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GEOMETRY OF UNITARY ORBITS

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INTRODUCTION

Let $\mathcal{A}$ be a complex C^*-algebra with unit and let U be its unitary group. In this paper we study the geometry of the unitary orbit $U(b) = \{ubu^* : u \in U\}$ of a fixed element $b \in \mathcal{A}$. First we prove that if $b \in \mathcal{A}$ is such that the C^*-subalgebra $C^*(b)$ is finite dimensional, then $U(b)$ is a C^∞-submanifold of $\mathcal{A}$, and that the map

$$\pi_b : U \longrightarrow U(b), \quad \pi_b(u) = ubu^*$$

defines a C^∞ principal bundle.

This map was studied in [2] and [6], for the case when $\mathcal{A} = L(H)$ the set of all bounded linear operators acting on a complex separable Hilbert space. For example, it is known that in this case, the condition that $C^*(b)$ is finite dimensional, is also necesary for $U(b)$ to be a submanifold of $L(H)$.

In [4] Corach, Porta and Recht introduced a connection in the manifold of systems of projectors of a Banach algebra—roughly speaking, similarity orbits of normal operators with finite spectrum when the algebra is $L(H)$—and were able to compute certain invariants of the connection, such as the parallel transport, curvature and torsion tensors, geodesics and exponential mapping.

Our aim in this paper is to follow these lines in the study of unitary orbits. We introduce a connection in the principal bundle π_b (for suitable elements $b \in \mathcal{A}$) and compute the invariants.

1. PRELIMINARY AND NOTATIONS

Let $\mathcal{A}$ be a C^*-algebra with unit and U its unitary group. By A_{ah} we denote the (real) Banach space of skewsymmetric elements of $\mathcal{A}$, i.e.

$$\mathcal{A}_{ah} = \{a \in \mathcal{A} : a^* = -a\}.$$

For $c \in \mathcal{A}$, let δ_c be the inner derivation, $\delta_c(x) = xc - cx$, which, unless otherwise stated, will be considered with domain in $\mathcal{A}_{ah}$. Analogously, $L_{ah}(H)$ denotes the set of skewsymmetric operators. We denote $U_b = \{u \in U : ub = bu\}$ the holonomy group of $b \in \mathcal{A}$.

We recall a result concerning the geometric structure of unitary orbits in Hilbert spaces, which can be found in [2] and [6].

THEOREM 1.1. *Let $b \in L(H)$. The following are equivalent:*

1) $U(b)$ is a C^∞-submanifold of $L(H)$.

2) $\pi_b : U(H) \to U(b)$ defines a C^∞-homogeneous space.

3) $\pi_b : U(H) \to U(b)$ has continuous local cross sections.

4) b is unitarily equivalent to an operator of the form

$$a \oplus (c \oplus c \oplus \ldots) = a \oplus c^{(\infty)}$$

where $a \in L(C^n)$, $c \in L(C^m)$, $n, m \in \mathbb{N}$.

5) The C^*-algebra $C^*(b)$ generated by b and I is finite dimensional.

6) $U(b)$ is norm closed in $L(H)$.

(3) through 6) was proved by Deckard and Fialkow in [6], the equivalence with 1) and 2) was shown in [2]).

Let us come back to the case of a general C^*-algebra $\mathcal{A}$.

REMARK 1.2. Let $b \in \mathcal{A}$ such that the C^*-algebra $C^*(b)$ generated by b has finite dimension, then there exist positive integers $n_1, \ldots, n_p$ and a $*$-isomorphism

$$\tau : C^*(b) \to M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_p}(\mathbb{C}).$$

(See [12] Chapter. 1, Section 1.1, pg. 1–14).

THEOREM 1.3. *If $b \in \mathcal{A}$ is such that $C^*(b)$ has a finite dimension, then the map π_b defines a C^∞-homogeneous space. In particular, $U(b)$ is a C^∞-submanifold of $\mathcal{A}$, with tangent space at $c \in U(b)$:*

$$TU(b)_c : \{xc - cx : x \in \mathcal{A}_{ah}\} = R\delta_c, \ c \in U(b).$$

Proof. We will construct a C^∞ map $\overline{\theta} : W \to U$ from a neighborhood of b in $\mathcal{A}$ into U such that $\overline{\theta}|_{W \cap U(b)}$ is a cross section for π_b. This will suffice to prove our statement (see [2], Proposition 2.1).

Let $n = \sum_{i=1}^{p} n_i$ ($n_i, \ldots, n_p$ as in 1.2). Consider the set of systems of projections

$$P_n = \{(p_1, \ldots, p_n) \in \mathcal{A}^n : p_i^2 = p_i^* = p_i,\ p_i p_j = 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{n} p_i = 1\}$$

Let $e^i_{jk} \in M_{n_i}(\mathbb{C})$ be the elementary matrix with the scalar 1 in the (j,k)-entry and zero elsewhere. Imbed canonically $M_{n_i}(\mathbb{C})$ into $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_p}(\mathbb{C})$. Then there exists a polynomial $p^i_{jk}(X,Y)$ in two non commuting variables X and Y such that $e^i_{jk} = p^i_{jk}(\tau(b), \tau(b^*))$. Consider $E^i_{ij} \in \mathcal{A}$, $E^i_{jk} = \tau^{-1}(e^i_{jk})$. Then $E^i_{jk} = p^i_{jk}(b, b^*)$. Let us define the maps:

$$\varphi : U(b) \to P_n,$$

$$\varphi(c) = (p^1_{11}(c,c^*), p^1_{22}(c,c^*), \ldots, p^1_{n_1,n_1}(c,c^*), \ldots, p^p_{11}(c,c^*), \ldots, p^p_{n_p,n_p}(c,c^*));$$

$\sigma : V \to U$, where V is a neighborhood of $\varphi(b)$ in P_n,

$$\sigma(q) = \sum_{i=1}^{n} \varphi_i(b) q_i [1 - (\varphi_i(b) - q_i)^2]^{-1/2},$$

and finally

$$\kappa : \varphi^{-1}(\varphi(b)) \to U,$$

given by

$$\kappa(c) = \sum_{i=1}^{p} \sum_{j=1}^{n_i} p^i_{j1}(c,c^*) E^i_{1j}.$$

In [1], it was shown that the map

$$\theta(c) = \sigma(\varphi(c))^* \kappa(\sigma(\varphi(c)) c \sigma(\varphi(c))^*)$$

defines a local cross section for π_b in a neighborhood of b for the case when the algebra is $L(H)$. The proof that this also holds for a general C^*-algebra is completely analogous.

