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COMPACT COMPOSITION OPERATORS NOT IN THE SCHATTEN CLASSES

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If φ is an analytic function mapping the unit disk $\mathbb{D}$ into itself and H is a Hilbert space of analytic functions on $\mathbb{D}$, the composition operator C_φ on H is defined by $(C_\varphi f)(z) = f(\varphi(z))$. In 1988, Donald Sarason [7] asked

Is there a compact composition operator on H^2 that is not in any Schatten class?

Shapiro [16] has shown that C_φ is compact on H^2 if and only if

$$\limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{\log(1/|w|)} = 0.$$

Here, N_φ is the Nevanlinna counting function: for φ defined on the disk and w a complex number,

$$N_\varphi(w) = \sum \log\left(\frac{1}{|z_n|}\right)$$

where the z_n are the solutions of $\varphi(z) = w$. Using related techniques, Zhu [19] has proved that if $p \geqslant 2$ then C_φ is in the Schatten p-class if and only if

$$\int_{\mathbb{D}} \left(\frac{N_\varphi(w)}{\log(1/|w|)}\right)^{p/2} \frac{\mathrm{d}A(w)}{(1-|w|)^2)^2} < \infty.$$

Contrasting this with Shapiro's condition for compactness strongly sugests an affirmative answer to Sarason's question. We will show that this is indeed the case, not only in H^2 but also in some weighted Dirichlet spaces with similar evaluation kernels, (c.f. Theorem 5).

For $\alpha > -1$, we let D_α denote the weighted Dirichlet space of analytic functions in the unit disk for which

$$\|f\|_{D_\alpha}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

This is a Hilbert space of analytic functions with the obvious inner product. For $f(z) = \sum a_n z^n$, the series $\sum (n+1)^{1-\alpha} |a_n|^2$ gives a coefficient norm equivalent to the above integral norm on D_α. The classical Dirichlet space is D_0, the Hardy space H^2 is D_1, and the Bergman space is D_2.

If φ is an analytic function mapping the unit disk to itself, we say that φ has a *finite angular derivative* at ς on the unit circle, which we will denote by $\varphi'(\varsigma)$, if there is ω on the unit circle (!) such that the difference quotient

$$\frac{\varphi(z) - \omega}{z - \varsigma}$$

has a finite limit as z tends to ς non-tangentially. Work of Julia and Carathéodory [3, pages 23-34] shows that this limit exists if and only if the non-tangential limit

$$L = \lim_{z \to \varsigma} \frac{1 - |\varphi(z)|}{1 - |z|} \tag{1}$$

is finite, in which case $|\varphi'(\varsigma)| = L$. MacCluer and Shapiro [13, Theorem 5.3] have shown that, for $\alpha > 0$, if φ is univalent and maps the unit disk into itself, then C_φ is compact on D_α if and only if φ has no finite angular derivatives. We will construct a univalent map φ that is continuous on the closed disk such that the image of the unit circle touches the unit circle only at the fixed points ± 1. Since we ensure that

$$\lim_{r \to 1^-} \frac{1 - |\varphi(\pm r)|}{1 - r} = \infty$$

φ has no finite angular derivatives and we conclude that C_φ is compact on D_α for $\alpha > 0$.

A Hilbert space operator A is said to be in the *Schatten p-class* for $0 < p < \infty$ if it is compact and the sum of the eigenvalues of $(A^*A)^{p/2}$ is finite [14]. The Schatten p-classes are (non-closed) ideals in the space of bounded operators on the Hilbert space and each Schatten class is dense in the compact operators. The proof that C_φ is not in any Schatten p-class is more delicate and depends on calculations involving the evaluation kernels. For w in the unit disk, we let K_w denote the function in D_α for which

$$\langle f, K_w \rangle = f(w)$$

for all functions f in D_α. Routine calculations give the explicit expression

$$K_w(z) = 1 + \sum_{n=1}^{\infty} \frac{\overline{w}^n}{n^2 B(n, \alpha+1)} z^n.$$

