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MEROMORPHIC CONTINUATION OF THE CUTOFF RESOLVENT FOR SYMMETRIC SYSTEMS OF NONCONSTANT DEFICIT

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0. INTRODUCTION

Many equations of the mathematical phisics have the form:

$$\partial_t u(t,x) = E(x)^{-1} \sum_{j=1}^{n} A_j \partial_{x_j} u(t,x) \text{ on } \mathbf{R}_t \times \mathbf{R}^n_x, \tag{0.1}$$

where $n \geqslant 2$, A_j's are constant Hermitian $(d \times d)$ matrices, $E(x) \in C^\infty(\mathbf{R}^n;\ \mathrm{Hom}\, \mathbf{C}^d)$ is a Hermitian positively definite $(d \times d)$ matrix which is a compact perturbation of I, the identity $(d \times d)$ matrix. As is well known, the solutions to (0.1) can be expressed by a unitary group $U(t) = \exp(tG)$ in the Hilbert space H_E which is the space $L^2(\mathbf{R}^n; \mathbf{C}^d)$ equiped with the scalar product

$$(f,g)_E = \int_{\mathbf{R}^n} f \cdot Eg \, dx,$$

where G is the skew-selfadjoint realization of the operator $E^{-1}\sum_{j=1}^{n} A_j \partial_{x_j}$ in H_E. Then, the resolvent $R(z) = (G - z)^{-1}$ is well defined for $z \in \mathbf{C}$, $\mathrm{Re}\, z > 0$, as a bounded operator from H_E into itself. One of the main problems concerning the operator G is the one of meromorphic continuation of the cutoff resolvent $R_\chi(z) = \chi R(z) \chi$ where $\chi \in C_0^\infty(\mathbf{R}^n)$ is a suitable function. The poles of such a continuation are called resonances and are objects of great interest. Set

$$A(\xi) = \sum_{j=1}^{n} A_j \xi_j,\ \xi \in \mathbf{R}^n \setminus 0,$$

The operator G is elliptic when $\mathrm{Rank}\, A(\xi) = d$ for all $\xi \in \mathbf{R}^n \setminus 0$. When Rank $A(\xi)$ =const for all ξ the operator G is said to be of constant deficit, and when Rank

$A(\xi)$ varies as ξ varies the G is said to be of nonconstant deficit. It is eay to see that G is of nonconstant deficit if and only if there are eigenvalues of $A(\xi)$ vanishing for some $\xi \in \mathbf{R}^n \setminus 0$. It is well known that when the G is elliptic and the space dimension n is odd the cutoff resolvent $R_\chi(z)$ admits a meromorphic continuation from $\{z \in \mathbf{C},\ \mathrm{Re}\, z > 0\}$ to the whole complex plane $\mathbf{C}$, while for even n the $R_\chi(z)$ can be meromorphically extended to the whole Riemann surface Λ of $\log z$ instead of the complex plane (see [2], [9]). Moreover, a similar continuation holds for the scattering matrix and the poles of this continuation coincide with the poles of $R_\chi(z)$. In [11] Weder has obtained similar results when the G is of constant deficit.

In the case of nonconstant deficit, however, the problem of meromorphic continuation of the cutoff resolvent, to our knowledge, is still open. In [8], under some natural assumptions on the vanishing eigenvalues of the $A(\xi)$, Tamura has proved the limiting absorption principle for G of nonconstant deficit. So, it is natural to expect that the mentioned problem can be solved under the same assumptions as in [8]. The purpose of this work is to prove this in the case when $E(x) = b(x)I$ where $b(x)$ is a scalar function. Note that even in this "simple" case the problem of meromorphic continuation of the scattering matrix is still open. To our knowledge, the only previous result concerning the meromorphic continuation of the cutoff resolvent for operators of nonconstant deficit is obtained by Rauch [6] for more general systems of nonconstant deficit but under the nontrapping hypothesis. To obtain a uniform decay of the local energy for such systems he proves that the cutoff resolvent admits a meromorphic continuation from $\{z \in \mathbf{C},\ \mathrm{Re}\, z \gg 1\}$ to Λ, the logarithmitc Riemann surface, in both cases of odd and even space dimensions. Note that in this situation it is also not clear whether the meromorphic continuation of the cutoff resolvent implies a meromorphic continuation of the scattering matrix.

