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DIFFERENTIAL BANACH ALGEBRA NORMS AND SMOOTH SUBALGEBRAS OF C^*-ALGEBRAS

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1. INTRODUCTION

It is frequently important in topology to consider additional structure such as a smooth or piecewise linear structure on certain topological spaces. A somewhat unconventional approach to defining such structures, which admits generalization to operator algebras, is to specify a certain dense $*$-subalgebra of the C^*-algebra of continuous functions. For example, a smooth structure on a manifold X may be defined by specifying the subalgebra $\mathcal{C}_0^\infty(X)$ of $\mathcal{C}_0(X)$. A piecewise-linear structure (triangulation) or a structure of an affine algebraic variety on X may be regarded as a choice of a suitable family of generators of $\mathcal{C}_0(X)$.

In the study of operator algebras, it has long been recognized that there are circumstances where it is natural to consider dense $*$-subalgebras of C^*-algebras (in particular in connection with cyclic cohomology or with the study of unbounded derivations on C^*-algebras.) If one adopts the familiar philosophy that C^*-algebras are generalizations of topological spaces, then dense subalgebras may be regarded as means to specify additional structure on the underlying "space". Recent developments in noncommutative differential geometry are based on this idea, [2], [4].

There are also other contexts (e.g. group algebras, [11], [12], [8]) where dense subalgebras appear naturally which do not at first seem to be closely related to the notion of a smooth structure, but which can sometimes fit into the same picture. While there has now been accumulated a number of "standard" examples of smooth subalgebras of C^*-algebras, which have been defined and studied in a largely ad hoc manner, there has been no general theory tying these examples together.

In this paper, we will begin the study of a reasonable abstract notion of a smooth subalgebra of a C^*-algebra which includes the "standard" examples. In analogy with the algebras $C^\infty(X)$, X a compact manifold, such an algebra should have the following properties:

(1) It should be closed under holomorphic functional calculus of all elements (i.e. should be a "local C^*-algebra" in the language of [1]) and under C^∞-functional calculus of self-adjoint elements

(2) It should be complete with respect to a locally convex algebra topology analogous to the usual topology on $C^\infty(X)$.

(3) Certain (derivationlike) natural linear and multilinear maps defined on generating subalgebras, such as the ones appearing in Hochschild and cyclic cohomology, should be continuous for this topology.

The principal examples of such algebras are of course the definition domains of closed derivations and their powers.

Our main contribution is the notion of a differential seminorm. A differential seminorm is an algebra "seminorm" T (i.e. $T(x) \geqslant 0$, $T(\lambda x) = |\lambda| T(x)$, $T(x+y) \leqslant T(x) + T(y)$, $T(xy) \leqslant T(x)T(y)$) with values in the convolution algebra $\ell^1(\mathbb{N})$. Such a map T is the sum of its degree 0 part and of the part of degree $\geqslant 1$, which is nilpotent for differential seminorms of finite order. The part of degree 0 is assumed to be a C^*-seminorm or to be continuous with respect to a C^*-seminorm. This implies a restriction on the growth of the differential seminorm T on products and exponentials. A discovery that we made is the role of the "logarithmic order" which contributes to the total order of the seminorm just like the number of "derivatives" that it contains (its order). The completion of a subalgebra $\mathfrak{A}$ of a C^*-algebra A with respect to a closable differential seminorm is a smooth subalgebra of A with the properties listed above.

Differential seminorms on $\mathfrak{A}$ arise from derivations and compositions of derivations but also from homomorphisms of $\mathfrak{A}$ into normed $\mathbb{N}$-graded algebras or from filtered structures. We show that the most general differential seminorm on $\mathfrak{A}$ can be obtained from a linear map with "non-negative curvature" from $\mathfrak{A}$ into an $\mathbb{N}$-graded Banach $*$-algebras whose degree 0 part is isomorphic to a C^*-algebra, and that every "flat" differential seminorm is induced by the powers of a derivation on an algebra containing $\mathfrak{A}$.

The main shortcoming of differential seminorms is the fact that they do not behave well under quotients. This is related to the fact that a quotient of a graded algebra is not graded in general. To circumvent this difficulty, we introduce derived norms as quotient norms for the total norms of differential seminorms, and study their properties.

We can introduce smooth subalgebras of C^*-algebras as subalgebras complete with respect to the topology defined by a suitable family of such seminorms. One feature of this approach is that it then makes sense to talk about the smooth subalgebra generated by an arbitrary dense $*$-subalgebra of a C^*-algebra. For every C^*-norm on an involutive algebra $\mathfrak{A}$, there will be a family of derived norms and one can naturally define the smooth completion $\mathcal{D}(\mathfrak{A})$ with respect to these norms. The completion will be invariant under holomorphic functional calculus for arbitrary elements, and under $\mathcal{C}^\infty$-functional calculus for normal elements (we also introduce the $\mathcal{C}^k$-completions, $\mathcal{C}^0(\mathfrak{A})$ being the C^*-completion of $\mathfrak{A}$, in a slightly different spirit). Another feature of the smooth completion is that every closable derivation of $\mathfrak{A}$ extends to a continuous derivation $\mathcal{D}(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A})$ and to a continuous derivation $\mathcal{C}^{k+1}(\mathfrak{A}) \to \mathcal{C}^k(\mathfrak{A})$. Smooth algebras behave in many respects like C^*-algebras. We study some of their basic properties in section 6.

Even though some of our definitions are still somewhat experimental, we believe that we essentially have developed the right framework for smooth algebras in the non-commutative setting. The groundwork laid in this paper is only a beginning of a reasonable theory of smooth structures on non-commutative algebras and leads to many intriguing and difficult questions which we have to leave open.

2. PRELIMINARIES ON GRADED AND FILTERED ALGEBRAS

All our algebras are over the complex numbers. When speaking of graded algebras in the following, we always mean $\mathbf{N}$-graded algebras, where $\mathbf{N}$ is the set of natural numbers, including 0. An $\mathbf{N}$-graded algebra is an algebra $\mathcal{B}$ equipped with projections p_k, $k \in \mathbf{N}$, $p_i p_j = 0$, $i \neq j$, onto subspaces $\mathcal{B}_k$ of $\mathcal{B}$ such that $\mathcal{B}_i \mathcal{B}_j \subset \mathcal{B}_{i+j}$ and such that $\sum_k p_k = \mathrm{id}$. This implies the important relation $p_k(xy) = = \sum_{i+j=k} p_i(x)p_j(y)$, $x, y \in \mathcal{B}$. Every $\mathbf{N}$-graded algebra $\mathcal{B}$ carries a canonical derivation δ defined by $\delta(x_n) = nx_n$, $x_n \in \mathcal{B}_n$, and a corresponding one-parameter automorphism group α_t, $t \in \mathbf{C}$, defined by $\alpha_t(x_n) = t^n x_n$, $x_n \in \mathcal{B}_n$. For each graded algebra $\mathcal{B}$, we denote by $\partial\mathcal{B}$ the ideal of elements x of degree $\geqslant 1$, i.e. for which $p_0(x) = 0$.

One can also consider graded Banach algebras, where one only requires that the (algebraic) direct sum of the $\mathcal{B}_j$ is dense. In that case we assume that $\sum_k \|p_k(x)\| < \infty$, $\forall x \in \mathcal{B}$, and moreover that $\|p_k\| \leqslant 1$ for all k and $\sum_k p_k = \mathrm{id}$ (norm convergence). In many of our examples, only finitely many $\mathcal{B}_j$'s are nonzero.

One of the simplest examples of a graded Banach algebra is the algebra $\ell^1(\mathbf{N})$ of summable complex-valued functions on $\mathbb{N}$, with the obvious grading. The sum and product with complex numbers of elements in $\ell^1(\mathbb{N})$ are defined pointwise, while the product $F * G$ which we will usually abbreviate to FG is the convolution product $(FG)_k = \sum_{i+j=k} F_i G_j$. The function $\mathbf{1}$ defined by $\mathbf{1}_0 = 1$, $\mathbf{1}_k = 0$, $k > 0$, is a unit for $\ell^1(\mathbb{N})$.

Let $\ell^1_+(\mathbb{N})$ denote the set of functions with values in $\mathbb{R}_+$ on $\mathbb{N}$. If F, G are real-valued functions in $\ell^1(\mathbb{N})$ we write $F \leqslant G$ if $F_k \leqslant G_k$ for all $k \in \mathbb{N}$. Thus $F \in \ell^1_+(\mathbb{N})$ iff $F \geqslant 0$. One has $\ell^1_+(\mathbb{N})\ell^1_+(\mathbb{N}) \subset \ell^1_+(\mathbb{N})$. Finally, there is a positive homomorphism $\int : \ell^1(\mathbb{N}) \to \mathbb{C}$, defined by $\int F = \sum_k F_k$.

The algebra $\ell^1(\mathbb{N})$ is isomorphic to the graded algebra of formal power series in one variable with complex ℓ^1-summable coefficients.

For each algebra $\mathfrak{A}$, there is a universal graded algebra $D\mathfrak{A}$ containing $\mathfrak{A}$ as a subalgebra and invariant under a linear operator d satisfying $d(xy) = xd(y) + d(x)y$ (but without assuming $d^2 = 0$!). Take the universal algebra generated by symbols $d^k(x)$ of degree k, that are linear in x and satisfy the relations $d^k(xy) = \sum_{i+j=k} d^i(x)d^j(y)$ where $k \geqslant 0$ (we think of $d^k(x)$ as representing $\frac{1}{k!}$ times the k-th power of d applied to x). Note that the relations respect the degree so that $D\mathfrak{A}$ is a graded algebra. If we set

$$d(d^k(x)) = (k+1)d^{k+1}(x),$$

$$d(d^{i_1}(x_1)d^{i_2}(x_2)\dots d^{i_k}(x_k)) = \sum_{1\leqslant j\leqslant k} d^{i_1}(x_1)\dots d(d^{i_j}(x_j))\dots d^{i_k}(x_k)$$

we obtain an operator $d : D\mathfrak{A} \to D\mathfrak{A}$ of degree 1, satisfying $d(xy) = xd(y) + d(x)y$. We also have the derivation δ of degree 0. One checks that $\delta d - d\delta = d$.

$D\mathfrak{A}$ is a graded quotient of the free graded algebra over the vector space $\mathfrak{A}$ which we denote by $G\mathfrak{A}$. The algebra $G\mathfrak{A}$ is by definition the universal algebra generated by symbols $d^k(x)$, $x \in \mathfrak{A}$, of degree k, that are linear in x with no further relations. It is the tensor algebra over $\mathbb{N}$ copies of the vector space $\mathfrak{A}$, one copy (corresponding to $d^k(\mathfrak{A})$) for each k in $\mathbb{N}$. Every element of $G\mathfrak{A}$ is a sum of elements of the form $d^{i_1}(x_1)d^{i_2}(x_2)\dots d^{i_k}(x_k)$, $x_j \in \mathfrak{A}$, $i_j \geqslant 0$. $G\mathfrak{A}$ is graded if we define the degree of such an element to be $i_1 + i_2 + \cdots + i_k$.

Besides the standard inclusion $\mathfrak{A} \to D\mathfrak{A}$ given by $x \to d^0(x)$ there is the "homomorphism" $e^d : x \to d^0(x) + d^1(x) + \dots$. This map is well defined only as a homomorphism into the quotient of $D\mathfrak{A}$ by the ideal of all elements of degree $\geqslant n$, where the series defining it is finite, or in the case where $D\mathfrak{A}$ is equipped and completed with respect to a norm for which the series $d^0(x) + d^1(x) + \dots$ converges for all

x. It has the following universal property: if $\varphi : \mathfrak{A} \to \mathcal{B}$ is a homomorphism of $\mathfrak{A}$ into an $\mathbb{N}$-graded algebra $\mathcal{B}$, then there is a unique graded homomorphism $\psi : D\mathfrak{A} \to \mathcal{B}$ such that $\varphi = \psi e^d := \sum \psi d^i$ (in particular, ψe^d converges). In fact, if $\varphi_n(x)$ denotes the part of degree n of $\varphi(x)$, we have $\varphi_k(xy) = \sum_{i+j=k} \varphi_i(x)\varphi_j(y)$ and we can define by $\psi(d^k(x)) = \varphi_k(x)$.