Here $B(p,q)$ is the Beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

where Γ is the Gamma function, and it follows that

$$\frac{1}{n^2 B(n,\alpha+1)} = \frac{\Gamma(n+\alpha+1)}{n^2\Gamma(n)\Gamma(\alpha+1)} =$$

$$= \frac{n+\alpha}{n\alpha}\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} = \left(\frac{1}{n} + \frac{1}{\alpha}\right)\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}.$$

It is not always convenient to work with the evaluation kernels K_w so we will use kernels k_w that grow at the same rate. The kernels k_w are evaluation kernels for an equivalent norm on D_α. For $\alpha > 0$,

$$k_w(Z) = \frac{1}{(1-\overline{w}z)^\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \overline{w}^n z^n.$$

Comparing the series representations for K_w and k_w, we see that for u and v in the disk with $\overline{u}v \geqslant 0$

$$0 < \frac{1}{\alpha} k_u(v) \leqslant K_u(v) \leqslant \left(1 + \frac{1}{\alpha}\right) k_u(v). \tag{2}$$

This estimate will be helpful because $K_u(v) = \langle K_u, K_v \rangle$.

In the construction of the example, we need an interpolation result for weighted Dirichlet spaces which is closely related to the work of Shapiro and Shields [15] (see also [5, section 3]). We first state the result in the form that we will use it and then derive the interpolation version as a corollary.

THEOREM 1. *Suppose that α is positive and that $\{r_n\}$ is an increasing sequence in $(0,1)$. Suppose further that*

$$\rho = \sup_n \left(\frac{1-r_{n+1}}{1-r_n}\right) < \left(1 + \frac{(\alpha+1)2^{\alpha+1}}{r_1^\alpha}\right)^{-2/\alpha}.$$

Let $\mathcal{W}$ denote the closed subspace of D_α spanned by $f_n = K_{r_n}/\|K_{r_n}\|$ for $n \geqslant 1$. If $\{x_n\} \in \ell^2$ then $\sum x_n f_n$ converges and, in fact, the operator $J : \ell^2 \to \mathcal{W}$ defined by

$$J(\{x_n\}) = \sum_{n=1}^{\infty} x_n f_n$$

is bounded and invertible. Moreover,

$$\mathcal{W} = \left\{ \sum_{n=1}^{\infty} x_n f_n : \{x_n\} \in \ell^2 \right\}.$$

Proof. The conclusion is equivalent to showing that the matrix A whose entries are $a_{ij} = \langle f_i, f_j \rangle$ is bounded and invertible (see [15, page 524]).

We see from inequality (2) that

$$0 < a_{ij} = \frac{\langle K_{r_i}, K_{r_j} \rangle}{\|K_{r_i}\| \, \|K_{r_j}\|} = \frac{K_{r_i}(r_j)}{\sqrt{K_{r_i}(r_i) K_{r_j}(r_j)}} \leqslant (\alpha+1) \frac{k_{r_i}(r_j)}{\sqrt{k_{r_i}(r_i) k_{r_j}(r_j)}}.$$

We let $s_n = 1 - r_n$, so that for $i < j$,

$$\frac{k_{r_i}(r_j)}{\sqrt{k_{r_i}(r_i) k_{r_j}(r_j)}} = \left(\frac{(1-(1-s_i)^2)(1-(1-s_j)^2)}{(1-(1-s_i)(1-s_j))^2} \right)^{\alpha/2} =$$

$$= \left(\frac{(2s_i - s_i^2)(2s_j - s_j^2)}{(s_i + s_j - s_i s_j)^2} \right)^{\alpha/2} =$$

$$= \left(\frac{s_j}{s_i} \right)^{\alpha/2} \left(\frac{(2-s_i)(2-s_j)}{(1 - s_j + s_j/s_i)^2} \right)^{\alpha/2} \leqslant$$

