\mathcal{L}$ in the strong operator topology. A much stronger "Kaplansky type" unit ball density property proved in [1] improves both of these results. Among the applications of the results in Section 2 we give below a perhaps more transparent proof of this. Some other applications are also given.

THEOREM 4.1 ([1]). *Let L and M be two non-trivial subspaces of a complex separable Hilbert space H satisfying $L \cap M = (0)$ and $L \vee M = H$ and let $\mathcal{L} = \{(0), L, M, H\}$. Every operator in the unit ball of $\mathrm{Alg}\,\mathcal{L}$ is the limit, in the strong operator topology, of a sequence of finite rank operators from the unit ball of $\mathrm{Alg}\,\mathcal{L}$.*

Proof. As indicated above, it is shown in [1] that we may assume that L and M are in generic position. (In [1] a much stronger result is proved, concerning "meshed products", but for the case of just two subspaces the work becomes considerably easier; nevertheless we omit it.) Thus, by Halmos' description, it is sufficient to prove the theorem with $H = K \oplus K$ and with L and M of the form $G(B)$ and $G(-B)$ with B a positive injective contraction on K (for which $I - B$ is also injective). Since the finite rank operators of $\mathrm{Alg}\,\mathcal{L}$ form an ideal of $\mathrm{Alg}\,\mathcal{L}$ and the norm is submultiplicative, we need only show that the identity on $K \oplus K$ is a limit of the type described.

By Theorem 2.2 choose a sequence (R_n) of finite rank operators such that s-lim R_nB = s-lim $BR_n = I$ and $\|R_nB\| \leqslant 1$, $\|BR_n\| \leqslant 1$ for every n. Put $S_n = \begin{pmatrix} R_nB & 0 \\ 0 & BR_n \end{pmatrix}$. By earlier remarks in this section (S_n) is a sequence if finite rank operators of $\mathrm{Alg}\,\mathcal{L}$. Clearly s-lim $S_n = I$ and $\|S_n\| \leqslant 1$ for every n. This completes the proof.

From the above result it follows, as pointed out to us by A. Katavolos, that each operator of $\mathrm{Alg}\,\mathcal{L}$ belonging to the von Neumann-Schatten class C_p ($1 \leqslant p \leqslant \infty$, where by C_∞ we mean the ideal of compact operators) is a limit in C_p norm of a sequence of finite rank operators of $\mathrm{Alg}\,\mathcal{L}$. This follows simply from the fact that if

$A_n \xrightarrow{s} A$ and $\|A_n\| \leqslant 1$ for every n, then $A_n K \to AK$ in C_p norm for every $K \in C_p$. The result in [8], referred to earlier, also follows since strong convergence implies ultraweak convergence on the unit ball of $\mathcal{B}(H)$.

As a further application we prove the following generalization of the well known fact that if T is an element of the ideal of trace class operators and X is any operator then $\operatorname{tr}(XT) = \operatorname{tr}(TX)$.

COROLLARY 4.2. *Let A be an injective closed densely defined linear transformation with dense range acting on a complex separable Hilbert space H and let S,T be trace class operators such that $\mathcal{R}(S) \subseteq \mathcal{D}(A)$ and $ASx = TAx$ for every $x \in \mathcal{D}(A)$. Then* $\operatorname{tr}(S) = \operatorname{tr}(T)$.

Proof. As the subspaces $L = H \oplus (0)$ and $M = G(A)$ are in generic position, Halmos' description shows that there exists a unitary $U: H \oplus H \to K \oplus K$ and a positive injective contraction B on K such that

$$UL = G(B) \quad \text{and} \quad UM = G(-B).$$

Since AS is everywhere defined (because $\mathcal{R}(S) \subseteq \mathcal{D}(A)$) and closed it is bounded. Consider the trace class operator $D = \begin{pmatrix} S & 0 \\ AS & -T \end{pmatrix}$. This operator sends L into M and M into L (since $AS = TA$ on $\mathcal{D}(A)$) so UDU^* sends $G(B)$ into $G(-B)$ and $G(-B)$ into $G(B)$. If UDU^* is written in the form $\begin{pmatrix} V & W \\ X & Y \end{pmatrix}$, it is easily verified (just as in the case of operators sending each of $G(B)$, $G(-B)$ into itself) that $BV = -YB$ and $X = -BWB$. Thus for the typical finite rank operator $\begin{pmatrix} CB & Z \\ BZB & BC \end{pmatrix}$ of $\operatorname{Alg}\{(0), G(B), G(-B), K \oplus K\}$ (here C and Z are finite rank operators) we have