Derivations into bimodules and compositions of derivations are best described by homomorphisms into graded algebras. Let us look at the example of a derivation. If δ is a derivation from $\mathfrak{A}$ into an $\mathfrak{A}$-bimodule M, then $\varphi(x) = (x, \delta(x))$ defines a homomorphism from $\mathfrak{A}$ into $\mathfrak{A} \oplus M$, and $\mathfrak{A} \oplus M$ is a graded algebra with multiplication $(x,m)(y,w) = (xy, xw + my)$. If ρ is an algebra norm on $\mathfrak{A}$ and M a normed $\mathfrak{A}$--bimodule with norm $\| \ \|_M$ satisfying $\|xm\|_M, \ \|mx\|_M \leqslant \rho(x)\|m\|$, then $\mathfrak{A} \oplus M$ becomes a graded normed algebra with the norm $\|(x,m)\| = \rho(x) + \|m\|_M$.

Another natural example of a graded algebra is the algebra $\mathcal{T}_n(B)$ of upper triangular matrices in $\mathrm{M}_n(B)$, B some C^*-algebra. The elements of degree k are those for which $a_{ij} = 0, \ j - i \neq k$. It is well known that derivations of B can also be described as homomorphisms into $\mathcal{T}_1(B)$.

An algebra with a decreasing filtration is an algebra $\mathcal{B}$ equipped with a decreasing family of subspaces $\mathcal{B}_k$ of $\mathcal{B}$ such that $\mathcal{B}_0 = \mathcal{B}, \ \mathcal{B}_i\mathcal{B}_j \subset \mathcal{B}_{i+j}$. Every algebra $\mathcal{B}$ with a decreasing filtration can be written as a quotient of an $\mathbb{N}$-graded algebra $\hat{\mathcal{B}}$. Take, for instance, $\hat{\mathcal{B}} = \mathcal{B} \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \ldots$ (with convolution product).

If $\mathcal{I}$ is an ideal in a graded algebra $\mathcal{B}$, then $\mathcal{B}/\mathcal{I}$ is canonically graded iff $p_k(\mathcal{I}) \subset \subset \mathcal{I}$ for all k. On the other hand, a filtered structure on $\mathcal{B}$ is conserved in the quotient by any ideal.

3. DIFFERENTIAL SEMINORMS AND SPECTRAL INVARIANCE

A C^*-(semi)normed algebra is an involutive algebra $\mathfrak{A}$ over $\mathbb{C}$ with a fixed C^*--(semi)norm, which will usually be denoted by $\| \ \|$. A C^*-normed algebra can also be viewed as a dense involutive subalgebra of a C^*-algebra (its completition). A morphism between C^*-seminormed algebras is by definition a C^*-seminorm decreasing $*$-homomorphism.

3.1 DEFINITION. *Let $\mathfrak{A}$ be C^*-(semi)normed. A differential seminorm on $\mathfrak{A}$ is a map $T : \mathfrak{A} \to \ell^1_+(\mathbb{N})$ such that $T_0(x) \leqslant C\|x\|$, for some constant $C > 0$, and such that for $x, y \in \mathfrak{A}$ and for $\lambda \in \mathbb{C}$*

(a) $T(\lambda x) = |\lambda| T(x), \ T(x+y) \leqslant T(x) + T(y)$

(b) $T(xy) \leqslant T(x)T(y)$ *(multiplication in $\ell^1_+(\mathbb{N})$).*

Each T_k is a seminorm. We say that T is a differential norm if $T(x) = 0$ implies $x = 0$. Adding a C^*-norm (if it exists) to T_0 turns any differential seminorm into a differential norm. The sup and sum of a finite family of differential seminorms is differential where $(\sup_\alpha T^\alpha)_i := \sup_\alpha T_i^\alpha$. Given a differential seminorm T and $t \in \mathbf{R}_+$, put $T(t)_j(x) := t^j T_j(x)$. Then $T(t)$ is again a differential seminorm.

3.2 EXAMPLE. Let $\mathfrak{A}$ be a subalgebra of a C^*-algebra A and δ a derivation of $\mathfrak{A}$. Given n, define T by $T_j(x) = \left(\frac{1}{j!}\right)\|\delta^j(x)\|$, $1 \leqslant j \leqslant n$. Then T is a differential norm on $\mathfrak{A}$. In the case where $\mathfrak{A}$ is the Schwartz space $\mathcal{S}(\mathbf{R})$ with the ordinary derivative as derivation, the T_j are the usual j-norms $\| \|_j$ on $\mathcal{S}(\mathbf{R})$, defined as the sup-norm of the j-th derivative of f. A similar construction works, given a finite family of derivations. Assume, to be specific, that we have two derivations δ_1, δ_2 of $\mathfrak{A}$. Then we define a differential seminorm of order 2 setting $T_0(x) = \|x\|$, $T_1(x) = \|\delta_1(x) + \delta_2(x)\|$, $T_2(x) = \|\delta_1\delta_2(x)\|$, where $\| \|$ is a C^*-norm on $\mathfrak{A}$.

As another example, take a p-summable quasihomomorphism, i.e. a pair α, β of $*$-homomorphisms $\mathfrak{A} \to \mathcal{L}(H)$, H a Hilbert space, such that $\alpha(x) - \beta(x) \in \mathcal{L}^p(H)$ for all $x \in \mathfrak{A}$. Then a differential seminorm on $\mathfrak{A}$ is defined by

$$T_0(x) = \max\{\|\alpha(x)\|, \|\beta(x)\|\}, \quad T_1(x) = \|\alpha(x) - \beta(x)\|_p.$$

3.3 PROPOSITION-DEFINITION. *Let T be a differential seminorm. Set $T_{\mathrm{tot}}(x) = \int T(x) = \sum_k T_k(x)$. Then T_{tot} is a submultiplicative seminorm, called the total seminorm of T.*

Proof. The map $\int : \ell^1(\mathbf{N}) \to \mathbf{C}$ is a positive homomorphism. Therefore

$$\int T(xy) \leqslant \int T(x) \int T(y). \qquad \blacksquare$$

Any differential seminorm T can be written as $T = T_0 + \partial T$ where (by abuse of notation), T_0 is the degree 0 part of T and ∂T is the "differential part" of T with $(\partial T)_0 = 0$. Clearly, $(\partial T)^n$ vanishes on the interval $0 \leqslant k \leqslant n-1$. If $T_i(x) \equiv 0$ for $i > n$, we say that the order of T is n. If T is any differential seminorm then its restriction to $0 \leqslant k \leqslant n$ is a differential seminorm of order n. If T is of finite order n then $\partial T(x)^{n+1} = 0$ for all x. If $T_0(a) = 0$, then $T_0(ay) = 0$ for all $y \in \mathfrak{A}$ and $T(ay)^{n+1} = 0$. Thus $J = \{a \in \mathfrak{A} | T_0(a) = 0\}$ is an ideal such that J^{n+1} is contained in the nullspace of T.

The inequality $T(xy) \leqslant T(x)T(y)$ in the definition of a differential seminorm is merely shorthand for the inequality $\partial T(xy) \leqslant C(\|x\|\partial T(y) + \partial T(x)\|y\|) + \partial T(x)\partial T(y)$.

3.4 DEFINITION. *The logarithmic order of T is defined to be $p = \log_2 L + 1$, where L is the minimal positive constant possible in the inequality*

$$\partial T(xy) \leqslant L(\|x\|\partial T(y) + \partial T(x)\|y\|) + \partial T(x)\partial T(y).$$

Note that, for a unital algebra, the logarithmic order is always non-negative and vanishes only if each T_k is dominated by a multiple of $\| \ \|$. For non-unital algebras the situation is different; see the comment in 6.5 (c) below.

Another useful way to write a differential seminorm is to use the isomorphism of the graded algebra $\ell^1(\mathbb{N})$ with an algebra of power series in one variable t. If T is a differential seminorm of degree n, then with each $x \in \mathfrak{A}$ we may associate the n-th order polynomial f_x defined by $f_x(t) = \sum_{0\leqslant i\leqslant n} T_i(x)t^i$. This gives a one-parameter family of submultiplicative seminorms on $\mathfrak{A}$. The defining property of a differential seminorm then corresponds to $f_x(0) \leqslant C\|x\|$ and to the fact that all derivatives at 0 of the function $f_x(t)f_y(t) - f_{xy}(t)$ are positive.

Differential seminorms extend to the unification of $\mathfrak{A}$ and to algebras of matrices over $\mathfrak{A}$ at the cost of raising their logarithmic order.

3.4 PROPOSITION *Let T be a differential seminorm on $\mathfrak{A}$ with $T_0 = C\| \ \|$.*

(a) *An extension of T to a differential seminorm $\tilde{T}$ on the unitification $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ is defined by $\tilde{T}_0 = (1+2C)\| \ \|$, $\lambda \in \mathbf{C}$ and $\partial\tilde{T}(\lambda 1 + x) = \partial T(x)$, $\lambda \in \mathbf{C}$ where $\| \ \|$ is the natural C^*-seminorm on $\tilde{\mathfrak{A}}$.*

(b) *An extension of T to a differential seminorm $\hat{T}$ on the algebra $M_n(\mathfrak{A})$ of matrices over $\mathfrak{A}$ is defined by $\partial\hat{T}((x_{ij})) = \sum_{ij} \partial T(x_{ij})$ and $\hat{T}_0 = nC\| \ \|$ where $\| \ \|$ is the natural C^*-seminorm on $M_n(\mathfrak{A})$.*

Proof. (a)

$$\partial\tilde{T}((\lambda 1 + x)(\mu 1 + y)) = \partial T(\lambda y + \mu x + xy)) \leqslant$$
$$\leqslant (\lambda + T_0(x))\partial T(y) + \partial T(x)(\mu + T_0(y)) + \partial T(x)\partial T(y)$$

One easily checks that $1+2C$ is the best possible constant G for which $|\lambda| + C\|x\| \leqslant$ $\leqslant G\|\lambda 1 + x\|$ (for $\|x\| \leqslant 2$, one has $\|1+x\| \geqslant 1$ and $1 + C\|x\| \leqslant 1 + 2C$, while, for $\|x\| \geqslant 2$, $\|1+x\| \geqslant \|x\| - 1$).

(b) Given $X = (x_{ij})$, $Y = (y_{ij}) \in \mathrm{M}_n(\mathfrak{A})$, one has

$$\partial\hat{T}(XY) = \sum_{ijk} \partial T(x_{ij}y_{jk}) \leqslant$$

$$\leqslant \sum_{ijk} T_0(x_{ij})\partial T(y_{jk}) + \sum_{ijk} \partial T(x_{ij})T_0(y_{jk}) + \sum_{ijk} \partial T(x_{ij})\partial T(y_{jk}) \leqslant$$

$$\leqslant nC\|X\|\sum_{jk}\partial T(y_{jk}) + \left(\sum_{ij}\partial T(x_{ij})\right) nC\|Y\| + \sum_{irsk}\partial T(x_{ir})\partial T(y_{sk}). \qquad \blacksquare$$

Let δ be a derivation of $\mathfrak{A}$ and T a differential seminorm on $\mathfrak{A}$. Then we can define a differential seminorm T' by setting $T'_k(x) = T_k(x) + T_{k-1}(\delta(x))$. If T is of finite order n then T' is of order $n+1$. This is a special case (see section 2) of the following

3.6. PROPOSITION. *Let $\varphi : \mathfrak{A} \to \mathcal{B}$ be a morphism of C^*-seminormed algebras, where $\mathcal{B}$ is $\mathbb{N}$-graded. Let φ_i be the part of degree i of φ, $\varphi(x) = \varphi_0(x) + \varphi_1(x) + \dots$, and let $T : \mathcal{B} \to \ell^1_+(\mathbb{N})$ be a submultiplicative seminorm (i.e. T is subadditive, submultiplicative and homogeneous) with values in $\ell^1_+(\mathbb{N})$ such that the restriction of T_0 to $\mathcal{B}_0$ is bounded by a multiple of a C^*-seminorm on $\mathcal{B}_0$. One defines a differential seminorm T' on $\mathfrak{A}$ by*

$$T'_k(x) = \sum_{i+j=k} T_i(\varphi_j(x)) \quad (T' \textit{is the "convolution product" } T * \varphi).$$

Proof.

$$T'_k(xy) = \sum_{i+j=k} T_i(\varphi_j(xy)) = \sum_{i+r+s=k} T_i(\varphi_r(x)\varphi_s(y)) \leqslant$$

$$\leqslant \sum_{a+b+r+s=k} T_a(\varphi_r(x))T_b(\varphi_s(y)) = \sum_{i+j=k} T'_i(x)T'_j(y). \qquad \blacksquare$$

In particular, every $\mathbb{N}$-graded normed $*$-algebra B, for which the norm on B_0 is bounded by a multiple of a C^*-seminorm, carries a canonical differential seminorm T^B defined by $T^B_k(x) = \|p_k(x)\|_B$.