$$\leqslant \left(\prod_{l=1}^{j-1} \frac{s_{l+1}}{s_l} \right)^{\alpha/2} \left(\frac{(2)(2)}{(1-s_1)^2} \right)^{\alpha/2} \leqslant$$

$$\leqslant \left(\rho^{\alpha/2} \right)^{j-i} \frac{2^\alpha}{r_1^\alpha}.$$

Denoting the strictly upper (lower) triangular portion of A by $A_U (A_L)$, that is, A_U is the matrix whose entries are a_{ij} for $i < j$ and 0 for $i \geqslant j$, we have $A = A_U + I + A_L$. We will use the Schur test [2, page 126] to show that $\|A_U + A_L\| < 1$ and conclude that A is bounded and invertible. The version of the Schur test we need is : *If B is a matrix and $M > 0$ so that $\sum_j |b_{ij}| \leqslant M$ for each i and $\sum_i |b_{ij}| \leqslant M$ for each j, then $\|B\| \leqslant M$.*

For A_U the row sums are

$$\sum_{j>i} \frac{\langle K_{r_i}, K_{r_j} \rangle}{\|K_{r_i}\| \, \|K_{r_j}\|} \leqslant (\alpha+1) \sum_{j>i} \frac{k_{r_i}(r_j)}{\sqrt{k_{r_i}(r_i) k_{r_j}(r_j)}} \leqslant$$

$$\leqslant \frac{(\alpha+1)2^\alpha}{r_1^\alpha} \sum_{j>i} \left(\rho^{\alpha/2} \right)^{j-i} = \frac{(\alpha+1)2^\alpha}{r_1^\alpha} \frac{\rho^{\alpha/2}}{1 - \rho^{\alpha/2}}.$$

The same bound is obtained for the column sums by an analogous calculation. Finally, the hypothesized inequality for ρ guarantees that this bound is smaller than 1/2.

Since $A_L = A_U^*$, it follows that $||A_L|| < 1/2$ also and that A is bounded and invertible. ■

As in [15], this can be interpreted as an interpolation result.

COROLLARY 2. *Let α be positive and $\{r_n\}$ be a sequence as in Theorem 1. If $\{w_n\}$ is a sequence of complex numbers such that*

$$\sum_{n=1}^{\infty} \frac{|w_n|^2}{||K_{r_n}||^2} < \infty$$

then there is a function f in D_α such that $f(r_n) = w_n$ for all n.

Proof. Since $J^{-1} : \mathcal{W} \longrightarrow \ell^2$ is a bounded operator, its adjoint $(J^{-1})^* : \ell^2 \longrightarrow$ $\longrightarrow \mathcal{W} \subset D_\alpha$ is a bounded operator. Now for any $x = (x_1, x_2, \ldots)$ in ℓ^2 (whose standard orthonormal basis is denoted $\{e_n\}$), we have

$$x_n = \langle x, e_n \rangle = \langle x, J^{-1}(f_n) \rangle = \langle (J^{-1})^*(x), f_n \rangle.$$

In particular, if we let $x_n = w_n/||K_{r_n}||$ and $f = (J^{-1})^*(x)$ in D_α, the above equation says that

$$\frac{w_n}{||K_{r_n}||} = \langle (J^{-1})^*(x), f_n \rangle = \langle f, f_n \rangle = \frac{f(r_n)}{||K_{r_n}||}.$$

Thus, $f(r_n) = w_n$ as required. ■

We are now ready to begin the construction of compact composition operators that are not in any Schatten p-class. The following theorem sets the stage by describing what we will construct. An analytic function h on $\mathbf{D}$ is a multiplier of D_α if the operator M_h defined by $M_h(f)(z) = h(z)f(z)$ is bounded. Although general conditions characterizing multipliers are delicate, all polynomials are multipliers.