Our approach follows the one used by Vainberg (see also [6]) to obtain meromorphic continuations of the cutoff resolvent in the case of nontrapping perturbations. Although the perturbation we consider is not of this type, the restriction on the matrix $E(x)$ enables us, by a suitable microlocalization, to reduce the study of the solutions to (0.1) to the study of the solutions to a pseudodifferential hyperbolic equation for which the propagation of the singularities is well studied and whose bicharacteristic curves are easily seen to be nontrapping.

The paper is organized as follows. In Section 1 we introduce our assumptions and state the main result. In Section 2 we state some well known results without proofs. In Sections 3 and 4 we prove the main theorem.

1. ASSUMPTIONS AND MAIN RESULTS.

Denote by H_b the space defined as H_E replacing $E(x)$ by $b(x)I$, where $b(x)$ is a scalar function satisfying the assumptions:

$$(1.1)\qquad \begin{cases} \text{i) } b(x) \in C^\infty(\mathbf{R}^n); \\ \text{ii) there exists a } b_0 > 0 \text{ so that } b(x) \geqslant b_0 \text{ for all } x \in \mathbf{R}^n; \\ \text{iii) there exists a } \rho > 0 \text{ so that } b(x) = 1 \text{ for } |x| \geqslant \rho. \end{cases}$$

In H_b consider the operator

$$G = b(x)^{-1} \sum_{j=1}^{n} A_j \partial_{x_j},$$

where the matrices A_j are as above. Set $R(z) = (G - z)^{-1}$ for $\operatorname{Re} z > 0$. Recall the definition of the matrix $A(\xi)$. It is clear that $A(\xi)$ can be written in the form:

$$A(\xi) = \sum_{j=1}^{m} \lambda_j(\xi) \Gamma_j(\xi)$$

where $m \leqslant d$, $\lambda_j(\xi)$'s are the nonidentically zero eigenvalues of the $A(\xi)$, $\Gamma_j(\xi)$'s are the corresponding orthogonal projections. Moreover, $\lambda_j(\xi)$, $\Gamma_j(\xi)$'s are continuous functions positively homogeneuous of degree one and zero, respectively. Let $\lambda_j(\xi)$, $j = 1, \ldots, k$, $k \leqslant m$, be the eigenvalues vanishining for some $\xi \in \mathbf{R}^n \setminus 0$, i.e. those ones for which

$$\Sigma_j = \{\xi \in \mathbf{R}^n \setminus 0 : \lambda_j(\xi) = 0\} \neq \emptyset.$$

Our next assumption is the following:

$$(1.2)\qquad \begin{cases} \text{Each } \Sigma_j,\ 1 \leqslant j \leqslant k, \text{ has an open conic neighbourhood } \Sigma_j' \\ \text{so that } \lambda_j(\xi),\ \Gamma_j(\xi) \in C^\infty(\Sigma_j') \text{ and } |\nabla_\xi \lambda_j(\xi)| \neq 0 \\ \text{for all } \xi \in \Sigma_j'. \end{cases}$$

Let $\chi(x) \in C_0^\infty(\mathbf{R}^n)$ be such that $\chi = 1$ for $|x| \leqslant \rho + 1$. Our main result is the following

THEOREM 1. *Under the assumptions* (1.1) *and* (1.2), *the cutoff resolvent* $R_\chi(z)$ *admits a meromorphic continuation from* $\{z \in \mathbf{C},\ \operatorname{Re} z > 0\}$ *to the logarithmic Riemann surface* Λ *with discrete set of poles which has no finite accumulation points.*

REMARK 1. Theorem 1 should hold under the assumption (1.2) for any matrix $E(x)$ which is a compact perturbation of the identity matrix.

REMARK 2. By using our method it is posible to prove Theorem 1 for a matrix $E(x)$ of the form $E(x) = \sum_{j=1}^{r} e_j(x) E_j$ where $e_j(x)$'s are scalar functions satisfying

(1.1), E_j's are constant orthogonal projections such that $\sum_{j=1}^{r} E_j = I$ and satisfying the conditions: $[E_i, A_j] \stackrel{\text{def}}{=} E_i A_j - A_j E_i = 0$ for any $i = 1, \ldots, r$, $j = 1, \ldots, n$. Unfortunately, this is not the case for many important systems of the form (0.1).