$$\operatorname{tr}\left(\begin{pmatrix} V & W \\ X & Y \end{pmatrix}\begin{pmatrix} CB & Z \\ BZB & BC \end{pmatrix}\right) = \operatorname{tr}(VCB + WBZB + XZ + YBC) =$$

$$= \operatorname{tr}((BV + YB)C + (BWB + X)Z) = 0.$$

But the map $\varphi(\cdot) = \operatorname{tr}\left(\begin{pmatrix} V & W \\ X & Y \end{pmatrix}(\cdot)\right)$ is continuous on the unit ball of $\mathcal{B}(K \oplus K)$ with the strong operator topology. We have shown that this map annihilates all the finite rank operators in the unit ball of $\operatorname{Alg}\{(0), G(B), G(-B), K \oplus K\}$, so it annihilates the identity which belongs to the strong operator closure of this set of operators by Theorem 4.1. Thus

$$\operatorname{tr}(S - T) = \operatorname{tr}\begin{pmatrix} S & 0 \\ AS & -T \end{pmatrix} = \operatorname{tr} D = \operatorname{tr} UDU^* = \operatorname{tr}\begin{pmatrix} V & W \\ X & Y \end{pmatrix} = 0.$$

as required. This completes the proof.

Before proving our final theorem we make a brief digression concerning the atomic Boolean subspace lattice $\mathcal{L} = \{(0), H \oplus (0), G(A), H \oplus H\}$ where $A \in \mathcal{B}(H)$ is an injective operator with dense range (we could consider the more general situation where A is an injective closed densely defined linear transformation with dense range, but we omit this to avoid technicalities). It is easily verified that

$$\operatorname{Alg}\mathcal{L} = \left\{ \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix} : AX + AZA = YA \right\}.$$

From this the finite rank operators of $\operatorname{Alg}\mathcal{L}$ can be described. If X, Y and Z are finite rank operators and $A(X + ZA) = YA$, then, by our remarks at the beginning of Section 3, there exists a finite rank operator C such that $X + ZA = CA$ and $Y = AC$. It follows that the finite rank operators of $\operatorname{Alg}\mathcal{L}$ are those operators of the form $\begin{pmatrix} (C-Z)A & Z \\ 0 & AC \end{pmatrix}$ with C and Z finite rank operators. These facts are used in the following theorem.

THEOREM 4.3. *Let A be an injective operator with dense range acting on a complex separable Hilbert space H and let S_j $(1 \leqslant j \leqslant m)$ be invertible operators on H $(m \geqslant 1)$. Let $K_0, K_1, \ldots, K_m$ be the following subspaces of $H^{(m+1)}$:*

$$K_0 = H \oplus (0) \oplus (0) \oplus \ldots \oplus (0)$$

and, for $1 \leqslant j \leqslant m$,

$$K_j = \{(x, S_1Ax, S_2Ax, \ldots, S_jAx, 0, 0, \ldots, 0) \ : \ x \in H\}.$$

Then $K_0, K_1, \ldots, K_m$ are the atoms of an atomic Boolean subspace lattice $\mathcal{L}$ on $H^{(m+1)}$ and there is a constant $M \geqslant 1$ such that the set of finite rank operators in the ball of radius M of $\operatorname{Alg}\mathcal{L}$ is dense in the unit ball of $\operatorname{Alg}\mathcal{L}$ in the strong operator topology.

Proof. First let us establish the required density property for $\operatorname{Alg}\mathcal{L}$ where $\mathcal{L}$ is the subspace lattice generated by the K_j $(0 \leqslant j \leqslant m)$. Using this we shall then show that $\mathcal{L}$ is an atomic Boolean algebra with atoms $K_0, K_1, \ldots, K_m$.

Applying Theorem 4.1 to the subspace lattice $\mathcal{M} = \{(0), H \oplus (0), G(A), H \oplus H\}$ there exists a sequence (D_n) of finite rank operators in the unit ball of $\operatorname{Alg}\mathcal{M}$ converging strongly to the identity $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ on $H \oplus H$. Each D_n has the form

$$D_n = \begin{pmatrix} (C_n - Z_n)A & Z_n \\ 0 & AC_n \end{pmatrix}$$

where C_n and Z_n are finite rank operators. Thus s-lim$(C_n - Z_n)A = I$, s-lim $Z_n = 0$ and s-lim $AC_n = I$. Hence the sequence (T_n) of finite rank operators on $H^{(m+1)}$ given by

$$T_n = \begin{pmatrix} (C_n - Z_n)A & Z_nS_1^{-1} & 0 & \dots & 0 \\ 0 & S_1AC_nS_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & S_2AC_nS_2^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & S_mAC_nS_m^{-1} \end{pmatrix}$$