THEOREM 3. *Let α be positive and suppose that $\{r_n\}$ is a sequence as in Theorem 1. Let $h(z) = z - r_1$ and let φ be an analytic function for which C_φ is bounded on D_α. If $\varphi(r_{n+1}) = r_n$ for $n \geqslant 1$, then $\mathcal{W}$ is an invariant subspace for $C_\varphi^* M_h^*$ and, moreover, $J^{-1}C_\varphi^* M_h^* J$ is a backward weighted shift on ℓ^2 with weight sequence $(r_{n+1} - r_1)||K_{r_n}||/||K_{r_{n+1}}||$.*

Proof. Routine calculations show that $M_h^*(K_w) = \overline{h(w)}K_w$ and that $C_\varphi^*(K_w) = K_{\varphi(w)}$. It follows for $n \geqslant 1$ that

$$C_\varphi^* M_h^*(K_{r_{n+1}}) = (r_{n+1} - r_1)K_{r_n} \quad \text{and} \quad C_\varphi^* M_h^*(K_{r_1}) = 0.$$

But $\mathcal{W}$ is spanned by the vectors K_{r_n}, so $\mathcal{W}$ is invariant for $C_\varphi^* M_h^*$. Now for $n \geqslant 1$

$$J^{-1} C_\varphi^* M_h^* J(e_{n+1}) = J^{-1} C_\varphi^* M_h^* \left(\frac{K_{r_{n+1}}}{\|K_{r_{n+1}}\|} \right) =$$

$$= \frac{(r_{n+1} - r_1)}{\|K_{r_{n+1}}\|} J^{-1}(K_{r_n}) = (r_{n+1} - r_1) \frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|} e_n$$

and $J^{-1} C_\varphi^* M_h^* J(e_1) = 0$ as we were to prove. ∎

If C_φ is in the Schatten p-class, which is a self adjoint ideal, then so is $J^{-1} C_\varphi^* M_h^* J$. A weighted shift W on ℓ^2 defined by $W(e_{n+1}) = \omega_n e_n$ is in the Schatten p-class if and only if $\sum |\omega_n|^p < \infty$. We will construct a univalent mapping φ that maps the disk into itself and a sequence r_n satisfying the hypotheses of Theorem 3 so that C_φ is a compact, but the weighted shift obtained in Theorem 3, and hence C_φ, is not in any Schatten p-class.

We let $\psi(t)$, $-\infty < t < \infty$, be the curve

$$\psi(t) = \begin{cases} 1 + (1 - \mathrm{e}^{-\mathrm{e}^{\mathrm{e}}})|t|, & \text{if } |t| \leqslant \mathrm{e}^{\mathrm{e}^{\mathrm{e}}} \\ |t|/\log\log\log|t|, & \text{if } |t| \geqslant \mathrm{e}^{\mathrm{e}^{\mathrm{e}}} \end{cases} \tag{3}$$

and Ω be the domain

$$\Omega = \{w : -\psi(\operatorname{Re} w) < \operatorname{Im} w < \psi(\operatorname{Re} w)\}.$$

Note that Ω is symetric about both the real and imaginary axes. A calculation shows that $\psi'(t)$ is positive but non-increasing on $[0, \infty)$, and so Ω is star-shaped about the origin.

Let us denote by $\sigma(z)$ the unique Riemann mapping from the unit disk $\mathbb{D}$ onto Ω for which $\sigma(0) = 0$ and $\sigma'(0) > 0$. Then σ is an increasing map of the interval $(-1, 1)$ onto the real axis in Ω.

For each positive integer n we denote by r_n the point of $(0, 1)$ for which $\sigma(r_n) = \mathrm{e}^n$. Clearly, $r_n \to 1$ as $n \to \infty$. The critical calculation concerns the rate at which the convergence occurs.