The following lemma contains the basic estimate which allows us to control the growth of a differential seminorm on products.

3.7 LEMMA. *Let $a(k)$, $k = 0,1,2,\dots$ be elements of $\partial\ell^1_+(\mathbb{N})$ such that $a(k) \leqslant \leqslant Ga(k-1) + a(k-1)^2$ for $k \geqslant 1$ where G is a positive constant. Then*

$$a(k) \leqslant G^k a(0) + kG^{2k}a(0)^2 + \dots + k^{(n-1)}G^{nk}a(0)^n + \dots$$

(since $(a(0)^n)_j = 0$ for $n > j$, the sum on the right hand side is finite in each component j).

Proof. Assume, by induction, that $a(k) \leqslant \sum_j k^{j-1}G^{jk}a(0)^j$. Then

$$a(k+1) \leqslant G\left(\sum_j k^{j-1}G^{jk}a(0)^j\right) + \left(\sum_j k^{j-1}G^{jk}a(0)^j\right)^2 = \sum \mu_j a(0)^j$$

where

$$\mu_j = k^{j-1}G^{jk+1} + \sum_s k^{s-1}G^{sk}k^{(j-s)-1}G^{(j-s)k} = k^{j-1}G^{jk+1} + (j-1)k^{j-2}G^{jk} \leqslant$$

$$\leqslant G^{j(k+1)}(k^{j-1} + (j-1)k^{j-2}) \leqslant G^{j(k+1)}(k+1)^{j-1}. \qquad \blacksquare$$

Here is a typical application of Lemma 3.7. We say that a subset B of $\mathfrak{A}$ is T-bounded if $\|x\| \leqslant 1$ for all $x \in B$ and if $\{T(x)|x \in B\}$ is bounded in $\ell^1_+(\mathbb{N})$.

3.8 LEMMA. *Let T be a differential seminorm of order n on $\mathfrak{A}$ with logarithmic order $p = \log_2 L + 1$ and let B be a T-bounded set with $R = \sup\{\partial T(x)|x \in B\}$. Let $s = 2^k$ and $x_1, \ldots, x_s \in B$. Then*

$$\partial T(x_1 x_2 \ldots x_s) \leqslant (2L)^k R + k(2L)^{2k} R^2 + \cdots + k^{(n-1)}(2L)^{nk} R^n$$

Proof. Let $x \in \mathfrak{A}$ with $\|x\| = 1$. Set $a(j) = \sup\{\partial T(x_1 \ldots x_{2^j})|x_i \in B\}$. Then

$$\partial T(x_1 \ldots x_{2^j}) \leqslant L\|x_1 \ldots x_{2^{j-1}}\| \partial T(x_{2^{j-1}+1} \ldots x_{2^j}) +$$
$$+ L\partial T(x_1 \ldots x_{2^{j-1}})\|x_{2^{j-1}+1} \ldots x_{2^j}\| + \partial T(x_1 \ldots x_{2^{j-1}})\partial T(x_{2^{j-1}+1} \ldots x_{2^j}) \leqslant$$
$$\leqslant 2La(j-1) + (a(j-1))^2$$

whence

$$a(j) \leqslant 2La(j-1) + a(j-1)^2.$$

Thus, by Lemma 3.7

$$\partial T(x_1 x_2 \ldots x_s) \leqslant (2L)^k a(0) + k(2L)^{2k} a(0)^2 + \cdots + k^{n-1}(2L)^{nk} a(0)^n. \qquad \blacksquare$$

3.9 DEFINITION. *Let T be a differential seminorm. We say that the order of T is $\leqslant k$ if, for each i and for each T-bounded sequence $\{x_s\}$,*

$$\limsup_{s \to \infty} \left(\frac{\log(T_i(x_1 x_2 \ldots x_s))}{\log s} \right) \leqslant k.$$

3.10 PROPOSITION. *Let T be a differential seminorm of order n and logarithmic order p. Then the total order of T is $\leqslant np$.*

Proof. If B is T-bounded, then so is $B^2 = \{xy|x, y \in B\}$. Let $\{x_j\}$ be a sequence in B and $2^k \leqslant s < 2^{k+1}$. Then $x_1 x_2 \ldots x_s$ can be written as $y_1 y_2 \ldots y_{2^k}$ with $y_i \in B \cup B^2$. Thus, by 3.8, $\partial T(x_1 x_2 \ldots x_s) \leqslant (2L)^k + k(2L)^{2k} R^2 + \cdots + k^{(n-1)}(2L)^{nk} R^n$, where R is the sup of ∂T over $B \cup B^2$. $\blacksquare$

Cauchy sequences, completion etc. for differential seminorms are defined in the obvious way. A sequence in $\mathfrak{A}$ is a Cauchy sequence for T if it is a Cauchy sequence for each T_i. Let $\mathfrak{A}_T$ be the completion of $\mathfrak{A}$ with respect to a differential norm T of order n. T extends to a differential norm on $\mathfrak{A}_T$ and $\mathfrak{A}_T$ is a Banach algebra for the total norm T_{tot}.

Let A be the C^*-completion of $\mathfrak{A}$ with respect to $\| \ \|$ and assume that $T_0 = C\| \ \|$, $C > 0$. There is a natural continuous map $\mathfrak{A}_T \to A$. The kernel $\mathcal{I}$ of this map consists of all $x \in \mathfrak{A}_T$ for which $T_0(x) = 0$, $\mathcal{I} = \{x | T_0(x) = 0\} = \{x | T(x)^n = 0\}$. Therefore $T(x_1x_2 \ldots x_n) \leqslant T(x_1) \ldots T(x_n) = \partial T(x_1) \ldots \partial T(x_n) = 0$ for $x_1, x_2, \ldots x_n \in \mathcal{I}$ whence $\mathcal{I}^n = \{0\}$.

We say that T is closable if, for any T-Cauchy sequence (x_k) with $\|x_k\| \to 0$, also $T(x_k) \to 0$. Thus T is closable if and only if $\mathcal{I} = \{0\}$ so that $\mathfrak{A}_T \subset A$.

In any case the map $\mathfrak{A}_T \to A$ induces an isomorphism in K-theory. In fact $\mathrm{K}(\mathfrak{A}_T) \cong \mathrm{K}(\mathfrak{A}_T/\mathcal{I})$ since a nilpotent algebra has trivial K-theory and $\mathrm{K}(\mathfrak{A}_T/\mathcal{I}) \cong \mathrm{K}(A)$ since $\mathrm{M}_n(\mathfrak{A}_T/\mathcal{I}) \subset \mathrm{M}_n(A)$ is dense and invariant under holomorphic functional calculus for all n by 3.12.

Remark. If the order of T is 1, then there is a unique closable maximal differential seminorm $\hat{T}$ dominated by T on $\mathfrak{A}$. Take $\hat{T}_0 = T_0$, and for $\hat{T}_1$ the quotient seminorm associated with T_1 on $\mathfrak{A}/J$. In the case of higher order, one obtains in general only a derived seminorm $\hat{T}$, see below.

3.11 Definition. *Let $\mathfrak{A}$ be a C^*-seminormed algebra with C^*-seminorm $\| \ \|$ and let α be a seminorm on $\mathfrak{A}$. We say that α is analytic if for each finite set F in $\mathfrak{A}$, with $\|y\| < 1$ for all $y \in F$, and for each sequence $\{x_s\}$ in F,*

$$\limsup_{s \to \infty} \left(\frac{\log(\alpha(x_1x_2 \ldots x_s))}{s} \right) \leqslant 0.$$

This is the case if and only if the radius of convergence of the complex power series $\sum_s \alpha(x_1x_2 \ldots x_s)z^s$, $z \in \mathbb{C}$, is > 1. Since every finite set F as above can be lifted to a set with the same property, it is clear that any quotient seminorm for an analytic seminorm, in the quotient by a $\| \ \|$-closed ideal, is again analytic.

The seminorms associated with differential seminorms of finite order are analytic by 3.10.

3.12 Proposition. *Let $\mathcal{B}$ be a unital C^*-normed algebra with C^*-norm $\| \ \|$ which is complete with respect to an analytic algebra norm α. Then for each $x \in \mathcal{B}$, the spectral radius of x with respect to α is less or equal to $\|x\|$. Therefore, the spectrum of each $x \in \mathcal{B}$ equals the spectrum of x in the C^*-completion B of $\mathcal{B}$.*

In particular, if T is a differential norm on $\mathfrak{A}$ and $\varphi : \mathfrak{A}_T \to A$ is as above, then the spectrum of x in $\mathfrak{A}_T$ coincides with the spectrum of $\varphi(x)$ in A (so that $\mathfrak{A}_T$ is closed under functional calculus by holomorphic functions in A).

Proof. If $\|x\| < 1$, then $\alpha(x^n) < 1$ for large n, so that the spectral radius of x with respect to α is $\leqslant 1$. Given $w \in B$ with inverse u in B, there is $u' \in B$ such that $\|1 - wu'\| < \frac{1}{2}$. This shows that 1 is not in $\mathrm{Sp}_B(1 - wu')$, so that 0 is not in $\mathrm{Sp}_B wu'$ and w is invertible in B. ■

Now let T be a differential seminorm of order n and of logarithmic order p, $p = \log_2 L + 1$. We assume that $1 \in \mathfrak{A}$ and that $\mathfrak{A}$ is complete with respect to T. Then clearly $T(e^x) \leqslant T(1) + T(x) + \frac{1}{2!}T(x)^2 + \ldots = (T(1) - \mathbf{1}) + e^{T(x)}$, whence $\partial T(e^x) \leqslant e^{T(x)}$.

If $f(z) = \sum a_s z^s$ is a complex power series which absolutely converges for $|z| < r$, then for each $x \in \mathfrak{A}$ with $\|x\| < r$, the series $f(x) = \sum a_s x^s$ converges and $T(f(x)) \leqslant \sum |a_s| T(x)^s$.

3.13 Proposition. *Let x be a selfadjoint element of $\mathfrak{A}$ and*

$$K = \sup\left\{ \left(\frac{1}{t_0}\right)^p \partial T(e^{ixt_0}) \mid \frac{1}{2} \leqslant t_0 \leqslant 1 \right\} \in \ell^1_+(\mathbb{N}).$$

Then

$$\partial T(e^{ixt}) \leqslant t^p K + (\log_2 t + 1)(t^p K)^2 + \cdots + (\log_2 t + 1)^{(n-1)}(t^p K)^n$$

for all $t \geqslant 1$. Moreover K can be estimated by $K \leqslant 2^p g(T(x))\partial T(x) \in \partial\ell^1_+(\mathbb{N})$ where g is the power series representing $\frac{(e^z - 1)}{z}$.