2. PRELIMINARIES.

Given two Hilbert spaces X and Y, $\mathcal{L}(X,Y)$ will denote the space of all linear bounded operators acting from X into Y. Let $P(t) \in \mathcal{L}(X,Y)$, $t \in \mathbf{R}^+$, be such that $P(t)f \in L^1_{\text{loc}}(\mathbf{R}^+; Y)$, $\forall f \in X$, and $\|P(t)\|_{\mathcal{L}(X,Y)} \leqslant C$, $\forall t \in \mathbf{R}^+$ with $C > 0$ independent of t. Now we can introduce the Fourier-Laplace trnsform $\widehat{P}(z) \in \mathcal{L}(X,Y)$ of $P(t)$ as follows:

$$\widehat{P}(z) = \int_0^\infty e^{-tz} P(t)\, dt \quad \text{for } \operatorname{Re} z > 0.$$

The proof of the following proposition can be found in [6], [9].

PROPOSITION 2. *Let $P(t)$ be as above and assume in aditional that $P(t)$ admits an analytic continuation to the region $\{t \in \mathbf{C},\ |t| \geqslant T\}$ for some $T > 0$, which has the following expansion at $t = \infty$:*

$$P(t) = \sum_{j=0}^{\infty} t^{-n-j} P_j \quad \text{for } t \in \mathbf{C},\ |t| \to \infty, \tag{2.1}$$

with operators $P_j \in \mathcal{L}(X,Y)$. Then the Fourier-Laplace transform $\widehat{P}(z)$ of $P(t)$ admits an analytic continuation to Λ with values in $\mathcal{L}(X,Y)$ which has the following expansion at $z = 0$:

$$\widehat{P}(z) = P'(z) + P''(z) z^{n-1} \log z \tag{2.2}$$

where $P'(z)$ and $P''(z)$ are holomorphic $\mathcal{L}(X,Y)$-valued functions.

Denote by $\mathcal{F}$ the d-dimensional Fourier transform. Given a $s \geqslant 0$, introduce the Sobolev space

$$H^s = \left\{ f \in L^2(\mathbf{R}^n; \mathbf{C}^d) : \|f\|_s = \left\| \mathcal{F}^{-1} (1 + |\xi|^2)^{s/2} \mathcal{F} f \right\|_{L^2} < \infty \right\}.$$

We shall write H instead of H^0. Let G_0 be the skew-selfadjoint realization of the operator $\sum_{j=1}^{n} A_j \partial_{x_j}$ in H. Denote by Π_0 the orthogonal projection onto $(\operatorname{Ker} G_0)^\perp$.

It is easy to see that $\Pi_0 = \Gamma(D_x) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\Gamma(\xi)\mathcal{F}$ where $\Gamma(\xi) = \sum_{j=1}^{m} \Gamma_j(\xi)$. Set $U_0(t) = \exp(tG_0)$, $U_0'(t) = U_0(t)\Pi_0$, $U_{0,\chi}'(t) = \chi U_0'(t)\chi$. Using the representation

$$U_0'(t) = \sum_{j=1}^{m} \mathcal{F}^{-1} e^{it\lambda_j(\xi)} \Gamma_j(\xi)\mathcal{F}$$

together with the assumption (1.2), one can prove the following proposition by the methods developed in [6], [9].

PROPOSITION 3. *There exists a $T_0 > 0$ so that $U_{0,\chi}'(t) \in \mathcal{L}(H, H^s)$ for any s and for $t \geqslant T_0$ and admits an analytic continuation to the region $\{t \in \mathbf{C},\ |t| \geqslant T_0\}$ with values in $\mathcal{L}(H, H^s)$, $\forall s$, which has an expansion of the form (2.1) at $t = \infty$.*