converges in the strong operator topology to the identity on $H^{(m+1)}$. Direct computation shows that, for every n, T_n leaves each K_j $(0 \leqslant j \leqslant m)$ invariant, so $T_n \in \operatorname{Alg}\mathcal{L}$. Put $M = \max\{\|S_1\|, \|S_1^{-1}\|, \|S_1\|\|S_1^{-1}\|, \|S_2\|\|S_2^{-1}\|, \dots, \|S_m\|\|S_m^{-1}\|\}$. As $\|D_n\| \leqslant 1$ we have

$$\|S_jAC_nS_j^{-1}\| \leqslant \|S_j\|\|AC_n\|\|S_j^{-1}\| \leqslant M \quad (2 \leqslant j \leqslant m)$$

and

$$\left\| \begin{pmatrix} (C_n - Z_n)A & Z_nS_1^{-1} \\ 0 & S_1AC_nS_1^{-1} \end{pmatrix} \right\| = \left\| \begin{pmatrix} I & 0 \\ 0 & S_1 \end{pmatrix} D_n \begin{pmatrix} I & 0 \\ 0 & S_1^{-1} \end{pmatrix} \right\| \leqslant$$

$$\leqslant \max\{1, \|S_1\|\} \max\{1, \|S_1^{-1}\|\} \|D_n\| \leqslant M.$$

As $\|T_n\|$ is the maximum of the norms of $\begin{pmatrix} (C_n - Z_n)A & Z_nS_1^{-1} \\ 0 & S_1AC_nS_1^{-1} \end{pmatrix}$ and of the $S_jAC_nS_j^{-1}$ $(2 \leqslant j \leqslant m)$, it follows that $\|T_n\| \leqslant M$. To summarize, (T_n) is a sequence of finite rank operators of $\operatorname{Alg}\mathcal{L}$, each with the norm at most M, converging strongly to the identity on $H^{(m+1)}$. As the set of finite rank operators of $\operatorname{Alg}\mathcal{L}$ is an ideal of $\operatorname{Alg}\mathcal{L}$ and the norm on $\mathcal{B}(H^{(m+1)})$ is submultiplicative, the required density property follows.

It remains to show that $\mathcal{L}$ is an atomic Boolean algebra with atoms K_0, $K_1, \dots, K_m$. Note that since S_kA is injective $(1 \leqslant k \leqslant m)$, $K_i \cap K_j = (0)$ whenever $i \neq j$ and $0 \leqslant i, j \leqslant m$. Since S_kA has dense range $(1 \leqslant k \leqslant m)$, $\bigvee_0^m K_j = H^{(m+1)}$.

For each n, the finite rank operator T_n above is a finite sum of rank one operators each belonging to $\operatorname{Alg}\mathcal{L}$. To prove this it is enough to show that, for every rank one operator $R \in \mathcal{B}(H)$ the operators F, G on $H^{(m+1)}$ given by $F = \operatorname{diag}(RA, S_1ARS_1^{-1}, S_2ARS_2^{-1}, \dots, S_mARS_m^{-1})$ and

$$G = \begin{pmatrix} -RA & RS_1^{-1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

are each finite sums of rank one operators of $\operatorname{Alg}\mathcal{L}$. For G this is easy to do: G is itself a rank one operator of $\operatorname{Alg}\mathcal{L}$. Now $F = \sum_{j=0}^{m} F_j$ where $F_0 = -G$, where

$$F_m = \begin{pmatrix} 0 & 0 & \dots & 0 & RS_m^{-1} \\ 0 & 0 & \dots & 0 & S_1ARS_m^{-1} \\ 0 & 0 & \dots & 0 & S_2ARS_m^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & S_mARS_m^{-1} \end{pmatrix}$$

and where, for $1 \leqslant j \leqslant m-1$,

$$F_j = \begin{pmatrix} 0 & 0 & \dots & 0 & RS_j^{-1} & -RS_{j+1}^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & S_1ARS_j^{-1} & -S_1ARS_{j+1}^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & S_2ARS_j^{-1} & -S_2ARS_{j+1}^{-1} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & S_jARS_j^{-1} & -S_jARS_{j+1}^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

where the first j columns (and the last $m-1-j$) of F_j are zero. It is not too difficult to verify that each F_j $(1 \leqslant j \leqslant m)$ is a rank one operator of $\operatorname{Alg}\mathcal{L}$.