Proof. Let $k = [\log_2 t + 1]$ be the integer part of $\log_2 t + 1$ and $s = \log_2 t - k$ so that $t = 2^{k+s} = t_0 2^k$ where $\frac{1}{2} \leqslant t_0 = 2^s \leqslant 1$. Given $x = x^*$ in $\mathfrak{A}$, put $a(k) = \partial T(e^{ixt_0 2^k})$. We have $a(k) \leqslant 2La(k-1) + a(k-1)^2$ since $\|e^{ixt_0 2^k}\| \leqslant 1$. Thus by Lemma 3.7

$$\partial T(e^{ixt}) \leqslant (2L)^k \partial T(e^{ixt_0}) + k(2L)^{2k}\partial T(e^{ixt_0})^2 + \cdots + k^{(n-1)}(2L)^{nk}\partial T(e^{ixt_0})^n.$$

With $p = \log_2 L + 1$ one has $(2L)^k = 2^{kp} = \left(\frac{t}{t_0}\right)^p$. It follows that

$$\partial T(e^{ixt}) \leqslant \left(\frac{t}{t_0}\right)^p \partial T(e^{ixt_0}) + k\left(\frac{t}{t_0}\right)^{2p} \partial T(e^{ixt_0})^2 + \cdots + k^{(n-1)}\left(\frac{t}{t_0}\right)^{np} \partial T(e^{ixt_0})^n =$$

$$= t^p K(t_0) + k(t^p K(t_0))^2 + \cdots + k^{(n-1)}(t^p K(t_0))^n$$

where $K(t_0) = \left(\frac{1}{t_0}\right)^p \partial T(e^{ixt_0})$. Let K be the sup of $K(t_0)$ in $\frac{1}{2} \leqslant t_0 \leqslant 1$ which exists by the inequality $T(e^x) \leqslant (T(1) - 1) + e^{T(x)}$. We finally obtain

$$\partial T(e^{ixt}) \leqslant t^p K + k(t^p K)^2 + \cdots + k^{(n-1)}(t^p K)^n \leqslant$$

$$\leqslant t^p K + (\log_2 t + 1)(t^p K)^2 + \cdots + (\log_2 t + 1)^{(n-1)}(t^p K)^n$$

for all $t \geqslant 1$. ■

3.14 Remark. In the special case where the order of T is 1 and $\partial T(1) = 0$ we obtain the following estimate:

$$\partial T(e^{ixt_0}) \leqslant (e^{t_0 T(x)})_1 = t_0 \partial T(x) + \frac{t_0^2}{2}(2T_0(x)\partial T(x)) + \frac{t_0^3}{3!}(3T_0(x)^2 \partial T(x)) + \ldots \leqslant$$

$$\leqslant t_0(1 + e^{t_0 L\|x\|})\partial T(x)$$

Thus $\partial T(e^{ixt}) = T_1(x) \leqslant 2^{(p-1)}(1 + e^L)t^p \partial T(x)$ whenever $\|x\| \leqslant 1$, $t \geqslant 1$, whence $\partial T(e^{ix}) \leqslant 2^{(p-1)}(1 + e^L)(1 + \|x\|^{p-1})\partial T(x)$ for all x.

The estimate in 3.13 shows that the order and the logarithmic order of a differential seminorm contribute equally to the growth of $T(e^{ixt})$. This is neither an accident nor due to a limitation of our estimates. In fact, the following two examples show that there are seminorms that may be regarded as the leading term in a differential seminorm of order n and logarithmic order 1, or, at the same time, as the leading term of another differential seminorm of order 1 and logarithmic order n.

3.15 Example. Let Ω be a bounded open subset of $\mathbf{R}^n$ with smooth boundary and X its closure. Let $\mathfrak{A}$ denote the algebra consisting of restrictions, to X, of C^∞--functions on $\mathbf{R}^n$ and let $f \in \mathfrak{A}$. Given $k \in \mathbf{N}$, define $\rho_k(f) = \max\{\|D^\alpha f\| \mid |\alpha| \leqslant k\}$ where $\|\ \|$ is the sup-norm taken over X. Then as before ρ_k may be embedded in a differential norm of order k. But, using Sobolev's inequalities [7, Part 1, 10.1], it follows that $T_1 = \rho_k$ also defines a differential norm of order 1 if we set $T_0(f) = C\|f\|$ where C is a constant that depends on k and X.

3.16 Example. Let $\mathfrak{A}$ be a $*$-algebra acting on some Hilbert space H. Let D be a selfadjoint operator on H and E_λ, $\lambda \in \mathbf{R}$, the projection operators onto the spectral subspaces of D for the interval $(-\infty, \lambda]$. Assume that $\mathfrak{A} = \bigcup \mathfrak{A}_k$ where

$$\mathfrak{A}_k = \{x \in \mathfrak{A} | xE_\lambda(H),\ x^* E_\lambda(H) \subset E_{\lambda+k}(H),\ \forall \lambda\}$$

There are two natural ways to associate differential seminorms on $\mathfrak{A}$ with these data.

(a) Consider $\delta(x) = [D, x]$ as a derivation of $\mathfrak{A}$ into $\mathcal{L}(H)$ and take as above $T_j(x) = (\frac{1}{j!})\|\delta^j(x)\|$, $1 \leqslant j \leqslant n$. T is a differential norm of order n and logarithmic order 1.

(b) First define $\gamma_k(x) = \sup\{\|E^\perp_{\lambda+k} x E_\lambda\|, \|E^\perp_{\lambda+k} x^* E_\lambda\| \,|\, \lambda \in \mathbf{R}\}$. One has

$$\|E^\perp_{\lambda+k} xyE_\lambda\| = \|E^\perp_{\lambda+k} x(E^\perp_{\lambda+\frac{k}{2}} + E_{\lambda+\frac{k}{2}})yE_\lambda\| \leqslant$$

$$\leqslant \|E^\perp_{\lambda+k} xE^\perp_{\lambda+\frac{k}{2}} yE_\lambda\| + \|E^\perp_{\lambda+k} xE_{\lambda+\frac{k}{2}} yE_\lambda\| \leqslant \|x\|\gamma_{\frac{k}{2}}(y) + \gamma_{\frac{k}{2}}(x)\|y\|$$

whence $\gamma_k(xy) \leqslant \|x\|\gamma_{\frac{k}{2}}(y) + \gamma_{\frac{k}{2}}(x)\|y\|$. Now, set $\rho_m(x) = \sup\{k^m\gamma_k(x) | k \in \mathbf{R}_+\}$. One finds

$$k^m\gamma_k(xy) \leqslant 2^m \left(\|x\| \left(\frac{k}{2}\right)^m \gamma_{\frac{k}{2}}(y) + \left(\frac{k}{2}\right)^m \gamma_{\frac{k}{2}}(x)\|y\|\right)$$

so that $\rho_m(xy) \leqslant C(\|x\|\rho_m(y) + \rho_m(x)\|y\|)$ with $m = \log_2 C$. Defining T' by $T'_0(x) = C\|x\|$, $T'_1(x) = \rho_m(x)$, gives a differential norm of order 1 and logarithmic order $m+1$ on $\mathfrak{A}$. We leave it to the reader to verify that, in the case where $H = \ell^2(\mathbf{Z})$, $D(\xi_k) = k\xi_k$ (ξ_k, $k \in \mathbf{Z}$ the orthonormal basis) and $\mathfrak{A}$ is the $*$-algebra generated algebraically by the bilateral shift operator U, $U(\xi_k) = \xi_{k+1}$, the two differential seminorms induce equivalent total norms T_{tot} and T'_{tot} on $\mathfrak{A}$, if $n = m + 1$.

4. REPRESENTATION OF DIFFERENTIAL SEMINORMS

We have seen above that every homomorphism φ from $\mathfrak{A}$ into a graded Banach algebra B with B_0 isomorphic to a C^*-algebra, gives rise to a differential seminorm on $\mathfrak{A}$ by setting $T_i(x) = \|p_i(\varphi(x))\|_B$ where p_i denotes the projection $B \to B_i$ and $\| \;\|_B$ the norm of B. Let us consider the following more general situation, where one obtains a differential seminorm on $\mathfrak{A}$ in an obvious way. Let $\varphi : \mathfrak{A} \to B$ be a linear map from $\mathfrak{A}$ into a graded Banach algebra B such that $\|p_k(\varphi(xy))\|_B \leqslant \|p_k(\varphi(x)\varphi(y))\|_B$ for all k, and such that $\|p_0(\varphi(x))\|_B \leqslant C\|x\|$ for some constant $C \geqslant 1$. If we set $T_i(x) = \|p_i(\varphi(x))\|_B$, $i \geqslant 1$, $T_0(x) = C\|x\|$, we obtain

$$T_k(xy) = \|p_k(\varphi(xy))\|_B \leqslant \|p_k(\varphi(x)\varphi(y))\|_B =$$

$$= \left\| \sum_{i+j=k} p_i(\varphi(x))p_j(\varphi(y)) \right\| \leqslant \sum_{i+j=k} T_i(x)T_j(y)$$

for $k \geqslant 1$, so that T is a differential seminorm.

Let us analyze this situation in more detail. For this, let T^B denote the map $B \to \ell^1(\mathbf{N})$ defined by $T^B(x)_k = \|p_k(x)\|_B$. One has $T^B(xy) \leqslant T^B(x)T^B(y)$. Given a linear map $\varphi : \mathfrak{A} \to B$, the norm curvature of φ is the function τ of two variables with values in $\ell^1(\mathbb{N})$ defined by $\tau(x,y) = T^B(\varphi(x))T^B(\varphi(y)) - T^B(\varphi(xy))$. We have

4.1 PROPOSITION. *Assume that the restriction of* $\|\ \|_B$ *to* B_0 *is equivalent to a* C^**-norm and that* $p_0 \circ \varphi$ *is continuous. Then* $T_i(x) = \|p_i(\varphi(x))\|_B$, $i \geqslant 0$, *gives a differential seminorm on* $\mathfrak{A}$ *if and only if the norm curvature of* φ *is positive,* i.e. *takes values in* $\ell^1_+(\mathbb{N})$.

Proof.

$$\sum_{i+j=k} T_i(x)T_j(y) - T_k(xy) = \sum_{i+j=k} \|p_i(\varphi(x))\|_B \|p_j(\varphi(y))\|_B - \|p_k(\varphi(xy))\|_B = \tau(x,y).$$

■

We will now show that every differential seminorm arises that way. In fact, we will prove much more precise results. For this , we use the free graded algebra $G\mathfrak{A}$ over the vector space $\mathfrak{A}$ and various quotients of it. By definition, $G\mathfrak{A}$ is the universal algebra generated by symbols $d^k(x)$, $x \in \mathfrak{A}$, of degree k, that are linear in x with no further relations. For a fixed multiindex $(i_1, i_2, \ldots, i_k)$, the subspace of type $(i_1, i_2, \ldots, i_k)$ is the subspace generated by all elements of the form $d^{i_1}(x_1)d^{i_2}(x_2)\ldots d^{i_k}(x_k)$. Then $G\mathfrak{A}$ is the direct sum of subspaces of different types. Every differential seminorm T on $\mathfrak{A}$ induces naturally an algebra seminorm $T^{\otimes}$ on $G\mathfrak{A}$ (the projective tensor product seminorm) defined by

$$T^{\otimes}(w) = \inf\left\{\sum T_{i_1}(x_1)T_{i_2}(x_2)\ldots T_{i_k}(x_k) \mid w = \sum d^{i_1}(x_1)d^{i_2}(x_2)\ldots d^{i_k}(x_k)\right\}$$

4.2 PROPOSITION. *Let* T *be any differential seminorm of order* n *on* $\mathfrak{A}$ *for which* T_0 *is equivalent to the* C^**-seminorm of* $\mathfrak{A}$. *There exists a graded Banach algebra* B, *with* B_0 C^**-equivalent, and a linear map* $\varphi : \mathfrak{A} \to B$ *such that*

(a) φ *has positive norm curvature and* $p_0 \circ \varphi$ *is a continuous* $*$*-homomorphism*

(b) $T_i(x) = \|p_i(\varphi(x))\|_B$ *for all* $i \geqslant 0$.

Proof. For B, we take a certain quotient of the algebra $G\mathfrak{A}$. We divide $G\mathfrak{A}$ by the relations $d^0(xy) - d^0(x)d^0(y)$, $x, y \in \mathfrak{A}$ and by the ideal of elements of degree $\geqslant n+1$. On this quotient we define a submultiplicative seminorm σ as the quotient seminorm for $T^{\otimes}$. Thus

$$\sigma(w) = \inf\left\{\sum T_{i_1}(x_1)T_{i_2}(x_2)\ldots T_{i_k}(x_k) \mid w \cong \sum d^{i_1}(x_1)d^{i_2}(x_2)\ldots d^{i_k}(x_k)\right\}$$

where $\cong$ means equality in the quotient.

If $1 \leqslant k \leqslant n$ and w is equivalent to $d^k(x)$ in $G\mathfrak{A}$, then $w = d^k(x) + R$, where R is a sum of elements in subspaces of type $(i_1, i_2, \ldots, i_k)$, with at least one i_j equal to 0 or with $i_1 + i_2 + \cdots + i_k \geqslant n+1$. This shows that the inf in the definition for $\sigma(d^k(x))$ is attained with $T_k(x)$ and thus that $\sigma(d^k(x)) = T_k(x)$. If w is equivalent to $d^0(x)$, then $w = R_0 + R_1$, where R_0 is in the subspace K consisting of sums of elements of type $(i_1, i_2, \ldots, i_k)$ with all $i_j = 0$, while R_1 is a sum of elements of type $(i_1, i_2 \ldots, i_k)$ with at least one $i_j \geqslant 1$. The multiplication map $m : K \to \mathfrak{A}$, K equipped with $T^{\otimes}$, $\mathfrak{A}$ equipped with T_0, is norm decreasing and $m(R_0) = x$. This shows that also $\sigma(d^0(x)) = T_0(x)$. Now, we further divide by the null space of σ and take for B the completion of this quotient, with the norm $\|\ \|_B$ induced by σ. Note that B_0 is isomorphic to the T_0-completion of $\mathfrak{A}$. Also, all the quotients respect the graded structure.