Lemma 4. *We have*

$$\lim_{n \to \infty} \frac{1 - r_{n+1}}{1 - r_n} = 0 \tag{4}$$

whereas, for each $q > 0$,

$$\sum_{n=1}^{\infty} \left(\frac{1 - r_{n+1}}{1 - r_n} \right)^q = \infty. \tag{5}$$

This lemma contains the estimates essential to the proof of the main theorem. We will state and prove the main theorem and postpone the proof of the lemma.

THEOREM 5. *Let φ be the univalent map of the unit disk into itself given by $\varphi(z) = \sigma^{-1}(e^{-1}\sigma(z))$. For $\alpha > 0$, the composition operator C_φ is compact on D_α but is not contained in any Schatten p-class.*

Proof. Since the domain Ω is star-shaped about the origin, for each z in the unit disk, $e^{-1}\sigma(z)$ is in Ω and φ is well defined. It is easily seen that φ is continuous on the closed disk and, since the finite boundaries of Ω and $e^{-1}\Omega$ do not intersect, that $\varphi(\partial\mathbf{D}) \cup \partial\mathbf{D}$ consists of the two points $+1$ and -1, which are fixed points of φ.

Using the univalence of φ, the change of variables formula for multiple integrals shows that C_φ is bounded on D_0 the unweighted Dirichlet space:

$$\|C_\varphi f\|^2 = |f(\varphi(0))|^2 + \int_{\mathbb{D}} |(f\circ\varphi)'(z)|^2 dA(z) =$$

$$= |f(0)|^2 + \int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 dA(z) =$$

$$= |f(0)|^2 + \int_{\varphi(\mathbb{D})} |f'(w)|^2 dA(w) < \|f\|^2.$$

Since C_φ is bounded on D_0, MacCluer and Shapiro's Theorem 5.3(b)[13, page 893] says that C_φ is compact on D_α for $\alpha > 0$ if and only if φ has no finite angular derivatives. Because finite angular derivatives can occur only at points whose image is on the circle, we must check ± 1, and by symmetry, only the point 1.

Note that by construction, the sequence $\{r_n\}$ is a backward iteration sequence for φ:

$$\varphi(r_{n+1}) = \sigma^{-1}(e^{-1}\sigma(r_{n+1})) = \sigma^{-1}(e^{-1}e^{n+1}) = \sigma^{-1}(e^n) = r_n.$$

By equation (4),

$$\lim_{n\to\infty} \frac{1-\varphi(r_{n+1})}{1-r_{n+1}} = \lim_{n\to\infty} \frac{1-r_n}{1-r_{n+1}} = \infty$$

and so, by (1), φ does not have a finite angular derivative at 1 which in turn proves that C_φ is compact on D_α for $\alpha > 0$.

If C_φ were in a Schatten p-class then, as the remarks following Theorem 3 show, the weighted shift $JC_\varphi^* M_h^* J^{-1}$ would also be in such a class. (Actually, the hypotheses of Theorem 3 and Theorem 1 might not be literally satisfied since the supremum of $(1-r_{n+1})/(1-r_n)$ may be too large. However, because of (4), we could renumber the sequence so that the hypotheses are satisfied. We will not burden the reader or ourselves by doing so.) To see that the weighted shift is not in the Schatten p-class, we use equation (2) and $q = p\alpha/2$ in equation (5) and calculate:

$$\sum_{n=1}^{\infty}\left((r_{n+1}-r_1)\frac{\|K_{r_n}\|}{\|K_{r_{n+1}}\|}\right)^p \geqslant$$

$$\geqslant (r_2-r_1)^p \sum_{n=1}^{\infty}\left(\frac{k_{r_n}(r_n)/\alpha}{(1+1/\alpha)k_{r_{n+1}}(r_{n+1})}\right)^{p/2} \geqslant$$

$$\geqslant \left(\frac{r_2-r_1}{\sqrt{1+\alpha}}\right)^p \sum_{n=1}^{\infty}\left(\frac{1-r_{n+1}^2}{1-r_n^2}\right)^{p\alpha/2} \geqslant$$

$$\geqslant \left(\frac{r_2-r_1}{\sqrt{2^\alpha(1+\alpha)}}\right)^p \sum_{n=1}^{\infty}\left(\frac{1-r_{n+1}}{1-r_n}\right)^{p\alpha/2} = \infty.$$