We will show that this map θ defined on a neighborhood of b in $U(b)$ can be extended to a map $\overline{\theta}$ on a neighborhood of b in $\mathcal{A}$, and this will complete the proof. Observe that φ and κ can be extended to $\overline{\varphi} : \mathcal{A} \to \mathcal{A}^n$ and $\overline{\kappa} : \mathcal{A} \to \mathcal{A}$ naturally. Put $\overline{\sigma} : V \to U$,

$$\overline{\sigma}(q) = \sum_{i=1}^{n} \varphi_i(b) q_i [1 - (\varphi(b) - q_i)(\varphi_i(b) - q_i^*)]^{1/2},$$

where V is a neighborhood of $\varphi(b)$ in $\mathcal{A}^n$, such that the element $1-(\varphi_i(b)-q_i)(\varphi_i(b)-$ $-q_i^*)$ have strictly positive spectrum.

Finally, take

$$\overline{\theta} : W \to U,\ \overline{\theta}(a) = \rho(\overline{\sigma}(\overline{\varphi}(a))^{*}\overline{\kappa}(\overline{\sigma}(\overline{\varphi}(a))a\overline{\sigma}(\overline{\varphi}(a))^{*})),$$

where ρ is the unitary part of the polar decomposition $\rho(\omega) = u$, for $\omega = u\lambda$, u unitary and λ positive, which is a C^{∞} map defined on the open set of invertible elements of $\mathcal{A}$. ■

REMARK 1.4. i) note that if $C^{*}(b)$ is finite dimensional, then $\ker\delta_b$ and $R\delta_b$ are complemented in $\mathcal{A}_{ah}$ and $\mathcal{A}$ respectively (see [2], §4).

ii) An analogous argument shows that if $C^{*}(b)$ is finite dimensional, then the map

$$\pi_{b,b^{*}} : \mathcal{A}^{-1} \to \mathcal{S}(b, b^{*}) = \{(gbg^{-1}, gb^{*}g^{-1}) \in \mathcal{A}^2 : g \in \mathcal{A}^{-1}\},$$
$$\pi_{b,b^{*}}(g) := (gbg^{-1}, gb^{*}g^{-1}),\ g \in \mathcal{A}^{-1}$$

defines an analitic homogeneous space. In particular the joint similarity orbit $\mathcal{S}(b, b^{*})$ is an analytic submanifold of $\mathcal{A}^2$, which is locally C^{∞}-diffeomorphic to $U(b) \times U(b)$ (see [1]).

2. A CONNECTION IN THE PRINCIPAL BUNDLE

Let $b \in \mathcal{A}$, and E^{i}_{jk}, $1 \leqslant i \leqslant p$, $1 \leqslant j,\ k \leqslant n_i$, as in 1.3. The principal bundle $\pi_b : U \to U(b)$ has structure group $U_b = \{v \in U : vb = bv\}$.

Using the system of projectors $\varphi(b)$, U_b consists of matrices of the form

$$v = \begin{pmatrix} \begin{matrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_1 \end{matrix} & & & \\ & \begin{matrix} v_2 & & 0 \\ & \ddots & \\ 0 & & v_2 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} v_p & & 0 \\ & \ddots & \\ 0 & & v_p \end{matrix} \end{pmatrix}$$

with v_i "unitary" in $E^{i}_{11}\mathcal{A}E^{i}_{11}$.

Given $\omega \in U$ and $X \in TU_{\omega} = \omega\mathcal{A}_{ah}$, we say that X is vertical at ω if and only if $d(\pi_b)_{\omega}(X) = 0$. And we denote by $\mathcal{V}_{\omega}$ the set of all vertical tangent vectors to U at ω,

$$\mathcal{V}_{\omega} = \{X \in TU_{\omega} : d(\pi_b)_{\omega}(X) = 0\} = \ker d(\pi_b)_{\omega}.$$

Observe that

$$\mathcal{V}_\omega = \omega\mathcal{V}_1 = \omega\{a \in \mathcal{A}_{ah} : ab = ba\},$$

because $d(\pi_b)_\omega = \delta_{\omega b \omega^*} \circ \tau_{\omega^*}$.

Also, we can describe $\mathcal{V}_1$ in terms of the elements E^i_{jk}:

$$\mathcal{V}_1 = \{X \in A_{ah} : E^i_{jj} X E^\ell_{kk} = 0 \text{ if } i \neq \ell \text{ or } j \neq k \text{ and } E^i_{1j} X E^i_{j1} = E^i_{11} X E^i_{11},$$

$$1 \leqslant i,\ \ell \leqslant p,\ 1 \leqslant j,\ k \leqslant n_i\}$$

Then

$$\mathcal{V}_\omega = \{x \in \omega\mathcal{A}_{ah} : p^i_{jj}(\omega b \omega^*, \omega b^* \omega^*) X E^\ell_{kk} = 0 \text{ if } i \neq \ell \text{ or } j \neq k \text{ and}$$

$$p^i_{1j}(\omega b \omega^*, \omega b^* \omega^*) X E^i_{j1} = p^i_{11}(\omega b \omega^*, \omega b^* \omega) X E^i_{11},\ 1 \leqslant i,\ \ell \leqslant p,\ 1 \leqslant j,\ k \leqslant n_i\}$$

We define now the orizontal spaces. Let

$$\mathcal{H}_1 = \{X \in \mathcal{A}_{ah} : E^i_{11} X E^i_{11} = 0,\ 1 \leqslant i \leqslant p\}$$

and in general, for $\omega \in U$

$$\mathcal{H}_\omega = \omega\mathcal{H}_1 = \{X \in \omega\mathcal{A}_{ah} : p^i_{11}(\omega b \omega^*, \omega b^* \omega^*) X E^i_{11} = 0,\ 1 \leqslant i \leqslant p\}$$

Then, it is obvious that $\mathcal{H}_\omega \oplus \mathcal{V}_\omega = \omega\mathcal{A}_{ah} = TU_\omega$. A tangent vector X lies in $\mathcal{H}_1$ if its matrix relative to the system of projectors $\varphi(b)$ is of the form

$$X = \left(\begin{array}{c|c|c|c} \begin{matrix} 0 & \\ & * \end{matrix} & & & \\ \hline & \begin{matrix} 0 & \\ & * \end{matrix} & & \\ \hline & & \ddots & \\ \hline & & & \begin{matrix} 0 & \\ & * \end{matrix} \end{array}\right)$$