The linear map $\varphi = e^d : \mathfrak{A} \to B$, defined by $e^d = d^0 + d^1 + \cdots + d^n$ (recall that d^i means the symbol and not the i-th power of d) has all the required properties. ■

4.3 Definition. Let T be a differential seminorm of order n on $\mathfrak{A}$. We say that T is *flat* if T can realized as $T_i(x) = \|p_i(\varphi(x))\|_B$, $i \geqslant 0$, for a $*$-homomorphism $\varphi : \mathfrak{A} \to B$ (curvature 0) from $\mathfrak{A}$ into a graded algebra B equipped with an algebra seminorm $\|\ \|_B$.

We say that T is *almost flat* if T can be realized as $T_i(x) = \|p_i(\varphi(x))\|_B$, $i \geqslant 0$, for a $*$-linear map $\varphi : \mathfrak{A} \to B$ from $\mathfrak{A}$ into a graded seminormed algebra B such that $p_k(\varphi(xy) - \varphi(x)\varphi(y)) = 0$ for $k < n$, $x, y \in \mathfrak{A}$.

Remark. A neccesary condition for a differential seminorm T to be flat is that

$$T_k\left(\sum_\alpha x_\alpha y_\alpha\right) \leqslant \sum_{i+j=k} T_i \otimes T_j \left(\sum_\alpha x_\alpha \otimes y_\alpha\right)$$

where $T_i \otimes T_j$ is the projective tensor seminorm on $\mathfrak{A} \otimes \mathfrak{A}$ defined by T_i and T_j. Another neccesary condition will be given in 4.7.

We now show that flat differential norms are induced by the powers of a derivation of a normed algebra D containing $\mathfrak{A}$.

4.4 Theorem. *Let T be differential seminorm of order n on $\mathfrak{A}$. The following are equivalent:*

(a) *T is flat.*

(b) *The quotient seminorm for $T^{\otimes}$ in the quotient $D\mathfrak{A}$ of $G\mathfrak{A}$ takes the value $T_k(x)$ on $d^k(x)$.*

(c) *There is a seminormed algebra D, that contains $\mathfrak{A}$ in such a way that the restriction of the seminorm $\|\ \|_D$ of D to $\mathfrak{A}$ is T_0, and a derivation δ of D, such that $T_i(x) = \frac{1}{i!}\|\delta^i(x)\|_D$, $x \in \mathfrak{A}$.*

Proof. (a)$\Rightarrow$(b): If T is realized by a homomorphism φ from $\mathfrak{A}$ into a graded seminormed algebra B, then, by the universal property of $D\mathfrak{A}$, φ factors through

$e^d : \mathfrak{A} \to D\mathfrak{A}$ and $\|\varphi_k(x)\|_B = T_k(x)$. On the other hand, by construction, the quotient seminorm for $T^{\otimes}$ in $D\mathfrak{A}$ necessarily majorizes any algebra seminorm $\|\ \|'$ on $D\mathfrak{A}$ for which $\|d^k(x)\|' = T_k(x)$, $0 \leqslant k \leqslant n$.

(b)$\Rightarrow$(c): We may take $D\mathfrak{A}$ equipped with the quotient seminorm of $T^{\otimes}$ for D, together with the universal derivation d of $D\mathfrak{A}$.

(c)$\Rightarrow$(a): From D, we construct the graded seminormed algebra $B = D \oplus D \oplus \cdots \oplus D$ (n times), with convolution product and with $\|x\|_B = \sum \|p_i(x)\|_D$. One has $T_k(x) = \|p_k(e^\delta(x))\|_B$ (with $e^\delta = \sum \frac{1}{i!}\delta^i : \mathfrak{A} \to B$, $\delta^i : \mathfrak{A} \to B_i$). ■

REMARK. In 4.4 (c), the derivation δ can be chosen closable if and only if T is closable.

4.5 PROPOSITION. *Every differential seminorm T of finite order majorizes a unique maximal flat differential seminorm S.*

Proof. Let σ be the quotient seminorm for $T^{\otimes}$ in $D\mathfrak{A}$ and set $S_k(x) = \sigma(d^k(x))$. ■

Flat differential seminorms on $\mathfrak{A}$ correspond exactly to certain submultiplicative seminorms on $D\mathfrak{A}$.

We say that two differential seminorms of order n are equivalent if they have equivalent total seminorms. Replacing T with an equivalent seminorm, we arrive at a considerable strengthening of 4.2.

4.6 THEOREM. *Every differential seminorm of order n is equivalent to an almost flat differential seminorm of order n.*

Proof. Given such a differential seminorm T, let $S_k = T_0 + T_1 + \cdots + T_k$. This is an equivalent differential seminorm. Since S_n is the total seminorm of T, any differential seminorm S' of order n with $S'_n = S_n$ and $S'_k \leqslant S_k$, $k < n$, will still be equivalent to T. On $G\mathfrak{A}$ we define the seminorm $S^{\otimes}$ as above. Consider its quotient seminorm $\|\ \|_B$ in the quotient B by the relations $d^k(xy) = \sum_{i+j=k} d^i(x)d^j(y)$ where $k \leqslant n-1$. If $w \in G\mathfrak{A}$ is equivalent to $d^n(x)$ modulo the ideal generated by these relations, then $w = d^n(x) + R$ with R an element in the sum of subspaces of type $(i_1, i_2, \ldots, i_k)$ with at least one $i_j \leqslant n-1$, so that the inf in the definition of the quotient seminorm is attained with $R = 0$ and $\|d^n(x)\|_B = S_n(x)$. As linear map $\mathfrak{A} \to B$ we take of course the composition φ of $d^0 + d^1 + \cdots + d^n : \mathfrak{A} \to G\mathfrak{A}$ with the quotient map $G\mathfrak{A} \to B$. One has $p_k(\varphi(xy) - \varphi(x)\varphi(y)) = d^k(xy) - \sum_{i+j=k} d^i(x)d^j(y) = 0$ in B for $k \leqslant n-1$ and $S_n(x) = \|d^n(x)\|_B = \|p_n(\varphi(x))\|_B$. Put $S'_k(x) = \|d^k(x)\|_B$. S' has the required properties. ■

If T is flat, then the logarithmic order of T does not contribute essentially to the growth of T on products.

4.7 PROPOSITION. *If T is a flat differential seminorm of order n, then the total order of T is $\leqslant n$.*

Proof. Let T be given by $T_i(x) = \frac{1}{i!}\|\delta^i(x)\|_D$ as in 4.4 (c) and let $T_0 = C\|\ \|$. We say that $\{i_1, \dots, i_r\}$ is a partition of the natural number k if $0 \leqslant i_j \leqslant k$ and $i_1 + \dots + i_r = k$. With this notation, one has for each sequence $\{x_j\}$ in $\mathfrak{A}$ and for natural numbers s and k with $s \geqslant k$,

$$\delta^k(x_1 x_2 \dots x_s) =$$

$$= \sum x_1 \dots x_{i_0} \delta^{j_1}(x_{i_0+1}) x_{i_0+2} \dots x_{i_0+i_1+1} \delta^{j_2}(x_{i_0+i_1+2}) \dots \delta^{j_k}(x_{s-i_k}) x_{s-i_k+1} \dots x_s$$

where the sum is taken over all partitions $\{j_1, \dots, j_k\}$ of k with $j_i \geqslant 1$ and over all partitions $\{i_0, \dots, i_k\}$ of $s - k$. If $\|x_s\| \leqslant 1$, for all s, and if $\{T(x_s)\}$ is bounded by $R \in \ell^1_+(\mathsf{N})$, then

$$\|\delta^k(x_1 x_2 \dots x_s)\|_D \leqslant \sum C^{k+1} R_{j_1} R_{j_2} \dots R_{j_k} \leqslant C^{k+1} s^k (R^k)_k$$

since there are $\leqslant s^k$ partitions $\{i_0, \dots, i_k\}$ of $s - k$. The assertion follows since $\|\delta^k(x)\|_D = 0$ for $k > n$. ∎

This shows that the differential seminorms in 3.15 and 3.16 are not flat. We end this section with a construction of (presumably non-flat) differential seminorms from filtered structures.

4.8 PROPOSITION. *Let B be a Banach $*$-algebra and J a closed ideal in B such that B/J is a C^*-algebra with the quotient norm. Let $\mathfrak{A}$ be an involutive subalgebra of B and denote the quotient norm in B/J^k by $\|\ \|_{J^k}$. Then $T_k(x) := := \|x\|_J + \|x\|_{J^2} + \dots + \|x\|_{J^{k+1}}$, $k \leqslant n$, defines a differential seminorm of order n on $\mathfrak{A}$.*

Proof. Let $x, y \in \mathfrak{A}$ and choose x_0, y_0 such that $x - x_0 \in J$, $y - y_0 \in J$ and such that $\|x_0\|$, $\|y_0\|$ approximate $\|x\|_J$ and $\|y\|_J$ respectively. Choose further $r_1, \dots, r_n$ and $s_1, \dots, s_n$ with $r_j, s_j \in J^j$ such that $\|x_k\|$, $x_k = x_0 + r_1 + \dots + r_k$, approximates $\|x\|_{J^{k+1}}$ for each k, while $\|y_k\|$, $y_k = y_0 + s_1 + \dots + s_k$, approximates $\|y\|_{J^{k+1}}$ and such that $x - x_k$, $y - y_k \in J^{k+1}$. Now modulo J^{k+1}

$$(x_0 + r_1 + \dots + r_k)(y_0 + s_1 + \dots + s_k) = x_0(y_0 + s_1 + \dots + s_k) + r_1(y_0 + s_1 + \dots + s_{k-1}) +$$

$$+ r_2(y_0 + s_1 + \dots + s_{k-2}) + \dots + r_k y_0 = \sum_{i+j=k} (x_i - x_{i-1}) y_j$$

where $x_{-1} = 0$. Putting $\hat{T}_i(x) = \|x\|_{J^{i+1}}$, $\|x_i\|_{J^{k+1}}$ approximates $\hat{T}_i(x)$ for $i \leqslant k$ and it follows that

$$\hat{T}_k(xy) = \|xy\|_{J^{k+1}} = \|(x_0 + r_1 + \cdots + r_k)(y_0 + s_1 + \cdots + s_k)\|_{J^{k+1}} \leqslant$$
$$\leqslant \sum_{i+j=k} (\hat{T}_i(x)\hat{T}_j(y) + \hat{T}_{i-1}(x)\hat{T}_j(y)) + \varepsilon = \sum_{i+j=k} \hat{T}_i(x)\hat{T}_j(y) + \sum_{i+j=k-1} \hat{T}_i(x)\hat{T}_j(y) + \varepsilon$$

whence the assertion. ■

5. DERIVED SEMINORMS

We now introduce derived norms as quotient norms for differential seminorms. The analog of this idea in the commutative case would be the following. Consider a compact space X equipped with an algebra of functions ("coordinate functions"). One would then take all embeddings of X as a closed subspace of a compact manifold Y and demand that differentiable functions on X extend, for each such embedding, to differentiable functions on Y if this is the case for all coordinate functions.

5.1 DEFINITION. Let $\mathfrak{A}$ be a C^*-normed algebra. We say that an algebra seminorm α on $\mathfrak{A}$ is a *derived seminorm* (of order $\leqslant k$) if there is a surjective map $\varphi : \mathcal{B} \to \mathfrak{A}$ of C^*-seminormed algebras and a differential seminorm T (of total order $\leqslant k$) on $\mathcal{B}$ such that α is the quotient seminorm for T_{tot}.

It is clear that $\mathcal{B}$ may be chosen to be a free algebra. In the case where $\mathfrak{A}$ is finitely generated, say with generators $x_1, \dots, x_n$, every derived norm on $\mathfrak{A}$ is obtained as a quotient seminorm for the total seminorm of a differential seminorm on the free C^*-normed algebra $\mathcal{F}_C$ on n generators $z_1, \dots, z_n$ with only the relation $\|z_i\| \leqslant C$, for some $C > 0$. In fact, any surjective morphism $\varphi : \mathcal{B} \to \mathfrak{A}$ of C^*-normed algebras gives rise to a surjective morphism $\mathcal{F}_C \to \mathcal{B}$ of C^*-normed algebras for sufficiently large C.

