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PARTIAL O^*-ALGEBRAS GENERATED BY TWO CLOSED SYMMETRIC OPERATORS

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1. INTRODUCTION

In this paper we shall study the structure, the standardness and the standard extensions of the partial O^*-algebra $\mathfrak{M}(A, B)$ generated by weakly commuting, symmetric operators A ans B defined a comon dense domain in a Hilbert space.

Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all linear maps X from $\mathcal{D}$ into $\mathcal{H}$ such that $\mathcal{D}(X^*) \supset \mathcal{D}$, and equip $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ with the usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \longrightarrow X^\dagger \equiv X^*|\mathcal{D}$ and the weak partial multiplication $X_1 \mathbin{\square} X_2 = X_1^{\dagger *} X_2$, defined whenever X_1 is a weak left multiplier of X_2 ($X_1 \in L^{\mathrm{w}}(X_2)$ or $X_2 \in R^{\mathrm{w}}(X_1^*)$), that is, iff $X_2\mathcal{D} \subset \mathcal{D}(X_1^{\dagger *})$ and $X_1^\dagger \mathcal{D} \subset \mathcal{D}(X_2^*)$. Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial $*$-algebra, that is, it is a vector space with the involution $X \longrightarrow X^\dagger$ (i.e., $(X+\lambda Y)^\dagger = X^\dagger + \bar{\lambda} Y^\dagger$, $X^{\dagger\dagger} = X$) such that

(i) $X \in L^{\mathrm{w}}(Y)$ iff $Y^\dagger \in L^{\mathrm{w}}(X^\dagger)$;

(ii) whenever $X \in L^{\mathrm{w}}(Y)$ and $X \in L^{\mathrm{w}}(Z)$, $X \in L^{\mathrm{w}}(\lambda Y + \mu Z)$ for each $\lambda, \mu \in \mathbf{C}$ and $X \mathbin{\square} (\lambda Y + \mu Z) = \lambda(X \mathbin{\square} Y) + \mu(X \mathbin{\square} Z)$;

(iii) whenever $X \in L^{\mathrm{w}}(Y)$, $(X \mathbin{\square} Y)^\dagger = Y^\dagger \mathbin{\square} X^\dagger$.

A *partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$* is a $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$; that is, $\mathcal{M}$ is a subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ such that $X^\dagger \in \mathcal{M}$ for each $X \in \mathcal{M}$, and $X_1 \mathbin{\square} X_2 \in \mathcal{M}$ whenever $X_1, X_2 \in \mathcal{M}$ and $X_1 \in L^{\mathrm{w}}(X_2)$. In [2,3] the commutativity of a partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ is defined as follows: $\mathcal{M}$ is said to be *commutative* if for $X, Y \in \mathcal{M}$, $X \in L^{\mathrm{w}}(Y)$ iff $Y \in L^{\mathrm{w}}(X)$, and then $X \mathbin{\square} Y = Y \mathbin{\square} X$. We now define a new commutativity of $\mathcal{M}$ as follows: $\mathcal{M}$ is said to be *weakly commutative* if $(X\xi|Y\eta) = (Y^\dagger\xi|X^\dagger\eta)$ for each $X, Y \in \mathcal{M}$ and $\xi, \eta \in \mathcal{D}$. It is easily shown that if $\mathcal{M}$ is weakly commutative, then it is commutative, and the notion of weakly commutativity is better than that of commutativity as shown in this paper. If $X^* = \overline{X^\dagger}$ for each $X \in \mathcal{M}$, then $\mathcal{M}$ is said to be *standard*. When $\mathcal{M}$ is an O^*-algebra, that is, $X\mathcal{D} \subset \mathcal{D}$ for each $X \in \mathcal{M}$, $\mathcal{M}$ is standard iff $A^* = \bar{A}$ for each $A^\dagger = A \in \mathcal{M}$ [11], but we remark that the two notions need not be equivalent for partial O^*-algebras. $\mathcal{M}$ is said to be *self-adjoint* if $\mathcal{D}^*(\mathcal{M}) \equiv \bigcap_{X \in \mathcal{M}} \mathcal{D}(X^*) = \mathcal{D}$, and *essentially self-adjoint* if $\mathcal{D}^*(\mathcal{M}) = \hat{\mathcal{D}}(\mathcal{M}) \equiv \equiv \bigcap_{X \in \mathcal{M}} \mathcal{D}(\bar{X})$. For further details about partial O^*-algebras, refer to [1–5,9].

Let S and T be closed symmetric operators in $\mathcal{H}$ and $\mathcal{D} \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ be a core for S and T satisfing $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting. We denote by $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ the minimal partial O^*-algebra on $\mathcal{D}$ containing $S|\mathcal{D}$ and $T|\mathcal{D}$. When $S\mathcal{D} \subset \mathcal{D}$ and $T\mathcal{D} \subset \mathcal{D}$, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ equals the polynomial algebra $\mathfrak{P}(S|\mathcal{D}, T|\mathcal{D})$ of $S|\mathcal{D}$ and $T|\mathcal{D}$, and in this case the self-adjointness and the standardness of $\mathfrak{P}(S|\mathcal{D}, T|\mathcal{D})$ were investigated in [7,8,11]. But, when $T\mathcal{D} \not\subset \mathcal{D}$, even the partial O^*-algebra $\mathfrak{M}(T|\mathcal{D})$ generated by $T|\mathcal{D}$ the structure is tricky as we saw in [3,5]. In this paper we shall study the partial O^*-algebra $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ which is more tricky than $\mathfrak{M}(T|\mathcal{D})$.

In Section 2 we shall show that the regular part (the polynomial part) $\mathfrak{R}(S|\mathcal{D}, T|\mathcal{D})$ and the singular part $\mathfrak{S}(S|\mathcal{D}, T|\mathcal{D})$ of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ are defined and

$$\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) = \mathfrak{R}(S|\mathcal{D}, T|\mathcal{D}) + \mathfrak{S}(S|\mathcal{D}, T|\mathcal{D}).$$

In Section 3 we shall study the standardness of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$. We define the *strong commutant* of S and T by

$$\{S, T\}' = \{C \in \mathcal{B}(\mathcal{H}); CS \subset SC \text{ and } CT \subset TC\}.$$

When S and T are self-adjoint,

$$\{S, T\}' = \{E_S(\lambda), E_T(\mu); -\infty < \lambda, \mu < \infty\}',$$

where $\{E_S(\lambda); -\infty < \lambda < \infty\}$ and $\{E_T(\mu); -\infty < \mu < \infty\}$ are the spectral resolutions of S and T, respectively. We say that the self-adjoint operators S and T are *strongly commuting* if $E_S(\lambda)E_T(\mu) = E_T(\mu)E_S(\lambda)$ for each $\lambda, \mu \in \mathbf{R}$; equivalently, $\{S, T\}''$ is commutative. We shall show that $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is standard iff S and T are strongly commuting self-adjoint operators and $\{S, T\}'\hat{\mathcal{D}}(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})) \subset \hat{\mathcal{D}}(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}))$, and in this case, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is weakly commutative, and it is isomorphic to a commutative partial $*$-algebra of polynomials with two variables.