Proposition 2.1. *The distribution $u \mapsto \mathcal{H}_u$ ($u \in U$) given above defines a connection in the principal bundle $\pi_b : U \to U(b)$.*

Proof. First observe that the distribution is equivariant relative to the action of the structure group U_b, that is:

$$\mathcal{H}_u \cdot v = \mathcal{H}_{uv}, \quad \text{if } v \in U_b.$$

To prove this it suffices to see that $\mathcal{H}_1 = v\mathcal{H}_1 v^*$, $v \in U_b$. This is clear, because if v commutes with b, it also commutes with b^* and with the algebra $C^*(b)$. In particular commutes with E^i_{11}, $1 \leqslant i \leqslant p$.

Now compute the projections h_u and v_u onto $\mathcal{H}_u$ and $\mathcal{V}_u$ respectively:

$$h_u, v_u \in L(TU_u), \ Rh_u = \mathcal{H}_u, \ Rv_u = \mathcal{V}_u \ \text{ and } \ h_u + v_u = Id_{TU_u}.$$

Then

$$v_u(X) = u \sum_{i=1}^{p} \sum_{j=1}^{n_i} E^i_{j1} u^* X E^i_{1j}, \ \text{ and } \ h_u(X) = X - v_u(X), \ X \in TU_u.$$

Clearly, the maps $u \mapsto h_u$ and $u \mapsto v_u$ are C^∞. ∎

Let $\gamma : [\alpha, \beta] \to U(b)$ be a curve of class C^1. We study now the horizontal liftings of γ. That is, for each fixed u_α in the fiber of $\gamma(\alpha)$, the unique curve $\Gamma : [\alpha, \beta] \to U$ of class C^1, such that

i) $\Gamma(t) b \Gamma(t)^* = \gamma(t)$,

ii) $\dot{\Gamma}(t) \in \mathcal{H}_{\Gamma(t)}$, $t \in [\alpha, \beta]$ $\left(\dot{\Gamma}(t) = \frac{d}{dt}\Gamma(t)\right)$,

iii) $\Gamma(\alpha) = u_\alpha$.

DEFINITION 2.2. Denote by $S : \delta_b(\mathcal{A}_{ah}) \to \mathcal{A}_{ah}$ the *horizontal relative inverse* of δ_b, that is, the unique operator S satisfying:

$$S\delta_b S = S, \ \delta_b S \delta_b = \delta_b, \ \text{and } R(S) = \mathcal{H}_1.$$

In other words, $S = (\delta_b|_{\mathcal{H}_1})^{-1}$ (notice that $\mathcal{H}_1$ is a suplement for $\ker \delta_b$ in $\mathcal{A}_{ah}$). Let $\overline{S}$ be the linear extension of S to $\mathcal{A}$, with $\overline{S}|M \equiv 0$, for a fixed suplement M of $R\delta_b$ in $\mathcal{A}$.

This operator S allows us to lift vector fields horizontally.

Let X be a C^∞-vector field defined on a neighborhood W of $\gamma(\alpha)$ in $U(b)$. Consider the C^∞-vector field $\tilde{X}$ defined on $\pi_b^{-1}(W)$ by

$$\tilde{X}_u = uS(u^* X_{ubu^*} u), \quad u \in \pi_b^{-1}(W).$$

It is clear that:

i) $\tilde{X}$ is π_b-related to X, that is

$$d(\pi_b)_u(\tilde{X}_u) = X_{\pi_b(u)}, \quad u \in \pi_b^{-1}(W),$$

ii) $\tilde{X}_u \in \mathcal{H}_u$, $u \in \pi_b^{-1}(W)$.

These two conditions characterize the vector field $\tilde{X}$, which is called the horizontal lifting of X. With this idea we can introduce the differential equations satisfied by the horizontal liftings of the curve γ, $\gamma(\alpha) = u_\alpha b u_\alpha^*$.

If γ is C^∞, then $\dot\gamma(t)$ defines a C^∞-vector field along γ. Suppose that X is a C^∞-vector field that extends $\dot\gamma$ to a neighborhood of γ in $U(b)$, $X_{\gamma(t)} = \dot\gamma(t)$, $t \in [\alpha, \beta]$. Let $\overline{X}$ be the horizontal lifting of X. It is well known that if Γ is an integral curve of $\tilde{X}$ such that $\Gamma(\alpha) = u_\alpha$, that is

$$\begin{cases} \dot\Gamma(t) = \tilde{X}_{\Gamma(t)} = \Gamma(t)\overline{S}(\Gamma^*(t)X_{\Gamma(t)b\Gamma(t)^*}\Gamma(t)),\ t \in [\alpha,\beta] \\ \Gamma(\alpha) = u_\alpha. \end{cases}$$

Then, Γ is the horizontal lifting of γ through u_α (see [7] Chapter 4, Section 1 and [10] Chapter 2, Section 2).

Observe that since Γ lifts γ, the above equation can be simplified to

$$\begin{cases} \dot\Gamma(t) = \Gamma(t)\overline{S}(\Gamma^*(t)\dot\gamma(t)\Gamma(t)),\ t \in [\alpha,\beta] \\ \Gamma(\alpha) = u_\alpha \end{cases} \quad (2.3)$$

THEOREM 2.4. *Let $\gamma : [\alpha, \beta] \to U(b)$ be a curve of class C^1, $\gamma(\alpha) = u_\alpha b u_\alpha^*$. Then, the horizontal lifting Γ of γ with $\Gamma(\alpha) = u_\alpha$ is the unique solution of (2.3).*

Proof. We know that $\Gamma(t)b\Gamma(t)^* = \gamma(t)$, $\dot\Gamma(t) \in \mathcal{H}_{\Gamma(t)}$, $t \in [\alpha, \beta]$, $\Gamma(\alpha) = u_\alpha$. Then $b = \Gamma^*(t)\gamma(t)\Gamma(t)$, $t \in [\alpha, \beta]$, so that

$$0 = [\Gamma^*(t)\gamma(t)\Gamma(t)]^{\cdot} = \dot\Gamma^*(t)\gamma(t)\Gamma(t) + \Gamma^*(t)\dot\gamma(t)\Gamma(t) + \Gamma^*(t)\gamma(t)\dot\Gamma(t)$$

then

$$\Gamma^*(t)\dot\gamma(t)\Gamma(t) = -[\dot\Gamma^*(t)\gamma(t)\Gamma(t) + \Gamma^*(t)\gamma(t)\dot\Gamma(t)].$$