An important immediate consequence of Definition 5.1 is the fact that a quotient norm of a derived seminorm, in a quotient by an ideal that is closed for the C^*-norm, is again in the same class. In special cases, it can be shown that the quotient norm of the total norm for a differential norm is again the total norm of a differential norm (possibly of higher order). This holds, in particular, for differential seminorms of order 1 and logarithmic order 1, using the estimate in 3.14.

Another useful consequence of this definition is the fact that any derived norm α on $\mathfrak{A}$ extends to a derived norm on the completion $\mathfrak{A}_\alpha$ of $\mathfrak{A}$ with respect to α. In fact, if α is the quotient norm for T_{tot} under the map $\pi : \mathcal{B} \to \mathfrak{A}$ and $\mathcal{B}_T$ is the completion of $\mathcal{B}$ with respect to T_{tot}, then π extends to a map $\mathcal{B}_T \to \mathfrak{A}_\alpha$ of norm $\leqslant 1$, for which α is the quotient norm.

The total order of a derived seminorm ρ does not seem to be necessarily finite. However, ρ clearly still is analytic (the quotient norm of any analytic seminorm is analytic) and remains analytic on the completion $\mathfrak{A}_\rho$. In particular, if in analogy with the analysis after 3.10, we consider the ideal $\mathcal{I} = \{x \in \mathfrak{A}_\rho | \, \|x\| = 0\}$ in $\mathfrak{A}_\rho$, then $\mathcal{I}$ is quasinilpotent: $\rho(x^n) \to 0$ for all $x \in \mathcal{I}$. Moreover $\mathfrak{A}_\rho$ and $\mathfrak{A}_\rho/\mathcal{I}$ have the same K-theory. The seminorm ρ is closable if and only if $\mathcal{I} = \{0\}$.

5.2 PROPOSITION. *Let $\mathcal{B}$ be the C^*-normed algebra with a derived seminorm ρ. Let $\varphi : \mathfrak{A} \to \mathcal{B}$ be a morphism of C^*-normed algebras. Then $\sigma(x) = \rho(\varphi(x))$ defines a derived seminorm on $\mathfrak{A}$.*

Proof. Assume that ρ is the quotient seminorm for a total seminorm ρ' on $\mathcal{B}'$ under a surjective map $\beta : \mathcal{B}' \to \mathcal{B}$ of C^*-seminormed algebras. We use the fibered product

$$\mathfrak{A} \bigoplus_{\varphi\beta} \mathcal{B}' = \{(a,b) \in \mathfrak{A} \oplus \mathcal{B}' | \varphi(a) = \beta(b)\}$$

$\mathfrak{A}$ is a quotient of this algebra under the C^*-normed map $\pi : (a,b) \to a$. On $\mathfrak{A} \bigoplus_{\varphi\beta} \mathcal{B}'$ there is a seminorm α defined by $\alpha((a,b)) = \rho'(b)$. The induced derived norm on $\mathfrak{A}$ is given by

$$\sigma(x) = \inf\{\alpha((x,y)) | \varphi(x) = \beta(y)\} = \inf\{\rho'(y) | \beta(y) = \varphi(x)\} = \rho(\varphi(x)). \qquad \blacksquare$$

5.3 PROPOSITION. *The sum of a finite family of derived seminorms is a derived seminorm.*

Proof. Assume that ρ_i is the quotient seminorm for a total seminorm ρ_i' on $\mathcal{B}_i$ under surjective maps $\beta_i : \mathcal{B}_i \to \mathcal{B}$ of C^*-seminormed algebras. We use again a fibered product

$$\mathcal{D} = \bigoplus_{\beta_i} \mathcal{B}_i = \{(x_i) | \beta_i(x_i) = \beta_j(x_j) \ \forall i,j\}$$

Clearly $\alpha((x_i)) = \sum_i \rho'(x_i)$ is the total seminorm of a differential seminorm on $\mathcal{D}$. The quotient seminorm under the natural map $\mathcal{D} \to \mathcal{B}$ is

$$\sigma(x) = \inf\left\{\sum \rho_i'(x_i) | \beta_i(x_i) = x\right\} = \sum \rho_i(x). \qquad \blacksquare$$

5.4 PROPOSITION. *Let δ be a derivation of $\mathcal{B}$ and ρ a derived seminorm on $\mathcal{B}$. Then $\rho + \rho \circ \delta$ is a derived seminorm on $\mathcal{B}$.*

Proof. Assume that ρ is a quotient seminorm for a total seminorm ρ' on $\mathcal{B}'$ under a surjective map $\beta : \mathcal{B}' \to \mathcal{B}$ of a C^*-seminormed algebras and that ρ' is associated with the differential seminorm T on $\mathcal{B}'$. Let $\hat{T}_i$ be the quotient seminorm for T_i on $\mathcal{B}$. If x', y' are preimages of $\delta(x), \delta(y)$ in $\mathcal{B}'$, respectively, then

$$\hat{T}_k(\delta(xy)) = \hat{T}_k(x\delta(y) + \delta(x)y) \leqslant T_k(xy' + x'y) \leqslant \sum_{i+j=k} T_i(x)T_j(y') + T_i(x')T_j(y)$$

whence

$$\hat{T}_k(\delta(xy)) \leqslant \sum_{i+j=k} T_i(x)\hat{T}_j(\delta(x)) + \hat{T}_i(\delta(x))T_j(y).$$

Thus we define a new differential seminorm R on $\mathcal{B}'$ setting $R_k(x) = T_k(x) + \hat{T}_{k-1}(\delta(x))$. The total seminorm for R is $\rho' + \rho \circ \delta$ which induces $\rho' + \rho \circ \delta$ as a quotient seminorm. ■

Let $\mathfrak{A}$ be a unital C^*-normed algebra, T a differential seminorm of total order $\leqslant k$ on $\mathfrak{A}$ and $\tau = T_{\mathrm{tot}}$ its total seminorm. Assume that $\mathfrak{A}$ is complete with respect to τ. Then, for each $x = x^*$ in $\mathfrak{A}$, $\limsup_n \left(\dfrac{\log(\tau(\mathrm{e}^{\mathrm{i}xn}))}{\log(n)}\right) \leqslant k$ whence, for all $\varepsilon > 0$, $\tau(\mathrm{e}^{\mathrm{i}xn}) \leqslant n^{k+\varepsilon}$ for sufficiently large n. For sufficiently large $t \in \mathbf{R}_+$, $t = n + s$, $n \in \mathbf{N}$, $0 \leqslant s \leqslant 1$, $\tau(\mathrm{e}^{\mathrm{i}xt}) = \tau(\mathrm{e}^{\mathrm{i}xs}\mathrm{e}^{\mathrm{i}xn}) \leqslant \tau(\mathrm{e}^{\mathrm{i}xs})n^{k+\varepsilon} \leqslant Kt^{k+\varepsilon}$ where $K = \sup\{\tau(\mathrm{e}^{\mathrm{i}xs}) \mid 0 \leqslant s < 1\}$. This estimate of course is still valid in any C^*-normed quotient of $\mathfrak{A}$ and therefore also holds for derived seminorms.

6. ENVELOPING SMOOTH ALGEBRAS AND FUNCTIONAL CALCULUS

We first consider a class of C^*-(semi)normed algebras that are closed under holomorphic functional calculus.

6.1 DEFINITION. *An analytic algebra is a C^*-seminormed algebra that is a Banach algebra with respect to an analytic (cf. Definition 3.11) algebra norm α.*

The spectral radius with respect to α equals the spectral radius with respect to the C^*-seminorm. Also $*$-homomorphisms between analytic algebras are always C^*-seminorm decreasing. In fact, if φ is such a homomorphism, then using 3.12

$$\|\varphi(y)\|^2 = \|\varphi(y^*y)\| = r(\varphi(y^*y)) \leqslant r(y^*y) = \|y^*y\| = \|y\|^2$$

where r denotes the spectral radius. The completion of a C^*-normed algebra with respect to a derived norm is analytic. If $\mathfrak{A}$ is analytic with the algebra norm α, then $\mathrm{M}_n(\mathfrak{A})$ is analytic with the algebra norm $\alpha((x_{ij})) = \sum_{ij} \alpha(x_{ij})$.

Let $\mathfrak{A}$ be an analytic Banach algebra. We denote by $P(\mathfrak{A})$ the set of all x in $\mathfrak{A}$ that can be lifted under any surjective $*$-homomorphism $\mathcal{B} \to \mathfrak{A}$ between analytic algebras to an element z in $\mathcal{B}$ with $\|z\| \leqslant 1$. Clearly, $P(\mathfrak{A})$ contains the open unit ball in $\mathfrak{A}$ (with respect to the C^*-seminorm) and is contained in the closed unit ball. If $\mathfrak{A}$ is a C^*-algebra, then $P(\mathfrak{A})$ coincides with the closed unit ball in $\mathfrak{A}$. In general, one has $P(\mathfrak{A}) = P(\mathfrak{A})^*$ and $P(\mathfrak{A}) \cdot P(\mathfrak{A}) \subset P(\mathfrak{A})$.

6.2 LEMMA. *Let $\mathfrak{A}$ be an analytic algebra. Then every element of $\mathfrak{A}$ which is a convex combination of unitaries in the augmented algebra $\tilde{\mathfrak{A}}$ is in $P(\mathfrak{A})$. If $\mathfrak{A}$ is unital, then $P(\mathfrak{A})$ contains the set of all convex combinations of unitaries in $\mathfrak{A}$.*

Proof. Let $\pi : \mathcal{B} \to \mathfrak{A}$ be a surjective $*$-homomorphism between analytic algebras and $\tilde{\pi} : \tilde{\mathcal{B}} \to \tilde{\mathfrak{A}}$ the canonical extension to the augmented algebras. It suffices to show that any unitary u in $\tilde{\mathfrak{A}}$ can be lifted to an element z with $\|z\| \leqslant 1$ in $\tilde{\mathcal{B}}$. The unitary $w = u \oplus u^*$ in $\mathrm{M}_2(\tilde{\mathfrak{A}})$ is homotopic to 1, and can therefore, by a standard argument using holomorphic functional calculus, be written as a product of finitely many exponentials $\mathrm{e}^{\mathrm{i}x_k}$, $k = 1, \ldots, m$, x_k selfadjoint in $\mathrm{M}_2(\tilde{\mathfrak{A}})$ (note that the spectrum of any element in $\mathrm{M}_2(\tilde{\mathfrak{A}})$ is equal to the spectrum with respect to the C^*-seminorm). If y_k are selfadjoint lifts for x_k in $\mathrm{M}_2(\tilde{\mathcal{B}})$, and $p = 1 \oplus 0$ in $\mathrm{M}_2(\tilde{\mathcal{B}})$, then $z = p\mathrm{e}^{\mathrm{i}y_1}\mathrm{e}^{\mathrm{i}y_2} \ldots \mathrm{e}^{\mathrm{i}y_m}p \in \tilde{\mathcal{B}}$ satisfies $\tilde{\pi}(z) = u$ and $\|z\| = 1$. The second assertion follows from the fact that every unitary in a unital algebra $\mathfrak{A}$ is a convex combination of two unitaries in $\tilde{\mathfrak{A}}$. ∎

6.3 PROPOSITION. *Let $\mathfrak{A}$ be a C^*-normed algebra which is a Banach algebra with respect to a derived algebra norm α of order $\leqslant k$. Then for each sequence $\{x_s\}$ which takes values in a finite subset F of $P(\mathfrak{A})$ and for all $\varepsilon > 0$*

$$\sum_s \alpha(x_1 x_2 \ldots x_s) s^{-(k+\varepsilon)} < \infty$$

Proof. Let α be the quotient norm for the total norm of a differential seminorm T on $\mathcal{B}$ under a surjective map $\mathcal{B} \to \mathfrak{A}$. Let $\hat{F}$ be a set of elements of $\|\ \| \leqslant 1$ that lifts F and $\{\hat{x}_s\}$ the corresponding lift for $\{x_s\}$. The assertion follows from the fact that $\lim_s (T_{\mathrm{tot}}(\hat{x}_1 \ldots \hat{x}_s) s^{-(k+\delta)}) = 0$ for each δ with $0 < \delta < \varepsilon$. ∎

Let now $\mathfrak{A}$ be a C^*-normed algebra which is also a Banach algebra with respect to an algebra norm α. Note that, if $\mathfrak{A}$ is C^*-normed (not just C^*-seminormed), then α is necessarily closable. Let F be a finite subset of the closed unit ball (with respect to the C^*-norm) of $\mathfrak{A}$. If α is analytic and $\{x_s\}$ is a sequence in F, then $\sum_s \lambda_s x_1 x_2 \ldots x_s$ converges absolutely, with respect to α, for each sequence $\{\lambda_s\}$ of complex numbers for which the radius of convergence of $\sum_s \lambda_s z^s$ is > 1. This means that $\mathfrak{A}$ is stable under functional calculus for certain power series in finitely many non-commuting variables.