In Section 4 we shall study self-adjoint extensions and standard extensions of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$. We shall define the notions of an *extension*, a *multiplicative-extension* and a *quasi-extension* of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$, and show that if S and T have self-adjoint extensions with a domain condition, then $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a self-adjoint quasi-extension; and $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a standard quasi-extension if and only if S and

T have strongly commuting self-adjoint extensions; and $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ has a standard multiplicative-extension if and only if S and T have strongly commuting self-adjoint extensions A and B, respectively and $\{A,B\}'$ is contained in the quasi-weak commutant $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})'_{\mathrm{qw}}$ of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$.

2. THE STRUCTURE OF $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$

Let S and T be closed symmetric operators in a Hilbert space $\mathcal{H}$ and $\mathcal{D}$ a dense subspace in $\mathcal{H}$ contained in $\mathcal{D}(S)\cap\mathcal{D}(T)$. Suppose $\mathcal{D}$ is a core for S and T, and $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting; that is, $(S\xi|T\eta)=(T\xi|S\eta)$ for each $\xi,\eta\in\mathcal{D}$. Let m_0 (resp. n_0) be the largest number in $k\in\mathbf{N}\cup\{\infty\}$ satisfying $\mathcal{D}\subset\mathcal{D}(S^k)$ (resp. $\mathcal{D}\subset\mathcal{D}(T^k)$), and let m_1 (resp. $m_2,\dots,m_{n_0}$) be the largest number in $k\in\mathbf{N}\cup\{0\}\cup\{\infty\}$ satisfying $\mathcal{D}\subset\mathcal{D}(S^kT)\cap\mathcal{D}(TS^k)$ (resp. $\mathcal{D}\subset$ $\subset\mathcal{D}(S^kT^2)\cap\mathcal{D}(T^2S^k),\dots,\mathcal{D}\subset\mathcal{D}(S^kT^{n_0})\cap\mathcal{D}(T^{n_0}S^k)$). Clearly we have

$$m_0\geqslant m_1\geqslant\cdots\geqslant m_{n_0}. \tag{2.1}$$

Furthermore, if $\mathcal{D}\subset\mathcal{D}(ST)\cap\mathcal{D}(TS)$, then

$$(ST\xi|\eta)=(T\xi|S\eta)=(S\xi|T\eta)=(TS\xi|\eta)$$

for all $\xi,\eta\in\mathcal{D}$, and so $ST\xi=TS\xi$ for each $\xi\in\mathcal{D}$. If $\mathcal{D}\subset\mathcal{D}(S^2T)\cap\mathcal{D}(TS^2)$, then

$$(S^2T\xi|\eta)=(ST\xi|S\eta)=(TS\xi|S\eta)=$$

$$=(S\xi|TS\eta)=(S\xi|ST\eta)=(TS^2\xi|\eta)$$

for all $\xi,\eta\in\mathcal{D}$, and so $S^2T\xi=TS^2\xi$ for each $\xi\in\mathcal{D}$. Repeating this, we can show

$$\text{if } \mathcal{D}\subset\mathcal{D}(S^kT^l)\cap\mathcal{D}(T^lS^k),\ \text{then } S^kT^l\xi=T^lS^k\xi \quad \text{for each } \xi\in\mathcal{D}. \tag{2.2}$$

We define the *strong power length* of $S|\mathcal{D}$ and $T|\mathcal{D}$ and the *strongly regular part* of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ by

$$\ell_{\mathrm{s}}(S,T,\mathcal{D})=[m_0,n_0,m_1,m_2,\dots,m_{n_0}],$$

$$\mathfrak{R}_{\mathrm{s}}(S|\mathcal{D},T|\mathcal{D})=$$

$$=\left\{\sum_{k=0}^{m_0}\alpha_{k0}S^k|\mathcal{D}+\sum_{k=0}^{m_1}\alpha_{k1}S^kT|\mathcal{D}+\cdots+\sum_{k=0}^{m_{n_0}}\alpha_{kn_0}S^kT^{n_0}|\mathcal{D};\alpha_{kl}\in\mathbf{C}\right\},$$

respectively. We next define the weak power length of $S|\mathcal{D}$ and $T|\mathcal{D}$. Let $m(0)$ (resp. $n(0)$) be the largest number in $k \in \mathbb{N} \cup \{\infty\}$ satisfying $\mathcal{D} \subset \mathcal{D}(S^{*k})$ (resp. $\mathcal{D} \subset \mathcal{D}(T^{*k})$), and let $m(1)$ (resp. $m(2), \dots m(n(0))$) be the largest number in $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ satisfying $\mathcal{D} \subset \mathcal{D}(S^{*k}T^{*}) \cap \mathcal{D}(T^{*}S^{*k})$ (resp. $\mathcal{D} \subset \mathcal{D}(S^{*k}T^{*2}) \cap \cap \mathcal{D}(T^{*2}S^{*k}), \dots, \mathcal{D} \subset \mathcal{D}(S^{*k}T^{*n(0)}) \cap \mathcal{D}(T^{*n(0)}S^{*k})$). Then we have

$$
\begin{array}{ccccccccccc}
m(0) & \geqslant & m(1) & \geqslant & \cdots & \geqslant & m(n_0) & \geqslant & \cdots & \geqslant & m(n(0)) \\
\mathbin{\rotatebox{90}{$\geqslant$}} & & \mathbin{\rotatebox{90}{$\geqslant$}} & & & & \mathbin{\rotatebox{90}{$\geqslant$}} & & & & \\
m_0 & \geqslant & m_1 & \geqslant & \cdots & \geqslant & m_{n_0}. & & & &
\end{array}
\tag{2.3}
$$

Furthermore, we can show in similar to (2.2) the following

(2.4) if $\mathcal{D} \subset \mathcal{D}(S^{*k}T^{*l}) \cap \mathcal{D}(T^{*l}S^{*k})$, then $S^{*k}T^{*l}\xi = T^{*l}S^{*k}\xi$ for each $\xi \in \mathcal{D}$.