3. PROOF OF THEOREM 1.

Introduce the sets $K_1 = \{(t,x) \in \mathbf{R}^{n+1} : t \geqslant T+1,\ |x| \leqslant \rho+3\}$ and $K_2 = \{(t,x) \in \mathbf{R}^{n+1} : t \geqslant T,\ |x| \leqslant \rho + 4\}$, where $T > 0$ is a parameter to be chosen latter on. Choose a function $\eta(t,x) \in C^\infty(\mathbf{R}^{n+1})$ so that $\eta = 0$ in K_1, $\eta = 1$ outside K_2 and all the derivates of η are bounded on $\mathbf{R}^{n+1}$. Set $F(t) = (\partial_t - G)\eta\Pi_0 U(t)$. Since $G\Pi_0 = b^{-1}G_0\Pi_0 = b^{-1}G_0 = G$, it is easy to see that $F(t) = L(t)U(t)$ with $L(t) = \eta_1\Pi_0 - \eta(I - \Pi_0)[b^{-1}, G_0]$ where $\eta_1 = \partial_t\eta - [G, \eta]$ is a matrix-valued function with entries of class $C^\infty(\mathbf{R}^{n+1})$ supported in $K_2 \setminus K_1$. Let $V(t)$ be the solution to the equation

$$(\partial_t - G_0)V(t) = F(t), \quad V(0) = 0. \tag{3.1}$$

By Duhamel's formula,

$$V(t) = \int_0^t U_0(t-s)F(s)\mathrm{d}s. \tag{3.2}$$

Setting $Q = G - G_0$ and writing (3.1) in the form

$$(\partial_t - G)(\eta\Pi_0 U(t) - V(t)) = QV(t),$$

we obtain by Duhamel's formula,

$$\eta\Pi_0 U(t) - V(t) = U(t)\Pi_0 + \int_0^t U(t-s)QV(s)\,\mathrm{d}s.$$

Taking Fourier-Laplace transform of this identity and using that $\widehat{U}(z) = R(z)$ we get

$$\eta\widehat{\Pi_0 U}(z) - \widehat{V}(z) = R(z)\Pi_0 + R(z)Q\widehat{V}(z) \quad \text{for } \operatorname{Re} z > 0. \tag{3.3}$$

Now it is easy to see that $R(z)(I - \Pi_0) = -(I - \Pi_0)z^{-1}$ for $\operatorname{Re} z > 0$. Hence, multiplying (3.3) by χ on the both sides and taking into account that $Q = \chi Q$, we obtain

$$R_\chi(z) = (\chi\eta\widehat{\Pi_0 U}(z)\chi - \chi\widehat{V}(z)\chi + \chi(I - \Pi_0)\chi z^{-1})(1 + Q\widehat{V}(z)\chi)^{-1} \tag{3.4}$$

for $\operatorname{Re} z > 0$. Since $\chi\eta(t,x) = 0$ for $t \geqslant T + 1$, the term $\chi\eta\widehat{\Pi_0 U}(z)\chi$ has an analytic continuation to the whole $\mathbb{C}$ with values in $\mathcal{L}(H, H)$. Also, the term $\chi(I - \Pi_0)\chi z^{-1}$ has an analytic continuation to the whole Λ with values in $\mathcal{L}(H, H)$. To deal with the other terms in (3.4) we need the following

Lemma 4. *For a suitable choice of the parameter T the function $\chi G_0\widehat{V}(z)\chi$ (resp. $\chi\widehat{V}(z)\chi$) admits an analytic continuation from $\{z \in \mathbb{C},\ \operatorname{Re} z > 0\}$ to Λ with values in $\mathcal{L}(H, H^1)$ (resp. $\mathcal{L}(H, H)$). Moreover, the continuation of the $\chi G_0\widehat{V}(z)\chi$ has an expansion of the form* (2.2) *at $z = 0$.*

Assuming for a moment that the conclusions of Lemma 4 are fulfilled, we shall complete the proof of Theorem 1. Since $Q\widehat{V}(z)\chi = (b^{-1}-1)\chi G_0\widehat{V}(z)\chi$, it follows from the above lemma that the $Q\widehat{V}(z)\chi$ has an analytic continuation to Λ with values in $\mathcal{L}(H, H^1)$ and hence, by Rellich's compactness theorem, with values in the compact operators in $\mathcal{L}(H, H)$. Then, by the analytic Fredholm theorem, $(1 + Q\widehat{V}(z)\chi)^{-1}$ exists as a meromorphic function on Λ with values in $\mathcal{L}(H, H)$, which has a discrete set of poles possessing no finite accumulation points different from $z = 0$. The fact that the poles cannot accumulate at $z = 0$ follows from the fact that the $Q\widehat{V}(z)\chi$ has an expansion of the form (2.2) at $z = 0$ and Theorem 2 in [6]. Thus, the conclusions of Theorem 1 follow from Lemma 4 and (3.4).