Observe that $\Gamma^*(t)\Gamma(t) = 1$ implies $\dot\Gamma^*(t)\Gamma(t) = -\Gamma^*(t)\dot\Gamma(t)$. Then

$$\Gamma^*(t)\dot\gamma(t)\Gamma(t) = -\dot\Gamma^*(t)\Gamma(t)b - b\Gamma^*(t)\dot\Gamma(t) = \delta_b(\Gamma^*(t)\dot\Gamma(t))$$

so that

$$\overline{S}(\Gamma^*(t)\dot\gamma(t)\Gamma(t)) = \overline{S}\delta_b(\Gamma^*(t)\dot\Gamma(t)) = \Gamma^*(t)\dot\Gamma(t).$$

This is because $\Gamma^*(t)\Gamma(t) \in \mathcal{H}_1$, and $\overline{S}\delta_b$ is a projection onto $\mathcal{H}_1$. The last equality is equivalent to (2.3).

It remains to prove that (2.3) has unique solution. To do so we will use a theorem of existence and uniqueness of solutions of non-linear differentiable equations (see for instance [5], Theorem 1.1, Chapter 7, p. 277). It suffices to prove that $(t, u) \mapsto$ $\mapsto u\overline{S}(u^*\dot\gamma(t)u)$ satisfies Lipschitz conditions on neighborhoods of all pairs $(s, \Gamma(s)) \in$

$\in [\alpha,\beta] \times \mathcal{A}$. Fix $s \in [\alpha,\beta]$ and take $W_s = [\alpha,\beta] \times \{u \in \mathcal{A} : \|u - \Gamma(s)\| \leqslant 1\}$. Let (t,u) and (t,v) in W_s. Then

$$\|u\overline{S}(u^*\dot\gamma(t)u) - v\overline{S}(v^*\dot\gamma(t)v)\| \leqslant \|\overline{S}(u^*\dot\gamma(t)u)\|\,\|u-v\| + \|v\|\,\|\overline{S}(u^*\dot\gamma(t)u) - \overline{S}(v^*\dot\gamma(t)v\| \leqslant$$

$$\leqslant \|\overline{S}\|\,\|u\|^2\|\dot\gamma(t)\|\,\|u-v\| + \|v\|\,\|\overline{S}\|\{\|u^*\dot\gamma(t)u - u^*\dot\gamma(t)v\| + \|u^*\dot\gamma(t)v - v^*\dot\gamma(t)v\|\}.$$

Since $\|u - \Gamma(s)\| \leqslant 1$, it follows that $\|u\| \leqslant 1 + \|\Gamma(s)\|$. The same happens with v, so that we finally arrive at

$$\|u\overline{S}(u^*\dot\gamma(t)u) - v\overline{S}(v^*\dot\gamma(t)v)\| \leqslant$$

$$\leqslant 3\|\overline{S}\|(1 + \|\Gamma(s)\|)^2 \max_{r\in[\alpha,\beta]} \|\dot\gamma(r)\| \cdot \|u - v\|.$$

■

DEFINITION 2.5. *For $c \in U(b)$, let*

1) $\mathcal{H}^c = u\mathcal{H}_1u^*$, where $u \in U$ is chosen so that $\pi_b(u) = c$.

2) $S_c : R\delta_c \to \mathcal{A}_{ah}$, $S_c = i \circ [\delta_c|_{\mathcal{H}^c}]^{-1}$, where i is the inclusion map of $\mathcal{H}^c$ into $\mathcal{A}_{ah}$.

Observe that $\mathcal{H}^c$ is well defined, for it does not depend on the particular choice of $u \in \pi_b^{-1}(c)$. This follows from the fact that $\mathcal{H}_1$ is invariant under conjugation by elements of U_b.

S_c is also well defined, since it is easy to see that $\mathcal{H}^c$ is a suplement for $\ker \delta_c$. The operator S_c provides a relative inverse for δ_c with the range $\mathcal{H}^c$, so that $\delta_c S_c = \mathrm{Id}_{R\delta_c}$ and $S_c\delta_c$ is a projection onto $\mathcal{H}^c$. Also, it is clear that $S_b = S$ and

$$S_c = \mathrm{ad}(u) \circ S \circ \mathrm{ad}(u^*), \quad \text{for } u \in \pi_b^{-1}(c)$$

Where $\mathrm{ad} : \mathcal{A} \to \mathcal{A}$, $\mathrm{ad}(u)(X) = uXu^*$, $X \in \mathcal{A}$.

LEMMA 2.6. *If $\gamma : [\alpha,\beta] \to U(b)$ is of class C^k, $1 \leqslant k \leqslant \infty$, then $S_\gamma(\dot\gamma) : [\alpha,\beta] \to \mathcal{A}$ is of class C^{k-1}.*

Proof. Observe that $S_\gamma(\dot\gamma)$ is well defined, since for each $t \in [\alpha,\beta]$, $\dot\gamma(t) \in \in TU(b)_{\gamma(t)} = R\delta_{\gamma(t)}$. Let $u : [\alpha,\beta] \to U$ be a lifting of γ, of class C_k (it is a general fact that it can be found in a principal bundle). Fix a supplement M for $R\delta_{\gamma(\alpha)}$ in $\mathcal{A}$. Then $u(t)Mu(t)^*$ is a supplement for $u(t)R\delta_{\gamma(\alpha)}u(t)^* = R\delta_{\gamma(t)}$, $t \in [\alpha,\beta]$.

For $t \in [\alpha,\beta]$, we define $\overline{S}_t : \mathcal{A} \to \mathcal{A}_{ah}$ as follows:

$$\overline{S}_t|R\delta_{\gamma(t)} = S_{\gamma(t)} \text{ and } \overline{S}_t|u(t)Mu(t)^* = 0.$$

Clearly, $\overline{S}_t = \mathrm{ad}(u(t)) \circ \overline{S}_\alpha \circ \mathrm{ad}(u(t)^*)$. Then $\overline{S}_t$ is of class C^k in the parameter t. So that $S_{\gamma(t)}(\dot\gamma(t)) = \overline{S}(\dot\gamma(t))$ is of class C^{k-1}. ■

We introduce now the linear equations of the horizontal liftings.