Hence, we can define the *weak power length* of $S|\mathcal{D}$ and $T|\mathcal{D}$ by

$$\ell_{\mathrm{w}}(S, T, \mathcal{D}) = [m(0), n(0), m(1), \dots, m(n(0))].$$

We now put

$$P_0(S, T, \mathcal{D}) =$$

$$= \left\{ p^{(0)}(\lambda, \mu) = \sum_{k=0}^{m(0)} \alpha_{k0}\lambda^k + \sum_{k=0}^{m(1)} \alpha_{k1}\lambda^k\mu + \cdots + \sum_{k=0}^{m(n(0))} \alpha_{kn(0)}\lambda^k\mu^{n(0)}; \alpha_{kl} \in \mathbb{C} \right\},$$

$$\mathfrak{R}(S|\mathcal{D}, T|\mathcal{D}) =$$

$$= \left\{ p^{(0)}(S, T) = \sum_{k=0}^{m(0)} \alpha_{k0}S^{*k}|\mathcal{D} + \sum_{k=0}^{m(1)} \alpha_{k1}S^{*k}T^{*}|\mathcal{D} + \cdots \right.$$

$$\left. \cdots + \sum_{k=0}^{m(n(0))} \alpha_{kn(0)}S^{*k}T^{*n(0)}|\mathcal{D}; \alpha_{kl} \in \mathbb{C} \right\},$$

$$\mathfrak{Q}_1(S|\mathcal{D}, T|\mathcal{D}) = \text{linear span of}$$

$$\{q^{(1)}(S, T) = p_1^{(0)}(S, T) \mathbin{\square} p_2^{(0)}(S, T); p_1^{(0)}, p_2^{(0)} \in P_0(S, T, \mathcal{D})$$

$$\text{and } p_1^{(0)}(S, T) \in L^{\mathrm{w}}(p_2^{(0)}(S, T))\},$$

$$\mathfrak{Q}_2(S|\mathcal{D}, T|\mathcal{D}) = \text{linear span of}$$

$$\{q_1^{(1)}(S, T) \mathbin{\square} q_2^{(1)}(S, T); q_1^{(1)}(S, T), q_2^{(1)}(S, T) \in \mathfrak{Q}_1(S|\mathcal{D}, T|\mathcal{D})$$

$$\text{and } q_1^{(1)}(S, T) \in L^{\mathrm{w}}(q_2^{(1)}(S, T))\},$$

$$\cdots .$$

Then we have

$$(2.5)\qquad \mathfrak{R}_s(S|\mathcal{D},T|\mathcal{D}) \subset \mathfrak{R}(S|\mathcal{D},T|\mathcal{D}) \subset \mathfrak{Q}_1(S|\mathcal{D},T|\mathcal{D}) \subset \cdots .$$

It is clear that $\mathfrak{R}_s(S|\mathcal{D},T|\mathcal{D}) = \mathfrak{R}(S|\mathcal{D},T|\mathcal{D})$ if S and T are self-adjoint. We put

$$\mathfrak{S}_1(S|\mathcal{D},T|\mathcal{D}) = \mathfrak{Q}_1(S|\mathcal{D},T|\mathcal{D}) - \mathfrak{R}(S|\mathcal{D},T|\mathcal{D}),$$

$$\mathfrak{S}_2(S|\mathcal{D},T|\mathcal{D}) = \mathfrak{Q}_2(S|\mathcal{D},T|\mathcal{D}) - \mathfrak{Q}_1(S|\mathcal{D},T|\mathcal{D}),$$

$$\cdots$$

$$\mathfrak{S}(S|\mathcal{D},T|\mathcal{D}) = \bigcup_{k=1}^{\infty} \mathfrak{S}_k(S|\mathcal{D},T|\mathcal{D}).$$

Then we have the following

THEOREM 2.1. *Let S and T be closed symmetric operators in $\mathcal{H}$ and $\mathcal{D} \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ be a core for S and T satisfying $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting. Then*

$$\mathfrak{M}(S|\mathcal{D},T|\mathcal{D}) = \bigcup_{k=1}^{\infty} \mathfrak{Q}_k(S|\mathcal{D},T|\mathcal{D}) = $$
$$= \mathfrak{R}(S|\mathcal{D},T|\mathcal{D}) + \mathfrak{S}(S|\mathcal{D},T|\mathcal{D}).$$

$\mathfrak{R}(S|\mathcal{D},T|\mathcal{D})$ is called the regular part of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ and $\mathfrak{S}(S|\mathcal{D},T|\mathcal{D})$ is called the singular part of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$.

REMARK 2.2. Even if $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is standard, the singular part of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ need not be empty unlike $\mathfrak{M}(T|\mathcal{D})$. Let $k_1, k_2, l_1, l_2 \in \mathbb{N}$ such that $0 \leqslant l_1, l_2 \leqslant n(0)$, $k_1 \leqslant m(l_1)$ and $k_2 \leqslant m(l_2)$. When $l_1 + l_2 > n(0)$, or $l_1 + l_2 \leqslant n(0)$ and $k_1 + k_2 > m(l_1 + l_2)$,

$$((S-\mathrm{i})^{k_1}(T-\mathrm{i})^{l_1}|\mathcal{D}) \mathbin{\square} ((S-\mathrm{i})^{k_2}(T-\mathrm{i})^{l_2}|\mathcal{D}) \notin \mathfrak{M}(S|\mathcal{D},T|\mathcal{D}),$$

but $(S^{k_1}T^{l_1}|\mathcal{D}) \mathbin{\square} (S^{k_2}T^{l_2}|\mathcal{D})$ is possible to be contained in $\mathfrak{S}_1(S|\mathcal{D},T|\mathcal{D})$ because $(S-\mathrm{i})^k(T-\mathrm{i})^l$ is closed but S^kT^l need not be closed for each $k, l \in \mathbb{N}$.

3. STANDARDNESS OF $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$

In this section we study the standardness of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ and show that $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is isomorphic to a partial $*$-algebra of polynomials with two variables when $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is standard.

For the standardness of $\mathfrak{M}(T|\mathcal{D})$ we have obtained the following result in ([3] Proposition 4.9):

LEMMA 3.1. *Let T be a closed symmetric operator in $\mathcal{H}$ and a subspace $\mathcal{D}$ in $\mathcal{H}$ be a core for T. Let n be the largest number in $k \in \mathbb{N} \cup \{\infty\}$ such that $\mathcal{D} \subset \mathcal{D}(T^k)$. Then the following statements are equivalent:*

(1) $\mathfrak{M}(T|\mathcal{D})$ is standard.

(2) $\mathfrak{M}(T|\mathcal{D}) = \mathfrak{P}_n(T|\mathcal{D}) \equiv \left\{ \sum_{k=0}^{n} \alpha_k T^k|\mathcal{D};\ \alpha_k \in \mathbb{C},\ k = 1, 2, \ldots, n \right\}$ and it is ess. self-adjoint.

(3) T is self-adjoint and $\hat{\mathcal{D}}(\mathfrak{M}(T|\mathcal{D})) = \mathcal{D}(T^m)$ for some $m \in \mathbb{N} \cup \{\infty\}$.

(4) $T|\mathcal{D}, T^2|\mathcal{D}, \ldots, T^n|\mathcal{D}$ are ess. self-adjoint.

We have the following result for the standardness of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$:

THEOREM 3.2. *Let S and T be closed symmetric operators in $\mathcal{H}$ and $\mathcal{D} \subset \mathcal{D}(S) \cap \cap \mathcal{D}(T)$ be a core for S and T satisfying $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting. Then the following statements are equivalent:*

(1) $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is standard.

(2) S and T are strongly commuting self-adjoint operators and $\{S,T\}'\hat{\mathcal{D}}(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})) \subset \hat{\mathcal{D}}(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}))$.