By a result of [7], since the set of finite sums of operators of rank at most one of $\operatorname{Alg}\mathcal{L}$ is strongly dense in $\operatorname{Alg}\mathcal{L}$, $\mathcal{L}$ is (completely) distributive. Observing the convention that $\bigvee_{\emptyset} = (0)$ (where $\emptyset$ is the empty set) and using the fact that $K_i \cap K_j = (0)$ whenever $i \neq j$, it now follows that, for every pair I, J of (possibly empty) subsets of $\{0, 1, 2, \dots, m\}$ we have

$$(*) \qquad (\bigvee_I K_i) \cap (\bigvee_J K_j) = \bigvee_{I \cap J} K_k$$

(since by distributivity, $(\bigvee_I K_i) \cap (\bigvee_J K_j) = \bigvee_{j \in J} (K_j \cap \bigvee_I K_i) = \bigvee_{j \in J} (\bigvee_{i \in I} K_j \cap K_i))$. The desired result now follows from $(*)$ by a result of [1]. However, for the convenience of the reader, we give a proof of this special case.

Let $\mathbb{P}$ denote the atomic Boolean algebra of all subsets of $\{0, 1, 2, \dots, m\}$ (partially ordered by inclusion). Consider the map $\psi : \mathbb{P} \to \mathcal{L}$ defined by $\psi(J) = \bigvee_J K_j$. This map ψ is a complete homomorphism, that is, for every family $\{J_\alpha\}_\Gamma$ of subsets of $\{0, 1, 2, \dots, m\}$ we have $\psi(\bigcup_\Gamma J_\alpha) = \bigvee_\Gamma \psi(J_\alpha)$ and $\psi(\bigcap_\Gamma J_\alpha) = \bigcap_\Gamma \psi(J_\alpha)$. Since also $\psi(\emptyset) = (0)$ and $\psi(\{0,1,2,\ldots,m\}) = H^{(m+1)}$, $\psi(\mathbb{P})$ is a sublattice on $H^{(m+1)}$. Clearly $\psi(\mathbb{P})$ contains every K_j $(1 \leqslant j \leqslant m)$ so $\psi(\mathbb{P}) = \mathcal{L}$ by the definition of $\mathcal{L}$. The map ψ is a lattice isomorphism of $\mathbb{P}$ onto $\mathcal{L}$, that is, it is a bijection satisfying $I \subseteq J \Leftrightarrow \psi(I) \subseteq \psi(J)$. (That $I \subseteq J$ if $\psi(I) \subseteq \psi(J)$ follows easily from the observation that, if $\psi(I) \subseteq \psi(J)$ then $K_i = K_i \cap (\bigvee_J K_j) = \bigvee_{j \in J} (K_i \cap K_j)$, for every $i \in I$.) The desired result now follows and the proof is complete.

It would be interesting if the constant M in the statement of the above theorem could be replaced by unity. As the proof shows, this is the case if each S_j $(1 \leqslant j \leqslant m)$ is unitary. Thus we have the following corollary.

COROLLARY 4.4. *Let A be an injective operator with dense range acting on a complex separable Hilbert space H and let U_j $(1 \leqslant j \leqslant m)$ be unitary operators on H $(m \geqslant 1)$. If $\mathcal{L}$ is the atomic Boolean subspace lattice on $H^{(m+1)}$ with atoms K_j $(0 \leqslant j \leqslant m)$ given by $K_0 = H \oplus (0) \oplus (0) \oplus \ldots \oplus (0)$ and, for $1 \leqslant j \leqslant m$,*

$$K_j = \{(x, U_1Ax, U_2Ax, \ldots, U_jAx, 0, \ldots, 0) \ : \ x \in H\},$$

then every operator in the unit ball of $\operatorname{Alg}\mathcal{L}$ is the limit, in the strong operator topology, of a sequence of finite rank operators in the unit ball of $\operatorname{Alg}\mathcal{L}$.

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Added in proof: Using duality theory, identifying $\mathcal{B}(H)$ as the dual of the set of trace class operators on H and identifying the latter as the dual of the set of compact operators on H (in the complex separable Hilbert space case), and also using the fact that for any Banach space X, the unit ball of the canonical image of X in X^{**} is weak* dense in the unit ball of X^{**}, it can be shown that if the set of finite linear combinations of rank one operators of Alg $\mathcal{L}$ is ultraweakly dense in Alg $\mathcal{L}$, then the same is true for the corresponding unit balls of $\mathcal{R}$ and Alg $\mathcal{L}$ (see M. Anoussis, A. Katavolos, M. S. Lambrou, On the reflexive algebra with two invariant subspaces, preprint, for discussion on this). Two conclusions are pertinent to the above work. Firstly, the apparently weaker density result of Papadakis [8] mentioned in Section 4, is actually equivalent to the unit ball conclusion of Theorem 4.1. Secondly, the constant M in Theorem 4.3 can indeed be replaced by unity since the strong density of the ball of radius M of $\mathcal{R}$ in the unit ball of Alg $\mathcal{L}$ implies the ultrastrong (and hence ultraweak) density of the (convex) set $\mathcal{R}$ in Alg $\mathcal{L}$.