Thus, C_φ is compact on D_α for each $\alpha > 0$ but it is not in any Schatten p-class. ■

We will base the calculation implicit in Lemma 4 on the hyperbolic metric and a length-area inequality. If $\mathcal{U}$ is a simply connected domain and $u(z)$ is a conformal mapping of $\mathcal{U}$ onto the unit disk $\mathbb{D}$ then the hyperbolic metric on $\mathcal{U}$ is the metric whose infinitesimal element of length is

$$\frac{|u'(z)|\,|dz|}{1-|u(z)|^2}$$

which independent of the choice of u. If z_1 and z_2 are in $\mathcal{U}$, we write $d(z_1, z_2; \mathcal{U})$ for the hyperbolic distance between them with respect to $\mathcal{U}$.

The hyperbolic metric is conformally invariant in that if τ is a conformal map of $\mathcal{U}$ onto another simply connected domain $\mathcal{V}$, and if z_1 and z_2 lie in $\mathcal{U}$, then

$$d(\tau(z_1), \tau(z_2); \mathcal{V}) = d(z_1, z_2; \mathcal{U})$$

In the unit disk, we have

$$d(0, z; \mathbb{D}) = \frac{1}{2}\log\frac{1+|z|}{1-|z|}.$$

Thus, if z_1 and z_2 are in $\mathcal{U}$ and u is the conformal map of $\mathcal{U}$ onto $\mathbb{D}$ for which $u(z_1) = 0$ and $u(z_2)$ is positive then, by conformal invariance,

$$d(z_1, z_2; \mathcal{U}) = \frac{1}{2}\log\frac{1+u(z_2)}{1-u(z_2)}.$$

We let $\mathcal{S}$ denote the strip $\mathcal{S} = \{w : |\operatorname{Im} w| < \pi/2\}$. Note that $s(z) = \frac{1}{2}\log(1+$ $+z)/(1-z)$ conformally maps $\mathbb{D}$ onto $\mathcal{S}$ with $s(0) = 0$ and $s'(0) > 0$. Thus, if τ is a conformal map of $\mathcal{U}$ onto $\mathcal{S}$ so that $\tau(z_1)$ and $\tau(z_2)$ are real, then

$$d(z_1, z_2; \mathcal{U}) = |\tau(z_1) - \tau(z_2)|$$

A useful reference to these matters is [1].

The heart of the calculation is in the following lemma.

LEMMA 5. *There is a positive integer n_0 so that if $n > n_0$ then*

$$\frac{1}{4}\log\log n < d(e^n, e^{n+1}; \Omega) < 4\log\log n. \tag{6}$$

Because of the conformal invariance of hyperbolic distance, (6) is equivalent to

$$\frac{1}{4}\log\log n < d(r_n, r_{n+1}; \mathbb{D}) < 4\log\log n. \tag{7}$$

Proof of Lemma 5. The bounds in (6) are applications of a lemma due to Hayman [10, page 170].

Suppose that $\mathcal{U}$ is a simply connected domain containing the rectangle

$$\mathcal{R} = \{z : a_1 - b < \operatorname{Re} z < a_2 + b, |\operatorname{Im} z| < b\}$$

where $a_1 < a_2$ and $b > 0$. Hayman proves that

$$d(a_1, a_2; \mathcal{U}) \leqslant \frac{\pi}{4b}(a_2 - a_1) + \frac{\pi}{2}. \tag{8}$$

On the other hand, Hayman shows that if $\mathcal{R} \subset \mathcal{U}$ but the line segments $\{z : : a_1 - b < \operatorname{Re} z < a_2 + b,\ |\operatorname{Im} z| = b\}$ are not in $\mathcal{U}$, then

$$d(a_1, a_2; \mathcal{U}) \geqslant \frac{\pi}{4b}(a_2 - a_1) - \frac{\pi}{2}. \tag{9}$$