In this case, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is weakly commutative and $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{w}} = \{S,T\}'$.

Proof. (1) $\Rightarrow$ (2). We put

$$A = \overline{S|\mathcal{D} + \mathrm{i}T|\mathcal{D}}.$$

Since $S|\mathcal{D}$, $T|\mathcal{D}$, $S|\mathcal{D} + \mathrm{i}T|\mathcal{D}$ and $S|\mathcal{D} - \mathrm{i}T|\mathcal{D}$ belong to the standard partial O^*-algebra $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$, it follows that S and T are self-adjoint operators, and $A = (S|\mathcal{D} - \mathrm{i}T|\mathcal{D})^* = \overline{S + \mathrm{i}T}$. For each $\xi \in \mathcal{D}(\overline{S + \mathrm{i}T})$ there exists a sequence $\{\xi_n\}$ in $\mathcal{D}$ such that $\lim_{n\to\infty} \xi_n = \xi$ and $\lim_{n\to\infty} (S + \mathrm{i}T)\xi_n = \overline{S + \mathrm{i}T}\xi$. Since $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting, we have

$$\|(S + \mathrm{i}T)\xi_n - (S + \mathrm{i}T)\xi_m\|^2 = \|S\xi_n - S\xi_m\|^2 + \|T\xi_n - T\xi_m\|^2 +$$

$$+ \mathrm{i}(T(\xi_n - \xi_m)|S(\xi_n - \xi_m)) - \mathrm{i}(S(\xi_n - \xi_m)|T(\xi_n - \xi_m)) =$$

$$= \|S\xi_n - S\xi_m\|^2 + \|T\xi_n - T\xi_m\|^2,$$

which implies that $\xi \in \mathcal{D}(S) \cap \mathcal{D}(T)$, $\lim_{n\to\infty} S\xi_n = S\xi$ and $\lim_{n\to\infty} T\xi_n = T\xi$. Hence, $S + \mathrm{i}T$ is closed, and so

$$A = (S|\mathcal{D} - \mathrm{i}T|\mathcal{D})^{**} = S + \mathrm{i}T. \tag{3.1}$$

Similarly, we have

$$A^* = (S|\mathcal{D} - iT|\mathcal{D})^{**} = \overline{S|\mathcal{D} - iT|\mathcal{D}} = S - iT. \tag{3.2}$$

It follows from (3.1) and (3.2) that A is normal. Therefore, S and T are strongly commuting self-adjoint operators. Since $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is standard, it follows that for $p_1^{(0)}, p_2^{(0)} \in P_0(S,T,\mathcal{D})$, $q^{(1)}(S,T) = \underline{p_1^{(0)}(S,T)} \,\square\, p_2^{(0)}(S,T)$ exists iff $p_2^{(0)}(S,T)\mathcal{D} \subset \subset \mathcal{D}(\overline{p_1^{(0)}(S,T)})$ and $\overline{p_1^{(0)}(S,T)\mathcal{D}} \subset \mathcal{D}(\overline{p_2^{(0)}}(S,T))$, and then $\mathcal{D} \subset \mathcal{D}(\overline{(p_1^{(0)}p_2^{(0)})(S,T)})$ and $q^{(1)}(S,T) = \overline{(p_1^{(0)}p_2^{(0)})(S,T)}|\mathcal{D}$. Repeating this argument, we can show that every element X of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is represented as

$$X = \overline{p(S,T)}|\mathcal{D}$$

for some polynomial $p(\lambda, \mu)$ of two variables, which implies

$$\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{w}} = \{S, T\}' = \{E_S(\lambda), E_T(\mu); -\infty < \lambda, \mu < \infty\}'.$$

Therefore, we have

$$\{S,T\}'\hat{\mathcal{D}} = \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{w}}\hat{\mathcal{D}} \subset \hat{\mathcal{D}},$$

where $\hat{\mathcal{D}} = \hat{\mathcal{D}}(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}))$, by the ess. self-adjointness of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

(2) $\Rightarrow$ (1). Since S and T are strongly commuting self-adjoint operators, it follows that $\{S,T\}'' = \{E_S(\lambda), E_T(\mu); -\infty < \lambda, \mu < \infty\}''$ is a commutative von Neumann algebra on $\mathcal{H}$. We put

$$\mathcal{M} = \{X \in \mathcal{L}^\dagger(\hat{\mathcal{D}}, \mathcal{H}); \bar{X} \text{ is affiliated with } \{S,T\}''\}.$$

We show that $\mathcal{M}$ is a weakly commutative standard partial O^*-algebra on $\hat{\mathcal{D}}$. Since $\mathcal{M}'_{\mathrm{w}} = \{S, T, \}'$ and $\mathcal{M}'_{\mathrm{w}}\hat{\mathcal{D}} \subset \hat{\mathcal{D}}$, it follows that $\mathcal{M}$ is a partial O^*-algebra on $\hat{\mathcal{D}}$. Furthermore, since $\{S,T\}''$ is commutative, it follows that $\mathcal{M}$ is weakly commutative, and $\mathcal{M}$ is standard, that is, $X^* = \overline{X^\dagger}$ for each $X \in \mathcal{M}$ [10]. Since $S|\hat{\mathcal{D}}, T|\hat{\mathcal{D}} \in \mathfrak{M}$, it follows that $\mathcal{M}(S|\hat{\mathcal{D}}, T|\hat{\mathcal{D}}) \subset \mathcal{M}$, so that $\mathfrak{M}(S|\hat{\mathcal{D}}, T|\hat{\mathcal{D}})$ is weakly commutative and standard. Furthermore, it follows that $\hat{\imath} : X \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) \longrightarrow \hat{X} = \bar{X}|\hat{\mathcal{D}} \in \in \hat{\mathfrak{M}}(S|\mathcal{D}, T|\mathcal{D})$ is a $*$-isomorphism [2,3], which implies $\hat{\mathfrak{M}}(S|\mathcal{D}, T|\mathcal{D}) = \mathfrak{M}(S|\hat{\mathcal{D}}, T|\hat{\mathcal{D}})$. Therefore, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is weakly commutative and standard. This completes the proof.

By Theorem 2.1 and Theorem 3.2 we shall decide the structure of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ when $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is standard.