If α is a derived norm of order $\leqslant k$ and F is contained in $P(\mathfrak{A})$, then $\sum_s \lambda_s x_1 x_2 \ldots x_s$ converges absolutely, with respect to α, for each sequence $\{\lambda_s\}$ in $\mathbf{C}$ for which $\sum_s |\lambda_s| s^{k+\varepsilon} < \infty$ for some $\varepsilon > 0$. This can, in particular, be applied to the case where the $\{\lambda_s\}$ are the Fourier coefficients of a continuous function on the n-torus $\mathbf{T}^n$. For simplicity, we treat only the case $n = 1$ explicitly. Let $C^k(S^1)$ denote the algebra of functions on S_1 that are k-times continuously differentiable and $C^{k+}(S^1)$ the algebra of continuous functions f on S^1 satisfying $\sum_{s\in\mathbf{Z}} |\hat{f}(s)||s^k| < \infty$. Let $\|f\|_k = \|f^{(k)}\|$ be the ordinary k-norm on $C^k(S^1)$ and $\|f\|_{(k+\varepsilon)+} = \sum_{s\in\mathbf{Z}} |\hat{f}(s)||s|^{k+\varepsilon}$ the natural norm on $C^{(k+\varepsilon)+}(S^1)$. Then

$$C^k(S^1) \supset C^{(k+\varepsilon)+}(S^1) \supset C^{k+1}(S^1)$$

and $\|f\|_k \leqslant \|f\|_{(k+\varepsilon)+} \leqslant C\|f\|_{k+1}$ for $f \in C^{k+1}(S^1)$ and $0 \leqslant \varepsilon < \frac{1}{2}$. In fact, $f^{(k)}(t) = \sum_{s\in\mathbf{Z}} (is)^k \hat{f}(s) e^{ist}$ implies $\|f\|_k \leqslant \|f\|_{k+}$, while, on the other hand

$$\|f\|_{(k+\varepsilon)+} = \sum_{s\in\mathbf{Z}} |\hat{f}(s)|\,|s|^{k+\varepsilon} = \sum_{s\in\mathbf{Z}} |\hat{f}(s)(is)^{k+1}||s|^{\varepsilon-1} \leqslant C\|f\|_{k+1}$$

by Cauchy-Schwartz, where C is the ℓ^2-norm of $\{|s|^{\varepsilon-1}\}_{s\in\mathbf{Z}}$ in $\ell^2(\mathbf{Z})$ for $0 \leqslant \varepsilon < \frac{1}{2}$. In the following, we think of functions on S^1 as periodic functions on $\mathbf{R}$ and identify spaces such as $C^k(S^1)$, $C^{k+}(S^1)$ with spaces of functions on the interval $[-\pi, \pi]$.

6.4 Proposition. *Let $\mathfrak{A}$ be a unital C^*-normed algebra which is complete with respect to a derived norm α of order $\leqslant k$.*

(a) *Let $x = x^* \in \mathfrak{A}$. Then $g(x) \in \mathfrak{A}$ for each C^{k+1}-function g on $\mathbf{R}$. If (without restriction of generality) $\|x\| < \pi$, then $g(x) \in \mathfrak{A}$ for each $g \in C^{(k+\varepsilon)+}(S^1)$, $\varepsilon > 0$, and the map from $C^{(k+\varepsilon)+}(S^1)$ to $\mathfrak{A}$, sending g to $g(x)$ is continuous with respect to α.*

(b) *Let $x_1, \ldots, x_n$ be a family of pairwise commuting selfadjoint elements in $\mathfrak{A}$. Then the closed subalgebra of $\mathfrak{A}$ generated by $x_1, \ldots, x_n$ contains all restrictions of C^p-functions on $\mathbf{R}^n$ to the joint spectrum of $x_1, \ldots, x_n$ for sufficiently large p (depending on n). In particular, $\mathfrak{A}$ is closed under functional calculus by C^∞-functions for normal elements.*

Proof. (a) Assume that $\|x\| < \pi$. Then $g(x) = \sum_{s\in\mathbf{Z}} \hat{g}(s) e^{isx}$ and, for $\varepsilon > 0$,

$$\alpha(g(x)) \leqslant \sum_{s\in\mathbf{Z}} |\hat{g}(s)| \alpha(e^{isx}) \leqslant C \sum_{s\in\mathbf{Z}} |\hat{g}(s)|(1 + |s|^{k+\varepsilon}) = C\|g\|_{(k+\varepsilon)+}$$

for some constant $C > 0$ depending on the one-point set $F = \{e^{ix}\}$.

(b) The argument is the same as in (a) using the finite set $F = \{e^{ix_1}, e^{ix_2}, \ldots, e^{ix_n}\}$.

■

6.5 REMARK. (a) Results on invariance under functional calculus similar to those of this section, but for special differential seminorms, appear already in [10], §3.

(b) If $K \subset \mathbf{R}$ is compact and $\mathcal{C}^\infty_K(\mathbf{R})$ denotes the $\mathcal{C}^\infty$-functions with support in K, then the argument in 6.4 (a) shows that *any* derived norm of order $\leqslant k$ on $\mathcal{C}^\infty_c(\mathbf{R})$ is dominated on $\mathcal{C}^\infty_K(\mathbf{R})$ by $C\| \, \|_{(k+\varepsilon)+}$ for sufficiently large $C > 0$ (take for x a function with compact support such that $x(t) = t$, for $t \in K$).

(c) If under the hypotheses of 6.4, $\mathfrak{A}$ is stable under functional calculus by the square root function, then α is equivalent to the C^*-norm, [6]. In particular, any derived norm of order 0 on a unital C^*-normed algebra is equivalent to the C^*-norm (also [9], Theorem 34.3). On non-unital algebras the situation is quite different. Take, for instance, the Schatten ideal $\mathcal{L}^p(H)$ of p-summable operators on the Hilbert space H. On $\mathcal{L}^p(H)$, $T_1(x) = \|x\|_p$ defines a differential seminorm of logarithmic order 0. This suggests that the relevant notion of order is the order of an extension of a differential seminorm to the augmented algebra.

6.6 DEFINITION. *A smooth C^*-normed algebra is a C^*-normed algebra which is a complete locally convex $*$-algebra with the topology given by the family of all its closable derived norms.*

If $\mathfrak{A}$ is a C^*-normed algebra, then its smooth envelope $\curlywedge\mathfrak{A}$ is the projective limit $\operatorname{inv\,lim}_\alpha \mathfrak{A}_\alpha$ taken over all closable derived norms α with arbitrary order (recall that the derived norms form a directed set by 5.3), where $\mathfrak{A}_\alpha$ denotes the completion of $\mathfrak{A}$ with respect to α.

Every smooth algebra $\mathfrak{A}$ is naturally a dense subalgebra of a C^*-algebra A. Every completion of a C^*-normed algebra with respect to a countable family of closable derived seminorms is a smooth algebra (by the open mapping theorem, every closable seminorm on this completion is continuous). If $\mathfrak{A}$ has finitely many generators $x_1, \ldots, x_k$, then the topology on $\curlywedge\mathfrak{A}$ is determined by a countable family of seminorms. In fact, we obtain such a countable family if, for each n, m, we take the quotient norms, under the natural map $\mathcal{F} \to \mathfrak{A}$, for the total norm of the sup of all differential seminorms T of order n on the free $*$-algebra $\mathcal{F}$ on k generators $x_1, \ldots, x_k$ such that $T_i(x_j) \leqslant m$, $i = 1, \ldots, k$, $j = 1, \ldots, k$. Thus, $\curlywedge\mathfrak{A}$ is a Fréchet algebra in this case.

It is an immediate consequence of Proposition 6.4 that smooth algebras are closed under functional calculus by $\mathcal{C}^\infty$-functions for normal elements and for finitely generated abelian $*$-subalgebras.

6.7 PROPOSITION. *Let $\mathfrak{A}$ be C^*-normed. There are natural C^*-norms on the unitification $\tilde{\mathfrak{A}}$ and the algebra of matrices $M_n(\mathfrak{A})$. If J is a closed $*$-ideal in the C^*--completion A of $\mathfrak{A}$, then there are natural C^*-norms on $J \cap \mathfrak{A}$ and $\mathfrak{A}/(J \cap \mathfrak{A})$. With these norms one has*

(a)

$$\curlywedge\tilde{\mathfrak{A}} \cong \widetilde{\curlywedge\mathfrak{A}} \qquad \curlywedge M_n(\mathfrak{A}) \cong M_n(\curlywedge\mathfrak{A})$$

In particular, $\widetilde{\curlywedge\mathfrak{A}}$ and $M_n(\curlywedge\mathfrak{A})$ are smooth algebras.

(b) *If $\mathfrak{A}$ is a smooth algebra, then $J \cap \mathfrak{A}$ is a dense subalgebra of J and $\mathfrak{A}/(J \cap \mathfrak{A})$ is a smooth algebra. If $\mathfrak{A}$ is metrizable then $J \cap \mathfrak{A}$ is a smooth algebra.*

Proof. (a) All derived norms on $\mathfrak{A}$ extend to derived norms (of higher order) on $\tilde{\mathfrak{A}}$ and on $M_n(\mathfrak{A})$ by 3.4. These norms are closable if the original norms are, since the completion of $\mathfrak{A}$ with respect to such a norm is C^*-normed (not just seminormed) if and only if the norm is closable.

(b) The quotient $\mathfrak{A}/(J \cap \mathfrak{A})$ is a smooth algebra since every derived norm on $\mathfrak{A}$ induces a derived norm on the quotient. Let us show that $J \cap \mathfrak{A}$ is dense in J. Let $x = x^* \in J$ and $\varepsilon > 0$. Choose $y = y^* \in \mathfrak{A}$ with $\|x - y\| < \frac{\varepsilon}{3}$ and let $f \in C_C^\infty(\mathbf{R})$ with $f(t) = 0$ for $|t| < \frac{\varepsilon}{3}$ and $|f(t) - t| < 2\frac{\varepsilon}{3}$ for all t in the spectrum of y. Then $\|f(y) - y\| < \frac{2\varepsilon}{3}$, so $\|f(y) - x\| < \varepsilon$, and $f(y) \in \mathfrak{A}$ by 6.4. If $\pi : A \to A/J$ is the quotient map, then $\|\pi(y)\| = \|\pi(y - x)\| \leqslant \|y - x\| < \frac{\varepsilon}{3}$, so $\pi(f(y)) = f(\pi(y)) = 0$, whence $f(y) \in J$. Since every closable seminorm on $J \cap \mathfrak{A}$ is continuous by the closed graph theorem, if $\mathfrak{A}$ is metrizable, it follows that $J \cap \mathfrak{A}$ is a smooth algebra. ■

Smooth algebras share many properties with C^*-algebras.

6.8 PROPOSITION. *Let $\varphi : \mathfrak{A} \to \mathcal{B}$ be a $*$-homomorphism between smooth algebras. Then*

(a) *φ is norm-decreasing for the C^*-norm.*

(b) *For any closable derived seminorm ρ on $\mathcal{B}$, $\rho \circ \varphi$ is a closable derived seminorm on $\mathfrak{A}$.*

(c) *φ is continuous for the smooth structure.*

(d) *If φ is injective, then φ is isometric for the C^*-norm.*

(e) *$\varphi(\mathfrak{A})$ is a smooth algebra (if φ is not necessarily injective).*

Proof. (a) This follows already from the fact that $\mathfrak{A}$ and $\mathcal{B}$ are analytic.

(b) By 5.2, combined with (a), $\rho \circ \varphi$ is a derived norm. Let us show that $\rho \circ \varphi$ is also closable. If (x_n) is a $\rho \circ \varphi$-Cauchy sequence, then $(\varphi(x_n))$ is a ρ-Cauchy sequence. If $\|\varphi(x_n)\| \to 0$, then $\rho \circ \varphi(x_n) \to 0$, since ρ is closable. Thus $\|x_n\| \to 0$ implies $\rho \circ \varphi(x_n) \to 0$, showing that $\rho \circ \varphi$ is closable.