THEOREM 2.7. *Let $\gamma : [\alpha, \beta] \to U(b)$ be a C^k curve with $\gamma(\alpha) = u_\alpha b u_\alpha^*$. With the above notations, the equation*

$$\begin{cases} \dot{\Gamma}(t) = S_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t), \ t \in [\alpha, \beta] \\ \Gamma(\alpha) = u_\alpha \end{cases} \tag{2.8}$$

has unique solution the horizontal lifting (of class C^k) of γ with initial point u_α.

Proof. It is clear that (2.8) has unique solution (see [5] and [11]). Let Γ be the solution. We will verify that:

i) $\Gamma(t) \in U,\ t \in [\alpha, \beta]$.

ii) Γ lifts γ.

iii) Γ is horizontal.

First, consider the linear equations

$$\text{a)} \begin{cases} \dot{\Delta} = -\Delta S_\gamma(\dot{\gamma}) \\ \Delta(\alpha) = u_\alpha^* \end{cases} \qquad \text{b)} \begin{cases} \dot{\Phi} = S_\gamma(\dot{\gamma})\Phi - \Phi S_\gamma(\dot{\gamma}) \\ \Phi(\alpha) = u_\alpha \end{cases}$$

It is clear that Γ^* is the solution for a), so that $\mathrm{d}(\Gamma^*(t)\Gamma(t))/\mathrm{d}t = 0$ and therefore $\Gamma^*(t)\Gamma(t) = 1,\ t \in [\alpha, \beta]$. Also, $\Gamma\Gamma^*$ and 1 are both solutions of b), then $\Gamma(t)\Gamma^*(t) = 1,\ t \in [\alpha, \beta]$.

Secondly, let us compute

$$\begin{aligned} [\Gamma^*(t)\gamma(t)\Gamma(t)]^\bullet &= \dot{\Gamma}^*(t)\gamma(t)\Gamma(t) + \Gamma(t)\dot{\gamma}(t)\Gamma(t) + \Gamma^*(t)\gamma(t)\dot{\Gamma}(t) = \\ &= -\Gamma^*(t)S_{\gamma(t)}(\dot{\gamma}(t))\gamma(t)\Gamma(t) + \Gamma^*(t)\dot{\gamma}(t)\Gamma(t) + \\ &\quad + \Gamma^*(t)\gamma(t)S_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t) = \\ &= \Gamma^*(t)[-S_{\gamma(t)}(\dot{\gamma}(t))\gamma(t) + \dot{\gamma}(t) + \gamma(t)S_{\gamma(t)}(\dot{\gamma}(t))]\Gamma(t) = \\ &= \Gamma^*(t)[\dot{\gamma}(t) - \delta_{\gamma(t)}(S_{\gamma(t)}(\dot{\gamma}(t)))]\Gamma(t). \end{aligned}$$

Since $\dot{\gamma}(t) \in R\delta_{\gamma(t)}$ and $\delta_{\gamma(t)}S_{\gamma(t)} = \mathrm{Id}_{R\delta_{\gamma(t)}}$, it follows that the expression above equals 0. Because $\Gamma^*(\alpha)\gamma(\alpha)\Gamma(\alpha) = b$, we obtain that $\Gamma^*(t)\gamma(t)\Gamma(t) = b,\ t \in [\alpha, \beta]$.

Finally, let us show that Γ is horizontal:

$$\dot{\Gamma}(t) = S_{\gamma(t)}(\dot{\gamma}(t))\Gamma(t) \in (RS_{\gamma(t)})\Gamma(t) = \mathcal{H}^{\gamma(t)} \cdot \Gamma(t) = \Gamma(t)\mathcal{H}_1,$$

for we have already proved that Γ lifts γ. ■

REMARK 2.9. If $\gamma : [\alpha, \beta] \to U(b)$ is a C^k curve, the horizontal lifting of γ which starts at a given $v \in \pi_b^{-1}(\gamma(\alpha))$ coincides with $v \cdot \Gamma_0$, where Γ_0 is the horizontal lifting of $v^*\gamma v$, with $\Gamma_0(\alpha) = 1$.

We will give now another characterization of the horizontal lifting. Before we do so, let us recall some facts and refer the reader to the results obtained by Corach, Porta and Recht in [4], concerning the geometry of the set P_n of systems of projectors.

For a fixed $p = (p_1, \ldots, p_n)$ the following bundle is considered

$$\pi_p : U \to P_n, \quad \pi_p(u) = (up_1u^*, \ldots, up_nu^*).$$

In this bundle they defined a connection, where the horizontal space at $u \in U$ is given by

$$\{X \in \mathcal{A}_{ah} : p_i u^* X p_i = 0, \ 1 \leqslant i \leqslant n\},$$

and the horizontal lifting of a curve $\omega : [\alpha, \beta] \to P_n$ with $\omega(\alpha) = p$ is a solution of

$$\text{(2.10)} \qquad \begin{cases} \dot{\Omega}(t) = \sum_{i=1}^{n} \dot{\omega}_i(t)\omega_i(t)\Omega(t), \ t \in [\alpha, \beta] \\ \Omega(\alpha) = 1 \end{cases}.$$

With the maps defined in 1.3, we have the following

THEOREM 2.11. *Let $\gamma : [\alpha, \beta] \to U(b)$ be a C^k-curve $\gamma(\alpha) = b$. Let $P = \varphi(b)$ and $\omega : [\alpha, \beta] \to U(p)$ given by $\omega(t) = \varphi(\gamma(t))$. If $\Omega : [\alpha, \beta] \to U$ is the solution of (2.10) for this ω, then the horizontal lifting Γ of γ with $\Gamma(\alpha) = 1$ is given by*

$$\Gamma(t) = \Omega(t)\kappa(\Omega^*(t)\gamma(t)\Omega(t)).$$

Proof. Clearly $\Gamma(\alpha) = 1$. Also, by construction it is straightforward to verify that $\Gamma(t)$ is unitary and lifts γ. It remains to see that Γ is horizontal. It is enough to prove that

$$p^i_{11}(\gamma(t), \gamma^*(t))\dot{\Gamma}(t)E^i_{11} = 0, \quad t \in [\alpha, \beta].$$