We first define a partial $*$-algebra of polynomials with two variables defined by $(S, T, \mathcal{D})$. Let $(S, T, \mathcal{D})$ be as in Theorem 2.1 and let $\ell_{\mathrm{w}}(S, T, \mathcal{D}) = [m(0), n(0), m(1), \ldots, m(n(0))]$. We put

$$P_0(S, T, \mathcal{D}) =$$

$$= \left\{ p^{(0)}(\lambda,\mu) = \sum_{k=0}^{m(0)} \alpha_{k0}\lambda^k + \sum_{k=0}^{m(1)} \alpha_{k1}\lambda^k\mu + \cdots + \sum_{k=0}^{m(n(0))} \alpha_{kn(0)}\lambda^k\mu^{n(0)};\, \alpha_{kl} \in \mathbf{C} \right\},$$

$$P_1(S,T,\mathcal{D}) = \text{ linear span of}$$

$$\Big\{ p_1^{(0)}p_2^{(0)};\, p_1^{(0)}, p_2^{(0)} \in P_0(S,T,\mathcal{D}) \text{ and}$$

$$\mathcal{D} \subset \mathcal{D}(\overline{(p_1^{(0)}p_2^{(0)})(S,T)}) \cap \mathcal{D}(\overline{(\overline{p_1^{(0)}p_2^{(0)}})(S,T)}) \Big\},$$

$$P_2(S,T,\mathcal{D}) = \text{ linear span of}$$

$$\Big\{ p_1^{(1)}p_2^{(1)};\, p_1^{(1)}, p_2^{(1)} \in P_0(S,T,\mathcal{D}) \text{ and}$$

$$\mathcal{D} \subset \mathcal{D}(\overline{(p_1^{(1)}p_2^{(1)})(S,T)}) \cap \mathcal{D}(\overline{(\overline{p_1^{(1)}p_2^{(1)}})(S,T)}) \Big\},$$

$$\cdots$$

Then

$$P_0(S,T,\mathcal{D}) \subset P_1(S,T,\mathcal{D}) \subset \cdots \subset P_k(S,T,\mathcal{D}) \subset \cdots .$$

We now put

$$P(S,T,\mathcal{D}) = \bigcup_{k=0}^{\infty} P_k(S,T,\mathcal{D}).$$

Then $P(S,T,\mathcal{D})$ is a commutative partial $*$-algebra with the involution $p \longrightarrow \bar{p}$ and the partial multiplication:

$$\text{the partial multiplication } p_1 \bullet p_2 \text{ exists}$$

$$\text{iff } \mathcal{D} \subset \mathcal{D}(\overline{(p_1p_2)(S,T)}) \cap \mathcal{D}(\overline{(\overline{p_1p_2})(S,T)}),$$

and then $p_1 \bullet p_2 = p_1p_2$. $P(S,T,\mathcal{D})$ is said to be a *partial $*$-algebra of polynomials defined by* $(S,T,\mathcal{D})$. For partial $*$-algebras, refer to [1,2,3].

THEOREM 3.3. *Let $(S,T,\mathcal{D})$ be as in Theorem 2.1. Suppose $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is standard. Then it is isomorphic to a partial $*$-algebra $P(S,T,\mathcal{D})$ of polynomials defined by $(S,T,\mathcal{D})$.*

Proof. It is clear that the map $p^{(0)} \in P_0(S,T,\mathcal{D}) \longrightarrow \overline{p^{(0)}(S,T)}|\mathcal{D} = p^{(0)}(S,T) \in$ $\in \mathfrak{R}(S|\mathcal{D},T|\mathcal{D})$ is a $*$-invariant linear bijection. We show that for $p_1^{(0)}, p_2^{(0)} \in P_0(S,T,\mathcal{D})$, $p_1^{(0)}(S,T) \,\square\, p_2^{(0)}(S,T)$ exists iff $\mathcal{D} \subset (\mathcal{D}(\overline{p_1^{(0)}p_2^{(0)}(S,T)})) \cap \mathcal{D}(\overline{(\overline{p_1^{(0)}p_2^{(0)}})(S,T)})$. In fact, suppose $\mathcal{D} \subset (\mathcal{D}(\overline{p_1^{(0)}p_2^{(0)})(S,T)}) \cap \mathcal{D}(\overline{(\overline{p_1^{(0)}p_2^{(0)}})(S,T)})$. Since S and T are strongly commuting self-ajoint operators by Theorem 3.2, we have

$$(\overline{p_1^{(0)}}(S,T)\eta | p_2^{(0)}(S,T)\xi) =$$

$$= \lim_{n\to\infty} \overline{(\overline{p_2^{(0)}p_1^{(0)}}(S,T)E_S([-n,n])E_T([-n,n])\eta|\xi)} =$$

$$= \overline{(\overline{p_2^{(0)}p_1^{(0)}}(S,T)\eta|\xi)} =$$

$$= (\eta|\overline{(p_1^{(0)}p_1^{(0)}}(S,T)\xi)$$

and

$$(p_2^{(0)}(S,T)\eta|\overline{p_1^{(0)}}(S,T)\xi) = (\eta|\overline{((\overline{p_2^{(0)}p_1^{(0)}})(S,T)\xi)}$$

for each $\xi,\eta \in \mathcal{D}$, which implies that $p_1^{(0)}(S,T) \,\square\, p_2^{(0)}(S,T)$ exists. The converse is trivial. Hence, the map $p^{(1)} \in P_1(S,T,\mathcal{D}) \longrightarrow \overline{p^{(1)}(S,T)}|\mathcal{D} \in \mathfrak{Q}_1(S|\mathcal{D},T|\mathcal{D})$ is a $*$-invariant linear bijection. Similarly, the map $p^{(k)} \in P_k(S,T,\mathcal{D}) \to \overline{P^{(k)}(S,T,\mathcal{D})}|\mathcal{D} \in \mathfrak{Q}_k(S|\mathcal{D},T|\mathcal{D})$ is a $*$-invariant linear bijection. By Theorem 2.1 the map $p \in P(S,T,\mathcal{D}) = \bigcup_{k=0}^{\infty} P_k(S,T,\mathcal{D}) \longrightarrow \overline{P(S,T)}|\mathcal{D} \in \mathfrak{M}(S|\mathcal{D},T|\mathcal{D}) = \bigcup_{k=0}^{\infty} \mathfrak{Q}_k(S|\mathcal{D},T|\mathcal{D})$ is a $*$-invariant linear bijection. Furthermore, it follows that for $p_1,p_2 \in P(S,T,\mathcal{D})$, $p_1 \bullet p_2$ exists iff $\mathcal{D} \subset \mathcal{D}(\overline{(p_1p_2)(S,T)}) \cap \mathcal{D}(\overline{(\overline{p_1p_2})(S,T)})$ iff $\overline{p_1(S,T)}|\mathcal{D} \,\square\, \overline{p_2(S,T)}|\mathcal{D}$ exists. Therefore, the map $p \longrightarrow \overline{p(S,T)}|\mathcal{D}$ is a $*$-isomorphism of the partial $*$-algebra $P(S,T,\mathcal{D})$ onto the partial O^*-algebra $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$. This completes the proof.

COROLLARY 3.4. *Let $(S,T,\mathcal{D})$ be as in Theorem 2.1. Suppose*

(i) $\mathcal{D} \subset \mathcal{D}(S^2)\cap\mathcal{D}(T^2)\cap\mathcal{D}(ST)\cap\mathcal{D}(TS)$ and $S^2|\mathcal{D}+T^2|\mathcal{D}$ is ess. self-adjoint,

(ii) $\{S,T\}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$.

Then $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is standard.