(c) follows from (b).

(d) φ extends to a $*$-homomorphism $\bar{\varphi} : A \to B$ between the C^*-completions. If J is the kernel of $\bar{\varphi}$, then, by 6.7 (b), $J \cap \mathfrak{A}$ is dense in J. Therefore, if φ is injective, $J = \{0\}$. Thus $\bar{\varphi}$ is isometric for the C^*-norm.

(e) follows from (a) and (d) together with 6.7 (b). ■

6.9 EXAMPLE As an application, we prove two results on cyclic cocycles. The existence of the map $\mathrm{K}_i(A) \to \mathbf{C}$ in (a) is due to Connes.

(a) ([3, Theorem 2.7]) Let τ be an n-trace (in the sense of [3]) on the C^*-normed algebra $\mathfrak{A}$. Then τ extends canonically to a continuous cyclic cocycle on $\mathfrak{A}$, and defines a map $\mathrm{K}_i(A) \to \mathbf{C}$, where A is the C^*-completion of $\mathfrak{A}$ and $i = n \bmod 2$.

(b) Let τ be a cyclic cocycle of dimension $2n$ on the smooth algebra $\mathfrak{A}$, which is positive in the sense that it is the $2n$-dimensional character of a positive trace on $(q\mathfrak{A})^{2n}$ (cf. [5]). Then τ is continuous for the smooth structure on $\mathfrak{A}$.

Proof. (a) By definition, an n-trace τ can be viewed as a closed trace on $\Omega^n\mathfrak{A}$ such that

$$|\tau((x_1da_1)(x_2da_2)\ldots(x_nda_n))| \leqslant C\|x_1\|\cdots\|x_n\|, \quad x_1,\ldots,x_n \in \mathfrak{A}$$

for each fixed n-tuple $a_1,\ldots,a_n$ in $\mathfrak{A}$ with a constant C depending on this n-tuple. For $a \in \mathfrak{A}$ and $a_2,\ldots,a_n$ fixed, let $p(a)$ denote the smallest such constant for $a_1 = a$. Then p is clearly the degree 1 part of a differential seminorm on $\mathfrak{A}$. One easily checks that it is closable. Thus τ can be extended by continuity successively, for $j \leqslant n$, to define $\tau((x_1da_1)(x_2da_2)\ldots(x_nda_n))$ with $a_1,\ldots,a_j$ in $\mathfrak{A}$. The map $\mathrm{K}_i(\mathfrak{A}) \to \mathrm{K}_i(A)$ induced by the inclusion is an isomorphism.

(b) By [5, Th. 15] there exists a pair of morphisms π, $\bar{\pi}$ of $\mathfrak{A}$ into a semifinite von Neumann algebra N with a semifinite trace T such that $qx = \pi(x) - \bar{\pi}(x)$ is in the ideal $\mathcal{L}^{2n}(N) \cap N$ of $2n$-summable operators in N, for all $x \in \mathfrak{A}$ and such that $\tau(x_0, x_1, \ldots, x_{2n}) = T(\pi x_0 qx_1 \ldots qx_{2n})$. Now, $R_0(x) = \|x\|$, $R_1(x) = \|qx\|_{2n}$ is a differential seminorm on $\mathfrak{A}$. It is closable since, if $x_s \to 0$ in C^*-norm and $qx_s \to z$ in $\mathcal{L}^{2n}N$, then $z = 0$. ■

Note, that in (a), we do not really need the closability of the differential seminorms associated with τ in order to prove the existence of the map $\mathrm{K}_i(A) \to \mathbf{C}$. In fact, if $\bar{\mathfrak{A}}$ denotes the completion of $\mathfrak{A}$ with respect to all, not necessarily closable, differential seminorms then $\mathrm{K}_i(\bar{\mathfrak{A}}) \to \mathrm{K}_i(A)$ is an isomorphism, too.

7. FLAT DERIVED NORMS

Let $\mathfrak{A}$ be a C^*-normed algebra and let $D\mathfrak{A}$ be defined as in sections 2 and 4. As above let also p_k be the projection of $D\mathfrak{A}$ onto its degree k part $D^k\mathfrak{A}$. A seminorm α on $D\mathfrak{A}$ is called C^*-graded if $\alpha(p_k(x)) \leqslant \alpha(x)$ for all x and all k, and if its restriction to $D^0\mathfrak{A} \cong \mathfrak{A}$ is bounded by $C\|\ \|$, for some $C > 0$. We make $D\mathfrak{A}$ into a locally convex algebra, with topology defined by the family of all C^*-graded algebra seminorms. (Another natural choice for the topology of $D\mathfrak{A}$, which may be better suited in certain contexts, would be to take all algebra seminorms which are induced by $*$-homomorphisms from $D\mathfrak{A}$ into some C^*-algebra). Denote by $\mathcal{D}\mathfrak{A}$ the completion of $D\mathfrak{A}$ in this structure and by $\mathcal{D}^{\leqslant k}\mathfrak{A}$ the quotient of $\mathcal{D}\mathfrak{A}$ by the ideal of elements of degree $\geqslant k+1$ (note that $\mathcal{D}\mathfrak{A}$ is still graded with continuous projections p_k). The homomorphism $\varphi = e^d : \mathfrak{A} \to \mathcal{D}^{\leqslant k}\mathfrak{A}$ is well defined.

Finally, we write $\mathfrak{A}_k$ for the closure of $\varphi(\mathfrak{A})$ in $\mathcal{D}^{\leqslant k}\mathfrak{A}$. Thus, $\mathfrak{A}_k$ is the completion of $\mathfrak{A}$ with respect to all flat (not necessarily closable) differential seminorms of order k. A C^*-graded algebra seminorm on $D^{\leqslant k}\mathfrak{A}$ induces a closable differential seminorm on $\mathfrak{A}$ if and only if the closure of $\varphi(\mathfrak{A})$ in $\mathcal{D}^{\leqslant k}\mathfrak{A}$ has trivial intersection with the completion $\partial\mathcal{D}^{\leqslant k}\mathfrak{A}$ of $\partial D^{\leqslant k}\mathfrak{A}$.

REMARK. Let β be a C^*-graded algebra seminorm on $D\mathfrak{A}$, T the differential seminorm on $\mathfrak{A}$ induced by β, and $\alpha(x) = \beta(e^d(x))$, $x \in \mathfrak{A}$. Then α is equivalent to T_{tot}. In fact,

$$\left(\frac{1}{k}\right) \sum_{1\leqslant j\leqslant k} \beta(p_j(x)) \leqslant \alpha(x) \leqslant \sum_{1\leqslant j\leqslant k} \beta(p_j(x)) = T_{\mathrm{tot}}(x)$$

7.1 DEFINITION. We denote by $\mathcal{C}^k\mathfrak{A}$ the quotient $\mathfrak{A}_k/(\mathfrak{A}_k \cap \partial\mathcal{D}^{\leqslant k}\mathfrak{A})$. Thus $\mathcal{C}^k\mathfrak{A}$ is naturally a subalgebra of the C^*-completion $\mathcal{C}^0\mathfrak{A}$ of $\mathfrak{A}$.

By 6.4, $\mathcal{C}^k\mathfrak{A}$ is closed under functional calculus by $\mathcal{C}^{(k+\epsilon)+}$-functions for self--adjoint elements.

7.2 PROPOSITION. *Let $\pi : \mathfrak{A} \to \mathcal{B}$ be a surjective morphism of C^*-normed algebras and α a C^*-graded algebra seminorm on $D\mathfrak{A}$. Then the quotient seminorm $\hat{\alpha}$ defined on $D\mathcal{B}$ by the induced surjective map $D\pi : D\mathfrak{A} \to D\mathcal{B}$ is again C^*-graded.*

Proof. We have, for $x \in D\mathcal{B}$,

$$\hat{\alpha}(p_k(x)) = \inf\{\alpha(z) | D\pi(z) = p_k(x)\} = \inf\{\alpha(p_k(z)) | D\pi(z) = p_k(x)\} =$$

$$= \inf\{\alpha(p_k(z)) | D\pi(z) = x\} \leqslant \inf\{\alpha(z) | D\pi(z) = x\} = \hat{\alpha}(x). \qquad \blacksquare$$

7.3 PROPOSITION. *Let $\pi : \mathfrak{A} \to \mathcal{B}$ be a surjective morphism of C^*-normed algebras. Then π induces a surjective map $\mathcal{D}\pi : \mathcal{D}\mathfrak{A} \to \mathcal{D}\mathcal{B}$ and a continuous map $\mathcal{C}^k\mathfrak{A} \to \mathcal{C}^k\mathcal{B}$ for each k.*

Proof. Every morphism of C^*-normed algebras π extends to a continuous map $\mathcal{D}\mathfrak{A} \to \mathcal{D}\mathcal{B}$ since every C^*-graded algebra seminorm ρ on $D\mathcal{B}$ induces a C^*-graded algebra seminorm ρ on $D\mathfrak{A}$ via $\rho(x) = \rho(D\pi(x))$. If π is surjective, the map is surjective by 7.2. Finally, $\mathcal{D}\pi$ maps $\mathfrak{A}_k$ into $\mathcal{B}_k$ and $\partial\mathcal{D}^{\leqslant k}\mathfrak{A}$ into $\partial\mathcal{D}^{\leqslant k}\mathcal{B}$, whence the proposition. ■

7.4 PROPOSITION. *Let δ be a closable derivation of the C^*-normed algebra $\mathfrak{A}$. Then, for each k, δ extends to a continuous derivation $\mathcal{C}^{k+1}\mathfrak{A} \to \mathcal{C}^k\mathfrak{A}$.*

Proof. For each flat differential seminorm T of order k on $\mathfrak{A}$, there is a flat differential seminorm S of order $k+1$ and a constant $C > 0$, such that $T_i(\delta(x)) \leqslant CS_{i+1}(x)$ for $0 \leqslant i \leqslant k$. In fact, δ induces a derivation, still denoted by δ, of $D\mathfrak{A}$ that commutes with d. If T is induced by the algebra seminorm α on $D\mathfrak{A}$, we may use the homomorphism $D\mathfrak{A} \to D\mathfrak{A}$ that sends dx to $(d+\delta)(x)$ and take $S_i(x) := \alpha\left(\frac{1}{i!}(d+\delta)^i(x)\right)$.

Thus, the map sending $x + dx + d^2x + \cdots + d^{k+1}x$ to $\delta x + d\delta x + \cdots + d^k\delta x$ extends to a continuous map (still denoted by δ) $\mathfrak{A}_{k+1} \to \mathfrak{A}_k$. If $w \in \mathfrak{A}_{k+1} \cap \partial\mathcal{D}^{\leqslant k+1}\mathfrak{A}$ then there is a sequence $\{z_n\}$ in $\mathfrak{A}$ such that $z_n \to 0$ for the C^*-norm of $\mathfrak{A}$, while $dz_n \to w^1$, where w^1 is the degree 1 part of w. Now, δz_n converges in $\mathfrak{A}$ to the degree 0 part of $\delta(w)$. On the other hand, if the original δ is closable, then this limit has to be 0, so that $\delta(w) \in \partial\mathcal{D}^{\leqslant k}\mathfrak{A}$. ■

REMARK. We also have $\mathrm{M}_n(\mathcal{C}^k\mathfrak{A}) \cong \mathcal{C}^k(\mathrm{M}_n(\mathfrak{A}))$. In fact, there is a natural map $D\mathrm{M}_n(\mathfrak{A}) \to \mathrm{M}_n(D\mathfrak{A})$ sending $d(m \otimes x)$ to $m \otimes dx$, $m \in \mathrm{M}_n$, $x \in \mathfrak{A}$. It extends by continuity to a map $\mathcal{D}\mathrm{M}_n(\mathfrak{A}) \to \mathrm{M}_n(\mathcal{D}\mathfrak{A})$ mapping $\mathcal{C}^k(\mathrm{M}_n(\mathfrak{A}))$ into $\mathrm{M}_n(\mathcal{C}^k(\mathfrak{A})$. Since the restriction of this map to $\mathcal{C}^k\mathfrak{A}$ is an isomorphism, the assertion follows. We also have $\mathcal{C}^k\tilde{\mathfrak{A}} \cong \widetilde{\mathcal{C}^k\mathfrak{A}}$ by a similar argument.

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