Since Γ lifts γ, we have

$$p^i_{jj}(\gamma(t), \gamma^*(t)) = \Omega(t)E^i_{jj}\Omega^*(t),$$

because Ω lifts ω. Let us compute

$$\dot{\Gamma}(t) = \dot{\Omega}(t)\kappa(\Omega^*(t)\gamma(t)\Omega(t)) + \Omega(t)[\kappa(\Omega^*(t)\gamma(t)\Omega(t))]^{\cdot}.$$

Observe that since Ω is horizontal (in the sense of the connection defined in [4]), then

$$p^i_{jj}(\gamma(t), \gamma^*(t))\dot{\Omega}(t)E^i_{jj} = 0 \quad t \in [\alpha, \beta], \ 1 \leqslant i \leqslant p, \ 1 \leqslant j \leqslant n_i.$$

On the other hand, looking at the definition of κ, it is easy to see that $\kappa(\Omega^*(t)\gamma(t)\Omega(t))E^i_{11} = E^i_{11}$ and

$$[\kappa(\Omega^*(t)\gamma(t)\Omega(t))]^{\cdot} E^i_{11} = 0, \quad t \in [\alpha, \beta], \ 1 \leqslant i \leqslant p.$$

With these two identities, it follows that

a) $$p^i_{11}(\gamma(t),\gamma(t)^*)\dot{\Omega}(t)\kappa(\Omega^*(t)\gamma(t)\Omega(t))E^i_{11} = \omega^i_{11}(t)\dot{\Omega}(t)E^i_{11} = 0,$$

$t \in [\alpha,\beta]$, $1 \leqslant i \leqslant p$.

b) $$p^i_{11}(\gamma(t),\gamma^*(t))\Omega(t)[\kappa(\Omega^*(t)\gamma(t)\Omega(t))]^* E^i_{11} = 0,$$

$t \in [\alpha,\beta]$, $1 \leqslant i \leqslant p$, so that the proof is concluded. ■

We will end this paragraph with the computation of the 1-form and curvature form of the connection.

The 1-form θ valued in the Lie algebra v_1 is given by

$$\theta(X)_u = \sum_{i=1}^{p}\sum_{j=1}^{n_i} u^* X_u E^i_{1j},$$

for X a vector field on U and $u \in U$.

And the curvature form Δ

$$\Delta(X,Y)_u = \frac{1}{2} d\theta([hX,hY])_u,$$

where X and Y are vector fields on U and $(hX)_u = h_u(X_u)$, $u \in U$. So we have

$$\Delta(X,Y)_u = -\frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{n_i} E^i_{j1} u^* \{X_u u^* Y_u - Y_u u^* X_u - X_u E^i_{11} u^* Y_u +$$
$$+ Y_u E^i_{11} u^* X_u\} E^i_{1j}.$$

3. LINEAR CONNECTION IN $U(B)$

The connection defined in the principal bundle $\pi_b : U \to U(b)$ determines a linear connection in the tangent bundle $TU(b)$ of $U(b)$. In this section we study this connection and calculate its curvature and torsion.

Recall the definitions of the spaces $\mathcal{H}^c$, $R\delta_c$ and $\ker\delta_c$ and of the linear operators $S_c : R\delta_c \to \mathcal{A}_{ah}$, $c \in U(b)$, given in the preceding paragraphs.

Observe that, repeating the proof of 2.9, it can be shown that for a given C^1-curve $\gamma : [\alpha,\beta] \to U(b)$, the solution of

(3.1) $$\begin{cases} \dot{\Gamma}(t) = S_{\gamma(t)}(\dot{\gamma}(y))\Gamma(t) \\ \Gamma(\alpha) = 1 \end{cases}$$

lifts γ in the following sense:

$$\Gamma(t)\gamma(\alpha)\Gamma(t)^* = \gamma(t), \quad t \in [\alpha, \beta].$$

(3.1) is the transport equation of the curve γ. When $\gamma(\alpha) = b$, its solution coincides with the horizontal lifting. But in general it is not an horizontal curve, although it is related to certain horizontal curves. If $\gamma(\alpha) = ubu^*$, let $\overline{\gamma}(t) = u^*\gamma(t)u$. If $\overline{\Gamma}$ is the horizontal lifting of $\overline{\gamma}$ with $\overline{\Gamma}(\alpha) = 1$, it is clear that $u\overline{\Gamma}u^*$ is the solution of (3.1).

DEFINITION 3.2. Let $\gamma : [\alpha, \beta] \to U(b)$ be a C^1-curve. We define the linear map

$$\tau(\gamma)_\alpha^t : TU(b)_{\gamma(\alpha)} \to TU(b)_{\gamma(t)},$$

$$\tau(\gamma)_\alpha^t = \Gamma(t)X\Gamma^*(t), \quad X \in TU(b)_{\gamma(\alpha)}, \quad t \in [\alpha, \beta],$$

where Γ is the solution of the transport equation of γ. This map is the parallel transport along γ.

It is well defined, since

$$TU(b)_{\gamma(t)} = R\delta_{\gamma(t)} = \Gamma(t)R\delta_{\gamma(\alpha)}\Gamma(t)^* = \Gamma(t)TU(b)_{\gamma(\alpha)}\Gamma(t)^*.$$

We introduce now the covariant derivative:

DEFINITIONS 3.3. 1) Let $X \in TU(b)_c$, Y a C^1-vector field on $U(b)$, we define

$$(D_X Y)_c = \lim_{t \to t_0} \frac{[\tau(\gamma)_{t_0}^t]^{-1} Y_{\gamma(t)} - Y_c}{t - t_0} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Gamma^*(t)Y_{\gamma(t)}\Gamma(t)]\Big|_{t=t_0},$$

where $\gamma : (t_0-\varepsilon, t_0+\varepsilon) \to U(b)$ is a C^1-curve adapted to X (i.e.: $\gamma(t_0) = c$, $\dot\gamma(t_0) = X$) and Γ is the solution of the transport equation of γ, with $\Gamma(t_0) = 1$.