Proof. Since S and T are weakly commuting and $\mathcal{D} \subset \mathcal{D}(ST)\cap\mathcal{D}(TS)$, we have $ST|\mathcal{D} = TS|\mathcal{D}$. Hence, we can show that the ess. self-adjointness of $(S^2+T^2)|\mathcal{D}$ implies the normality of $\overline{S|\mathcal{D}+\mathrm{i}T|\mathcal{D}}$, which futher implies that S and T are strongly commuting self-adjoint operators. Hence, $(\{S|\mathcal{D},T|\mathcal{D}\}'_{\mathrm{w}})' = \{E_S(\lambda),E_T(\mu); -\infty < \lambda,\mu < \infty\}''$ is commutative, and so it is shown in similar to the proof of Theorem 3.2 that $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is contained in the standard partial O^*-algebra $\mathcal{M} \equiv \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H});\ \bar{X}$ is affiliated with $(\{S|\mathcal{D},T|\mathcal{D}\}'_{\mathrm{w}})'\}$. Therefore, $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ is standard.

Suppose that symmetric operators A, B, N in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ satisfy the following conditions:

(i) N is ess. self-adjoint.

(ii) There exists a constant $\gamma > 0$ such that

$$\|A\xi\| \leqslant \gamma\|(N+\mathrm{i})\xi\| \quad \text{and} \quad \|B\xi\| \leqslant \gamma\|(N+\mathrm{i})\xi\|$$

for each $\xi \in \mathcal{D}$.

(iii) $A \,\square\, B = B \,\square\, A$, $A \,\square\, N = N \,\square\, A$ and $B \,\square\, N = N \,\square\, B$.

Then we have the following

COROLLARY 3.5. (1) *Suppose* $\{A, N\}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. *Then* $\mathfrak{M}(A, N)$ *is standard.*

(2) *Suppose* $\{A, B\}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. *Then* $\mathfrak{M}(A, B)$ *is standard.*

Proof. By ([14] Lemma 1, Proposition 2), any pairs of $\bar{A}$, $\bar{B}$ and $\tilde{N}$ are strongly commuting self-adjoint operators, and so it is proved in similar to the proof of Corollary 3.4 that $\mathfrak{M}(A, N)$ and $\mathfrak{M}(A, B)$ are standard.

We finally investigate the partial O^*-algebra $\mathfrak{M}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}})$ on the domain D_{mn}:

$$D_{\mathrm{mn}} = \{\xi \in \mathcal{H}; \xi \in \mathcal{D}(S^kT^l) \cap \mathcal{D}(T^lS^k) \text{ and }$$
$$S^kT^l\xi = T^lS^k\xi \text{ for } 1 \leqslant \forall k \leqslant m \text{ and } 1 \leqslant \forall l \leqslant n\},$$
$$m, n \in \mathbf{N} \cup \{\infty\}.$$

These domains D_{mn} have been considered by Schmüdgen [13] and Schmüdgen and Friedrich [15] for the study of a pair of closed symmetric operators.

PROPOSITION 3.6. *Let* S *and* T *be closed symmetric operators in* $\mathcal{H}$. *Suppose* D_{mn} *is a core for* S *and* T. *Then the following statements hold.*

(1) $$\mathfrak{P}_{\mathrm{mn}}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}}) \equiv \left\{\sum_{k=0}^{m}\sum_{l=0}^{n} \alpha_{kl}S^kT^l|D_{\mathrm{mn}}; \alpha_{kl} \in \mathbf{C}\right\} \subset$$
$$\subset \mathfrak{N}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}}) \subset \mathfrak{M}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}}).$$

(2) *Suppose* S *and* T *are self-adjoint. Then* $\mathfrak{M}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}})$ *is self-adjoint,*

(3) S *and* T *are strongly commuting self-adjoint operators iff* $\mathfrak{M}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}})$ *is standard.*

Proof. (1) This is trivial.

(2) Take an arbitrary $\xi \in \mathcal{D}^*(\mathfrak{M}(S|D_{\mathrm{mn}}, T|D_{\mathrm{mn}}))$. Since D_{mn} is a core for the self-adjoint operators S, we have $\xi \in \mathcal{D}(S)$, and so

$$(S\eta|S\xi) = (S^2\eta|\xi) = (\eta|(S^2|D_{\mathrm{mn}})^*\xi)$$

for each $\eta \in D_{\mathrm{mn}}$. Hence, $\xi \in \mathcal{D}(S^2)$ and $S^2\xi = (S^2|D_{\mathrm{mn}})^*\xi$. Repeating this, we have

$$\xi \in \mathcal{D}(S^m) \text{ and } S^m\xi = (S^m|D_{\mathrm{mn}})^*\xi. \tag{3.3}$$

Similarly, we have

$$\xi \in \mathcal{D}(T^n) \text{ and } T^n\xi = (T^n|D_{\mathrm{mn}})^*\xi. \tag{3.4}$$

Let $k, l \in \mathbf{N}$ with $0 \leqslant k \leqslant m$ and $0 \leqslant l \leqslant n$. Since

$$(S\eta|T^l\xi) = (T^lS\eta|\xi) = (\eta|(T^lS|D_{\mathrm{mn}})^*\xi)$$

for each $\eta \in D_{mn}$, it follows that $\xi \in \mathcal{D}(ST^l)$ and $ST^l\xi = (T^lS|D_{mn})^*\xi$. Repeating this, we have

$$\xi \in \mathcal{D}(S^kT^l) \text{ and } S^kT^l\xi = (T^lS^k|D_{mn})^*\xi. \tag{3.5}$$

Similarly, we have

$$\xi \in \mathcal{D}(T^lS^k) \text{ and } T^lS^k\xi = (S^kT^l|D_{mn})^*\xi. \tag{3.6}$$

By (3.5) and (3.6) we have $\xi \in D_{mn}$. Therefore, $\mathfrak{M}(S|D_{mn}, T|D_{mn})$ is self-adjoint.

(3) Suppose S and T are strongly commuting self-adoint operators. Then it is clear that $\{S,T\}'D_{mn} \subset D_{mn}$, which implies by Theorem 3.2 that $\mathfrak{M}(S|D_{mn}, T|D_{mn})$ is standard. The converse follows from Theorem 3.2.

4. SELF-ADJOINT EXTENSIONS AND STANDARD EXTENSIONS OF $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$

In this section we investigate self-adjoint extensions and standard extensions of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$. Let S and T be closed symmetric operators in $\mathcal{H}$ and $\mathcal{D} \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ be a core for S and T satisfying $S|\mathcal{D}$ and $T|\mathcal{D}$ are weakly commuting.