We consider $n > n_1 = e^e$ and let $b_n = e^{n-1}/\log\log(n-1)$. Now $e^n - b_n > e^{n-1}$ so equation (3) gives

$$\psi(e^n - b_n) > \psi(e^{n-1}) = e^{n-1}/\log\log(n-1) = b_n.$$

Thus, the rectangle $\mathcal{R}_n = \{z : e^n - b_n < \operatorname{Re} z < e^{n+1} + b_n \text{ and } |\operatorname{Im} z| < b_n\}$ is contained in Ω. Applying Hayman's inequality (8) gives that

$$d(e^n, e^{n+1}; \Omega) \leqslant \frac{\pi(e^{n+1} - e^n)}{4e^{n-1}/\log\log(n-1)} + \frac{\pi}{2} =$$

$$= \frac{\pi}{4}(e^2 - e)\log\log(n-1) + \frac{\pi}{2} \leqslant$$

$$\leqslant 4\log\log n$$

if $n \geqslant n_2 > n_1$ for an appropiate n_2.

We now establish the left-hand inequality in (6). Choose c_n so that $\psi(e^{n+1}+c_n) = c_n$ (which exist because $\psi' < 1-\delta$ for some positive δ). Since $e^{n+1} + b_{n+2} < e^{n+2}$ and $\psi(e^{n+2}) = b_{n+2}$ and ψ is an increasing function, it follows that $c_n < b_{n+2}$. We let $\mathcal{R}'_n = \{z : e^n - c_n < \operatorname{Re} z < e^{n+1} + c_n \text{ and } |\operatorname{Im} z| < c_n\}$ and $\mathcal{U}_n = \Omega \cup \mathcal{R}'_n$ so that Hayman's inequality (9) gives

$$d(e^n, e^{n+1}; \mathcal{U}_n) \geqslant \frac{\pi}{4c_n}(e^{n+1} - e^n) - \frac{\pi}{2} \geqslant$$

$$\geqslant \frac{\pi(e^{n+1} - e^n)}{4e^{n+1}/\log\log(n+1)} - \frac{\pi}{2} \geqslant$$

$$\geqslant \frac{1}{4}\log\log n$$

for n exceeding an appropriate $n_3 > n_1$. Since $\Omega \subset \mathcal{U}_n$, which implies that $d(e^n, e^{n+1}; \mathcal{U}_n) \leqslant d(e^n, e^{n+1}; \Omega)$, we obtain (6). ■

We are now ready to prove Lemma 4.

Proof of Lemma 4. To begin, we note that

$$\frac{1}{2}\left(\frac{1-r_n}{1-r_{n+1}}\right)\left(\frac{1+r_{n+1}}{1+r_n}\right) < \left(\frac{1-r_n}{1-r_{n+1}}\right) < \left(\frac{1-r_n}{1-r_{n+1}}\right)\left(\frac{1+r_{n+1}}{1+r_n}\right). \tag{10}$$

The first inequality in (10) gives that

$$\frac{1}{2}\log\left(\frac{1-r_n}{1-r_{n+1}}\right) > \frac{1}{2}\log\left(\frac{1+r_{n+1}}{1-r_{n+1}}\right) - \frac{1}{2}\log\left(\frac{1+r_n}{1-r_n}\right) - \frac{1}{2}\log 2 =$$

$$= d(0, r_{n+1}; \mathbb{D}) - d(0, r_n; \mathbb{D}) - \frac{1}{2}\log 2 =$$

$$= d(r_n, r_{n+1}; \mathbb{D}) - \frac{1}{2}\log 2.$$

Inequality (7) implies $d(r_n, r_{n+1}; \mathbb{D}) \to \infty$ as $n \to \infty$ so (4) holds.