2) If $Z(t)$ is a C^1-vector field along the curve $\gamma(Z(t) \in TU(b)_{\gamma(t)})$, let

$$\frac{\mathrm{D}Z}{\mathrm{d}t}\Big|_{t_0} = \mathrm{D}_{\dot\gamma(t_0)} Z.$$

REMARK 3.4. If we abbreviate $[\![x, y]\!] = xy - yx$, $x, y \in \mathcal{A}$, then if X and Y are C^1-vector fields

$$(\mathrm{D}_X Y)_c = XY_c + [\![Y_c, S_c(X_c)]\!]$$

and for $Z(t)$ a C^1-vector field along γ

$$\left.\frac{\mathrm{D}Z}{\mathrm{d}t}\right|_{t_0} = \left.\frac{\mathrm{d}Z(\gamma(t))}{\mathrm{d}t}\right|_{t=t_0} + [\![Z_{\gamma(t_0)}, S_{\gamma(t_0)}(\dot\gamma(t_0))]\!].$$

This notion of covariant derivative introduces a linear connection in the tangent bundle of $U(b)$,

$$TU(b) = \{(c, X) \in \mathcal{A}^2 : c \in U(b),\ X \in R\delta_c\}.$$

We can conpute the geodesic curves of this connection.

THEOREM 3.5. *Let $X \in TU(b)_c$. The unique geodesic $\psi : \mathbf{R} \to U(b)$ verifying $\psi(0) = c$ and $\dot\psi(0) = X$ is given by*

$$\psi(t) = \mathrm{e}^{tS_c(X)} c\, \mathrm{e}^{-tS_c(X)}.$$

In particular, it follows that $U(b)$ is complete.

Proof. First suppose that $\varphi : [-\varepsilon, \varepsilon] \to U(b)$ is a geodesic curve with $\varphi(0) = c$ and $\dot\varphi(0) = X$. For each $t_0 \in [-\varepsilon, \varepsilon]$, we have that

$$\left.\frac{\mathrm{D}}{\mathrm{d}t}(\dot\varphi(t))\right|_{t_0} = 0.$$

That is, if Γ_{t_0} is the solution of

$$\begin{cases} \dot\Gamma_{t_0} = S_{\varphi(t)}(\dot\varphi(t)) \cdot \Gamma_{t_0}(t),\ t \in [-\varepsilon, \varepsilon] \\ \Gamma_{t_0}(t_0) = 1 \end{cases}$$

then $\frac{\mathrm{d}}{\mathrm{d}t}\{\Gamma_{t_0}^*(t) \cdot \dot\varphi(t) \cdot \Gamma_{t_0}(t)\} |_{t_0} = 0$. Note that for each $t_0 \in (-\varepsilon, \varepsilon)$,

$$S_{\varphi(t)}(\dot\varphi(t)) = \mathrm{ad}(\Gamma_{t_0}) \circ S_{\varphi(t_0)} \circ \mathrm{ad}(\Gamma_{t_0}^*)(\dot\varphi(t)).$$

Therefore

$$0 = S_{\varphi(t_0)}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\{\Gamma_{t_0}^*(t)\dot\varphi(t)\Gamma_{t_0}(t)\} |_{t_0}\Big) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} S_{\varphi(t_0)}(\Gamma_{t_0}^*(t)\dot\varphi(t)\Gamma_{t_0}(t)) |_{t_0}.$$

So, we have that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\{\mathrm{ad}(\Gamma_{t_0}^* \circ S_{\varphi(t_0)}(\dot\varphi(t))\} |_{t_0} =$$

$$= \dot\Gamma_{t_0}^*(t_0) S_{\varphi(t_0)}(\dot\varphi(t_0)) + S_{\varphi(t_0)}(\dot\varphi(t_0))\dot\Gamma_{t_0}(t_0) +$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}\{S_{\varphi(t)}(\dot\varphi(t))\} |_{t_0}.$$

Since

$$\dot{\Gamma}_{t_0}^* S_{\varphi(t_0)}(\dot{\varphi}(t_0)) + S_{\varphi(t_0)}(\dot{\varphi}(t_0))\dot{\Gamma}_{t_0}(t_0) =$$
$$= -S_{\varphi(t_0)}(\dot{\varphi}(t_0))^2 + S_{\varphi(t_0)}(\dot{\varphi}(t_0))^2 = 0$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\{S_{\varphi(t)}(\dot{\varphi}(t))\}\mid_{t_0} = 0, \quad \text{and this is true for all } t_0 \in (-\varepsilon, \varepsilon).$$

Therefore $S_{\varphi(t)}(\dot{\varphi}(t)) = S_c(X)$, $t \in (-\varepsilon, \varepsilon)$. Let us compute the solution of the transport equation of φ, $\Phi : (-\varepsilon, \varepsilon) \to U$, with $\Phi(0) = 1$. By the fact proved above, it verifies that $\dot{\Phi}(t) = S_c(X)\Phi(t)$. So it must be $\Phi(t) = \mathrm{e}^{tS_c(X)}$. So that $\varphi(t) = \Phi(t)\varphi(0)\Phi(t)^* = \mathrm{e}^{tS_c(X)}c\mathrm{e}^{-tS_c(X)}$, which can be extended to $\mathbf{R}$.

It remains to prove that all the curves of this type are geodesics, each one verifying that $\psi(0) = c$ and $\dot{\psi}(0) = X$. The first fact is obvious. The second:

$$\dot{\psi}(0) = \delta_c(S_c(X)) = X, \quad \text{since } X \in TU(b)_c.$$

Let us show that $\Phi(t) = \mathrm{e}^{tS_c(X)}$ is the solution of the transport eqution of ψ. This is equivalent to the fact that $\overline{\Phi}(t) = u^*\psi(t)u = \mathrm{e}^{tu^*S_c(X)u}$ is horizontal (since we already know that it lifts $u^*\psi u$).