4. PROOF OF LEMMA 4

First we shall study $\chi V(t)\chi$ and $\chi G_0 V(t)\chi$ for large t. To this end, we rewrite the equation (3.1) in the form:

$$(\partial_t - G_0)(V(t) - \eta\Pi_0 U(t)) = -Q\eta\Pi_0 U(t)$$

and by Duhamel's formula,

$$V(t) - \eta\Pi_0 U(t) = -U_0(t)\Pi_0 - \int_0^t U_0(t-s)Q\eta(s,x)\Pi_0 U(s)ds. \tag{4.1}$$

Multiplying (4.1) by χ on the both sides and taking into account that $Q = \chi Q = Q\chi$, $\chi\eta(t,x) = 0$ for $t \geqslant T+1$ and that $U_0(t) = U_0'(t) + (I - \Pi_0)$, we conclude that $\chi V(t)\chi = L_1(t) + L_2$ for $t \geqslant T+1$, where

$$L_1(t) = -U'_{0,\chi}(t) - \int_0^{T+1} U'_{0,\chi}(t-s)Q\eta\Pi_0 U(s)\chi\,\mathrm{d}s,$$

$$L_2 = \int_0^{T+1} \chi(I-\Pi_0)\left[G_0, b^{-1}\right]\eta\Pi_0 U(s)\chi\,\mathrm{d}s.$$

Obviously, $L_2 \in \mathcal{L}(H,H)$ and is independent of t. By Proposition 3 $L_1(t)$ admits an analytic continuation to the region $\{t \in \mathbf{C},\ |t| \geqslant T_1 = T_0 + T + 1\}$, which has an expansion of the form (2.1) at $t = \infty$. Hence, by Proposition 2 we conclude that the expression

$$\int_{T_1}^{\infty} \mathrm{e}^{-tz} L_1(t)\,\mathrm{d}t$$

has an analytic continuation from $\{z \in \mathbf{C},\ \operatorname{Re} z > 0\}$ to Λ with values in $\mathcal{L}(H,H)$. Furthermore, obviously

$$\int_{T_1}^{\infty} \mathrm{e}^{-tz} L_2\,\mathrm{d}t = L_2 z^{-1}\mathrm{e}^{-zT_1} \quad \text{for} \quad \operatorname{Re} z > 0$$

and hence it has an analytic continuation to Λ with values in $\mathcal{L}(H,H)$. On the other hand, clearly the expression

$$\int_0^{T_1} \mathrm{e}^{-tz}\chi V(t)\chi\,\mathrm{d}t$$

has an analytic continuation to the whole $\mathbf{C}$ with values in $\mathcal{L}(H,H)$. Thus, we have established that $\chi\widehat{V}(z)\chi$ admits an analytic continuation from $\{z \in \mathbf{C},\ \operatorname{Re} z > 0\}$ to Λ with values in $\mathcal{L}(H,H)$.

Choose a function $\chi_1 \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_1 = 1$ on $\operatorname{supp}\chi$. Multiplying (4.1) by χG_0 on the left and by χ on the right and taking into account that $G_0U_0(t) = G_0U_0'(t)$, as above we obtain

$$\chi G_0 V(t)\chi = -\chi G_0 U'_{0,\chi_1}(t)\chi - \int_0^{T+1} \chi G_0 U'_{0,\chi_1}(t-s)Q\eta\Pi_0 U(s)\chi\,\mathrm{d}s$$

for $t \geqslant T+1$. Now, by Proposition 3 we conclude that $\chi G_0 V(t)\chi$ has an analytic continuation to the region $\{t \in \mathbf{C},\ |t| \geqslant T_1\}$ with values in $\mathcal{L}(H,H^s)$, $\forall s$, which has an expansion of the form (2.1) at $t = \infty$. Hence, by Proposition 2 we deduce that the expression