DEFINITION 4.1. When a triple $(A, B, \mathcal{E})$ satisfies the following conditions:

(i) $\mathcal{E}$ is a subspace in $\mathcal{H}$ containing $\mathcal{D}$;

(ii) A and B are closed symmetric extensions of S and T, respectively, $\mathcal{E}$ is a core for A and B and $A|\mathcal{E}$ and $B|\mathcal{E}$ are weakly commuting;

(iii) $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})|\mathcal{D} = \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$,

$\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ is said to be an *extension* of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

Let $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ be an extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$. Then, the map $\iota : X \longrightarrow X|\mathcal{D}$ is a $*$-homomorphism of the partial O^*-algebra $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ onto the partial O^*-algebra $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ (i.e., ι is a $\dagger$-invariant linear map and if $X_1 \,\square\, X_2$ exists, then $\iota(X_1) \,\square\, \iota(X_2)$ exists and $\iota(X_1 \,\square\, X_2) = \iota(X_1) \,\square\, \iota(X_2)$) and one-to-one, but ι^{-1} is not necessarily a $*$-homomorphism. Hence, we need the definition of the stronger extension:

DEFINITION 4.2. An extension $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is said to be *multiplicative* if the map $\iota : X \in \mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}) \longrightarrow X|\mathcal{D} \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ is a $*$-isomorphism (i.e., both ι and ι^{-1} are $*$-homomorphism).

The closure $\tilde{\mathfrak{M}}(S|\mathcal{D}, T|\mathcal{D})$ and the full closure $\hat{\mathfrak{M}}(S|\mathcal{D}, T|\mathcal{D})$ of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ are multiplicative-extensions of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ and $\mathfrak{M}^{**}(S|\mathcal{D}, T|\mathcal{D})$ is an extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ [1, 2, 3]. We need here to weaken the notation of extension as seen later.

DEFINITION 4.3. When $(A, B, \mathcal{E})$ satisfies the conditions (i), (ii) of Definition 4.1 and

(iii)' $\mathfrak{R}_s(S|\mathcal{D}, T|\mathcal{D}) \subset \mathfrak{R}(A|\mathcal{E}, B|\mathcal{E})|\mathcal{D}$,

$\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ is said to be a *quasi-extension* of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

Suppose S and T are self-adjoint. For each $l = [m, n, m'_1, \dots, m'_n] \leqslant l_w(S, T, \mathcal{D})$ (simply, l_w) $=$ $[m(0), n(0), m(1), \dots, m(n(0))]$ (iff $m \leqslant m(0)$, $n \leqslant n(0)$, $m'_1 \leqslant \leqslant m(1), \dots, m'_n \leqslant m(n)$) we put

$$D_l(S,T) = \mathcal{D}(S^m) \cap \mathcal{D}(T^n) \cap \{\mathcal{D}(S^{m'_1}T) \cap \mathcal{D}(TS^{m'_1})\} \cap \dots$$

$$\dots \cap \{\mathcal{D}(S^{m'_n}T^n) \cap \mathcal{D}(T^n S^{m'_n})\},$$

$$D(S,T,\mathcal{D}) = D_{l_w}(S,T).$$

Then we have

$$\mathcal{D} \subset D(S,T,\mathcal{D}) \subset D_l(S,T). \tag{4.1}$$

THEOREM 4.4. (1) *Suppose S and T are self-adjoint. Then $\mathfrak{M}(S|D_l(S,T), T|D_l(S, T))$ is self-adjoint for each $l \leqslant l_w(S,T,\mathcal{D})$, and $\mathfrak{M}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))$ is a self--adjoint quasi-extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ with $\mathfrak{R}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))|\mathcal{D} = = \mathfrak{R}(S|\mathcal{D}, T|\mathcal{D})$.*

(2) *Suppose S and T have self-adjoint extensions A and B, respectively, such that $D_{l_s}(A, B)$ is a core for A and B, where $l_s \equiv l_s(S,T,\mathcal{D})$. Then $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a self-adjoint quasi-extension $\mathfrak{M}(A|D_{l_s}(A,B), B|D_{l_s}(A,B))$ with $\mathfrak{R}_s(S|\mathcal{D}, T|\mathcal{D}) \subset \subset \mathfrak{R}(A|D_{l_s}(A,B), B|D_{l_s}(A,B))|\mathcal{D} \subset \mathfrak{R}(S|\mathcal{D}, T|\mathcal{D})$.*

Proof. (1) Suppose S and T are self-adjoint. Then we can show in similar to the proof of Proposition 3.6 that $\mathfrak{M}(S|D_l(S,T), T|D_l(S,T))$ is self-adjoint. It follows from (4.1) that $D(S,T,\mathcal{D})$ is a core for S and T and $l_w(S,T,D(S,T,\mathcal{D})) = l_w(S,T,\mathcal{D})$, which implies that $\mathfrak{R}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))|\mathcal{D} = \mathfrak{R}(S|\mathcal{D}, T|\mathcal{D})$. Therefore, $\mathfrak{M}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))$ is a self-adjoint quasi-extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

(2) Since $D_{l_s}(A,B)$ is a core for A and B and $l_s \leqslant l_w(A,B,D_{l_s}(A,B)) \leqslant \leqslant l_w(S,T,\mathcal{D})$; it follows from (1) that $\mathfrak{M}(A|D_{l_s}(A,B), B|D_{l_s}(A,B))$ is self-adjoint and $\mathfrak{R}_s(S|\mathcal{D}, T|\mathcal{D}) \subset \mathfrak{R}(A|D_{l_s}(A,B), B|D_{l_s}(A,B)) \subset \mathfrak{R}(S|\mathcal{D}, T|\mathcal{D})$. This completes the proof.

For the study of standard extensions of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ we define the *weak commutant* and the *quasi-weak commutant* of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ by

$$\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_w = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi|\eta) = (C\xi|X^\dagger\eta)$$

$$\text{for all } X \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) \text{ and } \xi, \eta \in \mathcal{D}\},$$

$$\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}} = \{C \in \mathcal{B}(\mathcal{H}); (CX_1^\dagger \xi | X_2\eta) = (C\xi|(X_1 \,\square\, X_2)\eta)$$

$$\text{for all } X_1, X_2 \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) \text{ with } X_1 \in L^{\mathrm{w}}(X_2) \text{ and } \xi, \eta \in \mathcal{D}\}.$$

Clearly, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}} \subset \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{w}}$ and they are weakly closed $*$-invariant subspaces of $\mathcal{B}(\mathcal{H})$, but need not be algebras [3].

THEOREM 4.5. (1) *Suppose S and T are strongly commuting self-adjoint operators. Then $\mathfrak{M}(S|D_l(S,T), T|D_l(S,T))$ is standard for each $l \leqslant l_{\mathrm{w}}(S,T,\mathcal{D})$, and $\mathfrak{M}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))$ is a standard quasi-extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ with $\mathfrak{N}(S|D(S,T,\mathcal{D}), T|D(S,T,\mathcal{D}))|\mathcal{D} = \mathfrak{N}(S|\mathcal{D}, T|\mathcal{D})$.*

(2) *S and T have strongly commuting self-adjoint extensions if and only if $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a standard quasi-extension.*

(3) *S and T have strongly commuting self-adjoint extensions A and B, respectively and $\{A, B\}' \subset \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}}$ if and only if $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a standard multiplicative-extension.*

Proof. (1) This follows from Theorem 3.2 and Theorem 4.4.