Let $q > 0$ be given. From the second inequality in (10) we see that

$$\frac{1}{2}\log\left(\frac{1-r_n}{1-r_{n+1}}\right) < d(r_n, r_{n+1}; \mathbb{D}).$$

Therefore, we obtain from (7) that for $n \geqslant n_0$

$$\left(\frac{1-r_{n+1}}{1-r_n}\right) > \exp(-2d(r_n, r_{n+1}; \mathbb{D})) \geqslant$$

$$\geqslant \exp(-8 \log\log n) = \left(\frac{1}{\log n}\right)^8 .$$

There is an integer $N \geqslant n_0$ large enough that $1/\log n > n^{-1/(8q)}$ for $n > N$ and for this N,

$$\sum_{n=N}^{\infty} \left(\frac{1-r_{n+1}}{1-r_n}\right)^q \geqslant \sum_{n=N}^{\infty} \left(\frac{1}{\log n}\right)^{8q} \geqslant \sum_{n=N}^{\infty} \frac{1}{n} = \infty$$

as we were to show. ∎

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REFERENCES

1. AHLFORS, L. V., *Conformal Invariants*, McGraw-Hill, New York, 1973.
2. BROWN, A.; HALMOS, P. R.; SHIELDS, A. L., Cesaro operators, *Acta Sci. Math. (Szeged)*, **26**(1965), 125–137.
3. CARARTÉODORY, C., *Theory of functions of a complex variable. Vol 2*, Chelsea, New York, 1960.
4. COWEN, C. C., Iteration and the solution of functional equations for functions analytic in the unit disk, *Trans. Amer. Math. Soc.*, **265**(1981), 69–95.
5. COWEN, C. C., Composition operators on H^2, *J. Operator Theory*, **9**(1983), 77–106.
6. COWEN, C. C., An application of Hadamard multiplication to operators on weighted Hardy spaces, *Linear Alg. Appl.*, **133**(1990), 21–32.
7. COWEN, C. C., Composition operators on Hilbert spaces of analytic functions: A status report, in *Proceedings of Symposia in Pure Mathematics*, American. Math. Soc., Providence, R. I., **51**(1990) part I, 131–145.
8. COWEN, C. C.; POMMERENKE, Ch., Inequalities for the angular derivative of an analytic function in the unit disk, *J. London Math. Soc. (2)*, **26**(1982), 271–289.
9. DENJOY, A., Sur l'iteration des fonctions analytiques, *C. R. Acad. Sci. Paris Sér. A.*, **182**(1926), 255–257.
10. HAYMAN, W. K., The logarithmic derivative of multivalent functions, *Michigan J. Math.*, **27**(1980), 149–179.
11. KOENIGS, G., Recherches sur les intégrales des certaines equationes fonctionelles, *Ann. Sci. École Norm. Sup. (Sér 3)*, **1**(1884), supplément, 3–41.
12. LITTLEWOOD, J. E., On inequalities in the theory of functions, *Proc. London Math. Soc.*, **23**(1925), 481–519.
13. MACCLUER, B. D.; SHAPIRO, J. H., Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.*, **38**(1986), 878–906.
14. SCHATTEN, R., *Norm ideals of completely continuous operators*, Springer Verlag, Berlin, 1960.
15. SHAPIRO, H. S.; SHIELDS, A. L., On some interpolation problems for analytic functions, *Amer. J. Math.*, **83**(1961), 513–532.
16. SHAPIRO, J. H., The essential norm of a composition operator, *Ann. of Math.*, **125**(1987), 375–404.
17. SHAPIRO, J. H.; TAYLOR, P. D., Compact, nuclear, and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.*, **23**(1973), 471–496.

18. WOLFF, J., Sur l'iteration des functions, *C. R. Acad. Sci. Paris Sér. A.*, **182**(1926), 42–43, 200–201.

19. ZHU, K., Composition operators on the Hardy space, preprint, 1989.

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