$\overline{\Phi}^*(t)\dot{\overline{\Phi}}(t) = u^*\Phi(t)^*S_c(X)\Phi(t)u$. Since $S_c(X)$ and $\Phi(t)$ commute, this is equal to $u^*S_c(X)u$, which lies in $u^*\mathcal{H}^c u = \mathcal{H}_1$. Therefore, since $\Phi^*(t)\dot{\psi}(t)\Phi(t) = X$, $t \in \mathbf{R}$, it follows that $\frac{\mathrm{D}\dot{\psi}}{\mathrm{d}t} \equiv 0$. ∎

COROLLARY 3.6. *The exponential mapping of this connection is given by*

$$\exp : TU(b) \to U(b), \ \exp(c, X) = \mathrm{e}^{S_c(X)}c\mathrm{e}^{-S_c(X)}.$$

∎

REMARK 3.7. We can easily compute the torsion and curvature tensors. First, let X and Y be differentiable vector fields on $U(b)$, let $c \in U(b)$ and $[X, Y]$ the usual Lie bracket, then

$$T(X,Y)_c = (\mathrm{D}_X Y)_c - (\mathrm{D}_Y X)_c - [X,Y]_c.$$

It can be shown that

$$T(X,Y)_c = [\![Y_c, S_c(X_c)]\!] - [\![X_c, S_c(Y_c)]\!].$$

To compute the curvature $R(X,Y)Z$ on three differentiable vector fields X, Y and Z on $U(b)$,

$$\{R(X,Y)Z\}_c = [\mathrm{D}_X(\mathrm{D}_Y Z)]_c - [\mathrm{D}_Y(\mathrm{D}_X Z)]_c - (\mathrm{D}_{[X,Y]}Z)_c,$$

$c \in U(b)$, observe that if $\tilde{S}(Y)_d = S_d(Y_d)$, $d \in U(b)$, then

$$X\tilde{S}(Y)_c = [\![S_c(X_c), S_c(Y_c)]\!] + S_c((D_X Y)_c).$$

So that

$$\{R(X,Y)Z\}_c = [\![Z_c, [\![S_c(X_c), S_c(Y_c)]\!] + S_c(T(X,Y)_c)]\!].$$

REMARKS 3.8. 1) The constructions of $\mathcal{H}$, S_c and therefore of the transport and horizontal lifting equations, geodesics, etc., do not depend on the particular shape of $\mathcal{H}_1$, but on the invariance under the action of the isotropy group and on the other natural properties that enable it to define a connection on the principal bundle of π_b.

Therefore this machinery can be applied to join similarity and unitary orbits of n-tuple of elements of $\mathcal{A}$ which have differentiable structure (see [1]), whenever such an invariant subspace $\mathcal{H}_1$ can be found. All the invariants computed here can be calculated analogously in all the cases.

However, the existence of such a space is not always guaranteed. It can be found in the case of the joint similarity orbit of the pair (b, b^*), when $C^*(b)$ is finite dimensional. Also for joint unitary orbits with differentiable structure. But it is not the case for similarity orbits of one single element in general, as it can be easily shown taking the orbit of the 2×2 Jordan cell q_2 in $\mathbb{C}^{2\times 2}$.

2) This paper is based upon the study of geometry of the sets of idempotents and systems of projectors of a Banach algebra, developed in [4]. All the ideas contained here were either taken from there or communicated to us by its authors, to whom we are deeply grateful. Let us point out that in the case of similarity orbit of one idempotent q in a Banach algebra $\mathcal{B}$, the linear map $S_{q'}$ coincides with $\delta_{q'}$. So that if $\omega : [\alpha, \beta] \to \mathcal{S}(q) = \{gqg^{-1} : g \in \mathcal{B}^{-1}\}$ is a differentiable curve, then the analogous of the linear equation (2.8)

$$\begin{cases} \dot{\Omega}(t) = S_{\omega(t)}(\dot{\omega}(t))\Omega(t), \ t \in [\alpha, \beta] \\ \Omega(\alpha) = 1 \end{cases}$$

turns out to be

$$\begin{cases} \dot{\Omega}(t) = [\![\dot{\gamma}(t), \gamma(t)]\!]\Omega(t) \\ \Omega(\alpha) = 1 \end{cases}$$

which is in [4]. The sames happens with equation (2.3).

4. APPENDIX

The cross section introduced in 1.3 has in fact a plainer expression, it is straightforward to verify that

$$\theta(c) = \sum_{i=1}^{p}\sum_{j=1}^{n_i} p_{j1}^i(c,c^*)E_{11}^i[1-(E_{11}^i - p_{11}^i(c,c^*))^2]^{-\frac{1}{2}}E_{1j}^i \tag{4.1}$$

which makes sense if c is close enough to b. That is, there is a positive number r such that if $\|c-b\| < r$, then $\theta(c) \in \mathcal{U}$ and $\theta(c)b\theta(c)^* = c$.

Now let $x, y \in \mathcal{U}(b)$ such that $\|x-y\| < r$. Since $x = ubu^*$, $\|u^*yu - b\| < r$ and we can compute $\theta(u^*yu)$. Let us define $s_x(y) = u\theta(u^*yu)u^*$. It is easy to see that this definition does not depend on the choice of $u \in \pi_b^{-1}(x)$, moreover $s_x(y)$ is unitary and verifies $s_x(y)xs(y)^* = y$.

Also, $s_x(y)$ can be explicitely computed, using the expression of θ writen above:

$$s_x(y) = \sum_{i=1}^{p}\sum_{j=1}^{n_i} p_{j1}^i(y,y^*)p_{11}^i(x,x^*)[1-(p_{11}^i(x,x^*)-$$

$$-p_{11}^i(y,y^*))^2]^{-\frac{1}{2}}p_{1j}^i(x,x^*).$$

THEOREM 4.2. *The horizontal lifting Γ of γ, with $\Gamma(\alpha) = u_\alpha$ is the unique solution of the following linear differential equation:*

$$\begin{cases} \dot\Gamma(t) = \displaystyle\sum_{i=1}^{r}\sum_{j=1}^{n_i}\left[\frac{d}{dt}p_{j1}^i(\gamma(t),\gamma(t)^*)\right]p_{1j}^i(\gamma(t),\gamma(t)^*)\Gamma(t) \\ \Gamma(\alpha) = u_\alpha \end{cases}.$$

The proof is straightforward, it reduces to verify that the horizontal lifting satisfies the equation. ■

COROLLARY 4.3. *The unique geodesic $\psi : \mathbf{R} \to \mathcal{U}(b)$ with $\psi(0) = c \in \mathcal{U}(b)$ and $\dot\psi(0) = X \in R\delta_c$ is given by*

$$\psi(t) = \pi_c\left(e^{t\sum_{i=1}^{p}\sum_{j=1}^{n_i}[dp_{j1}^i)_{(c,c^*)}(X,X^*)]p_{1j}^i(c,c^*)}\right), \quad t \in \mathbf{R}.$$

■

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