(2) Suppose that S and T have strongly commuting self-adjoint extensions A and B, respectively. Since $\{E_A(\lambda)E_B(\mu); -\infty < \lambda, \mu < \infty\}\mathcal{H}$ is a core for A and B and it is contained in $D_{l_s}(A,B)$, where $l_s \equiv l_s(S,T,\mathcal{D})$, it follows that $D_{l_s}(A,B)$ is a core for A and B. Therefore, it follows from (1) and Theorem 4.4 that $\mathfrak{M}(A|D_{l_s}(A,B), B|D_{l_s}(A,B))$ is a standard quasi-extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

Conversely suppose $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a standard quasi-extension $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$. By Theorem 3.2 A and B are strongly commuting self-adoint extensions of S and T, respectively.

(3) Let A and B be strongly commuting self-adjoint extensions of S and T, respectively and $\{A, B\}' \subset \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}}$. We put

$$\mathcal{E} = \text{ linear span of } \{A, B\}'\mathcal{D},$$

$$\varepsilon(X)\left(\sum_{k=1}^{n} \alpha_k C_k \xi_k\right) = \sum_{k=1}^{n} \alpha_k C_k X \xi_k$$

$$\text{for } X \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) \text{ and } \sum_k \alpha_k C_k \xi_k \in \mathcal{E}.$$

Then ε is a $\dagger$-invariant linear map of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ onto the partial O^*-algebra $\varepsilon(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}))$ and ε^{-1} is a $*$-homomorphism. Since $\varepsilon(S|\mathcal{D}) = A|\mathcal{E}$ and $\varepsilon(T|\mathcal{D}) = = B|\mathcal{E}$, we have

$$\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}) \subset \varepsilon(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})). \tag{4.2}$$

Take arbitrary $X_1, X_2 \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ such that $X_1 \,\square\, X_2$ exists. Since $\{A,B\}' \subset \subset \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}}$, we have

$$(\varepsilon(X_1^\dagger)C_1\xi_1|\varepsilon(X_2)C_2\xi_2) = (C_2^* C_1 X_1^\dagger \xi|X_2\xi_2) =$$
$$= (C_2^* C_1\xi_1|(X_1 \,\square\, X_2)\xi_2) =$$
$$= (C_1\xi_1|C_2(X_1 \,\square\, X_2)\xi_2) =$$
$$= (C_1\xi_1|\varepsilon(X_1 \,\square\, X_2)C_2\xi_2)$$

and

$$(\varepsilon(X_2)C_1\xi_1|\varepsilon(X_1^\dagger)C_2\xi_1) = (C_1\xi_1|\varepsilon((X_1 \,\square\, X_2)^\dagger)C_2\xi_2)$$

for each $C_1, C_2 \in \{A,B\}'$ and $\xi_1, \xi_2 \in \mathcal{D}$. Hence, $\varepsilon(X_1)\square\varepsilon(X_2)$ exists and $\varepsilon(X_1 \square X_2) = = \varepsilon(X_1)\square\varepsilon(X_2)$. Therefore, ε is a $*$-isomorphism of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ onto $\varepsilon(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}))$, which implies that $\varepsilon^{-1}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}))$ is a partial O^*-algebra on $\mathcal{D}$ containing $S|\mathcal{D}$ and $T|\mathcal{D}$. Hence, $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D}) \subset \varepsilon^{-1}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}))$. By (4.2) we have

$$\varepsilon(\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})) = \mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}) \text{ and}$$

$$\varepsilon^{-1}(X) = X|\mathcal{D} \quad \text{for each } X \in \mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}). \tag{4.3}$$

Clearly, $\overline{\varepsilon(S|\mathcal{D})} = A$ and $\overline{\varepsilon(T|\mathcal{D})} = B$, and further, $\{A,B\}'\hat{\mathcal{D}}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})) \subset \subset \hat{\mathcal{D}}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}))$ since $\{A,B\}'\mathcal{E} \subset \mathcal{E}$. It hence follows from Theorem 3.2 that $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ is standard, which implies by (4.3) that $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$ is a standard, multiplicative-extension of $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$.

Conversely suppose $\mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ has a standard multiplicative-extension $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})$. By (2), A and B are strongly commuting self-adjoint extensions of S and T, respectively. Take arbitrary $C \in \{A,B\}'$ and $X_1, X_2 \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$ such that $X_1 \,\square\, \square X_2$ exists. Let ι be the map : $X \in \mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}) \longrightarrow X|\mathcal{D} \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})$. Since ι is a $*$-isomorphism, $\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})'_{\mathrm{w}} = \{A,B\}'$ and $\{A,B\}'\hat{\mathcal{D}}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E})) \subset \hat{\mathcal{D}}(\mathfrak{M}(A|\mathcal{E}, B|\mathcal{E}))$, we have

$$(CX_1^\dagger\xi|X_2\eta) = (C\iota^{-1}(X_1^\dagger)\xi|\iota^{-1}(X_2)\eta) =$$
$$= (\overline{\iota^{-1}(X_1^\dagger)}C\xi|\iota^{-1}(X_2)\eta) =$$
$$= (C\xi|\iota^{-1}(X_1 \,\square\, X_2)\eta) =$$
$$= (C\xi|(X_1 \,\square\, X_2)\eta),$$
$$(CX_2\xi|X_1^\dagger\eta) = (C\xi|(X_1 \,\square\, X_2)^\dagger\eta)$$

for each $\xi, \eta \in \mathcal{D}$. Therefore, $C \in \mathfrak{M}(S|\mathcal{D}, T|\mathcal{D})'_{\mathrm{qw}}$. This completes the proof.

REMARK 4.6. If $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ has a standard extension, then S and T have strongly commuting self-adjoint extensions A and B, respectively and $\{A,B\}' \subset \subset \mathfrak{M}(S|\mathcal{D},T|\mathcal{D})'_{\mathrm{w}}$. Conversely suppose S and T have strongly commuting self-adjoint extensions A and B respectively and $\{A,B\}' \subset \mathfrak{M}(S|\mathcal{D},T|\mathcal{D})'_{\mathrm{w}}$. Let ε be the map as in the proof of Theorem 4.5. Then $\varepsilon(\mathfrak{M}(S|\mathcal{D},T|\mathcal{D}))$ is a weakly commutative, standard partial O^*-algebra which is an extension of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ in the sense in ([3] 3.B.); that is , ε is a bijection and $X \subset \varepsilon(X)$ for each $X \in \mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$. But, $\mathfrak{M}(A|\mathcal{E},B|\mathcal{E}) \neq \varepsilon(\mathfrak{M}(S|\mathcal{D},T|\mathcal{D}))$ in general, and so $\mathfrak{M}(A|\mathcal{E},B|\mathcal{E})$ need be an extension of $\mathfrak{M}(S|\mathcal{D},T|\mathcal{D})$ in the sense of Definition 4.1.

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