$$\int_{T_1}^{\infty} e^{-tz} \chi G_0 V(t) \chi \, dt$$

has an analytic continuation from $\{z \in \mathbf{C},\ \operatorname{Re} z > 0\}$ to Λ with values in $\mathcal{L}(H, H^s)$, $\forall s$. Hence, to complete the proof of Lemma 4 it sufficies to show that $\chi S(z) \chi$ has an analytic continuation to $\mathbf{C}$ with values in $\mathcal{L}(H, H^1)$, where

$$S(z) = \int_0^{T_1} e^{-tz} G_0 V(t) \, dt \quad \text{for} \quad \operatorname{Re} z > 0.$$

Since $G_0 = \Pi_0 G_0$, in view of (3.1), we have

$$S(z) = \int_0^{T_1} e^{-tz} \Pi_0 (\partial_t V(t) - F(t)) \, dt =$$

$$(4.2) \qquad = e^{-zT_1} \Pi_0 V(T_1) + z \int_0^{T_1} e^{-tz} \Pi_0 V(t) \, dt -$$

$$- \int_0^{T_1} e^{-tz} \Pi_0 F(t) \, dt = e^{-zT_1} \Pi_0 V(T_1) + z S_1(z) - S_2(z),$$

where the last equality defines $S_1(z)$ and $S_2(z)$. In the same way as above it is easy to see that $\chi \Pi_0 V(T_1) \chi \in \mathcal{L}(H, H^1)$. Now the representation (4.2) shows that the desired result will be proved if we show that

$$(4.3) \qquad \begin{gathered} S_j(z)\chi \text{ has an analytic continuation to the whole } \mathbf{C} \\ \text{with values in } \mathcal{L}(H, H^1),\ j = 1, 2. \end{gathered}$$

In view of (3.1) we have

$$S_1(z) = \Pi_0 \int_0^{T_1} e^{-tz} \int_0^t U_0(t-s) \eta_1 \Pi_0 U(s) \, ds \, dt -$$

$$- \int_0^{T_1} e^{-tz} \int_0^t U_0(t-s) \Pi_0 \eta (I - \Pi_0) \left[b^{-1}, G_0 \right] U(s) \, ds \, dt =$$

$$= \Pi_0 S_3(z) + S_5(z),$$

and also,

$$S_2(z) = \Pi_0 \int_0^{T_1} e^{-tz} \eta_1 \Pi_0 U(t) \, dt -$$

$$-\int_0^{T_1} e^{-tz}\Pi_0\eta(I-\Pi_0)\left[b^{-1},G_0\right]U(t)\,dt =$$
$$= \Pi_0 S_4(z) + S_6(z),$$

where the last equalities define $S_j(z)$, $j = 3,\dots,6$. Now, clearly $\Pi_0\eta(I-\Pi_0) = [\Pi_0,\eta](I-\Pi_0)$. Recall that $\Pi_0 = \Gamma(D_x)$, $\Gamma(\xi)$ being introduced in Section 2. It follows easily from the assumption (1.2) that $\Gamma(\xi) \in C^\infty(\mathbb{R}^n\setminus 0)$. Hence $[\Gamma(D_x),\eta] \in \mathcal{L}(H,H^1)$, $\forall t$. Thus we conclude that $S_j(z)$, $j = 5,6$, are holomorphic in $\mathbb{C}$ with values in $\mathcal{L}(H,H^1)$. We are going to show that (4.3) holds with $S_j(z)$, $j = 3,4$. Fix a $\xi_0 \in \mathbb{R}^n$, $|\xi_0| = 1$, and let $\varphi(\xi) \in C^\infty(\mathbb{R}^n)$ vanish for $|\xi| \leqslant \frac{1}{2}$, be homogeneous of degree zero for $|\xi| \geqslant \frac{2}{3}$ and let $\varphi = 1$ in a neighbourhood of ξ_0. Define $S_j'(z)$, $j = 3,4$, by replacing in the definition of $S_j(z)$, $j = 3,4$, Π_0 by $(\varphi\Gamma_1)(D_x)$. Clearly, if we prove (4.3) with $S_j'(z)$, $j = 3,4$, no matter how small the support of φ is, the desired result will follow from a partition-of-unity argument and the identity $\Pi_0 = \sum_{j=1}^{m}\Gamma_j(D_x)$. It is sufficient to study $S_3'(z)$ only, since $S_4'(z)$ can be treated similarly. We shall consider two cases.

Case 1. $\xi_0 \notin \Sigma_1$. Fix a $z_0 \in \mathbb{C}$ with $\operatorname{Re} z_0 > 0$. We have

$$S_3'(z) = \int_0^{T_1} e^{-tz}\int_0^t U_0(t-s)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}(b\partial_s U(s) - z_0 U(s))\,ds\,dt =$$
$$= \int_0^{T_1} e^{-tz}\int_0^t U_0(t-s)\partial_s(\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}bU(s))\,ds\,dt +$$
$$+ \int_0^{T_1} e^{-tz}\int_0^t U_0(t-s)(-\partial_s\eta_1)(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}bU(s)\,ds\,dt -$$
$$-z_0\int_0^{T_1} e^{-tz}\int_0^t U_0(t-s)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}U(s)\,ds\,dt.$$

Denoting the first term in the right-hand side of the last identity by $S_3''(z)$, we have

$$S_3''(z) = \int_0^{T_1} e^{-tz}\partial_t\left(\int_0^t U_0(t-s)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}bU(s)\,ds\right)dt -$$
$$-\int_0^{T_1} e^{-tz}U_0(t)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}b\,dt =$$

$$= e^{-zT_1} \int_0^{T_1} U_0(T-s)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}bU(s)\,ds +$$

$$+z\int_0^{T_1} e^{-tz} \int_0^{t} U_0(t-s)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}bU(s)\,ds\,dt -$$

$$- \int_0^{T_1} e^{-tz}U_0(t)\eta_1(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1}b\,dt.$$

Now take φ so that $\operatorname{supp}\varphi \cap \Sigma_1 = \emptyset$. We are going to show that

$$(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1} \in \mathcal{L}(H, H^1). \tag{4.4}$$

Then the desired result in this case will follow from the above representations of $S_3'(z)$ and $S_3''(z)$. It is easy to see that

$$(\varphi\Gamma_1)(D_x)(G_0-z_0)^{-1} = \mathcal{F}^{-1}\varphi(\xi)(i\lambda_1(\xi)-z_0)^{-1}\Gamma_1(\xi)\mathcal{F}. \tag{4.5}$$

Clearly, $|\lambda_1(\xi)| \geqslant C_1|\xi|$ on $\operatorname{supp}\varphi$, and this implies

$$|\varphi(\xi)(i\lambda_1(\xi)-z_0)^{-1}| \leqslant C_2|\xi|^{-1} \quad \text{for} \quad |\xi| \gg 1. \tag{4.6}$$

Now (4.4) follows from (4.5) and (4.6) at once.

Case 2. $\xi_0 \in \Sigma_1$.

Take φ so that $\operatorname{supp}\varphi \subset \Sigma_1'$. By the assumption (1.2), there exists a j, $1 \leqslant j \leqslant n$, so that $\partial_{\xi_j}\lambda_1(\xi_0) \neq 0$. Without loss of generality we can suppose that $j = 1$ and $\partial_{\xi_1}\lambda_1(\xi_0) > 0$. Now, taking $\operatorname{supp}\varphi$ small enough we can arrange

$$\partial_{\xi_1}\lambda_1(\xi) \geqslant c_1 \quad \text{on} \quad \operatorname{supp}\varphi$$

for some constant $c_1 > 0$. Hence, we can take a function $a(\xi) \in C^\infty(\mathbf{R}^n)$, homogeneous of degree one for $|\xi| \geqslant \frac{1}{2}$, so that $a(\xi) = \lambda_1(\xi)$ on $\operatorname{supp}\varphi$ and

$$\partial_{\xi_1}a(\xi) \geqslant c_1 \quad \text{for all} \quad \xi \in \mathbf{R}^n. \tag{4.7}$$

Clearly, the operator $G_1 = ib(x)^{-1}a(D_x)I$ generates a unitary group $U_1(t) = \exp(tG_1)$ in H_b. We have

$$Z(t) \stackrel{\text{def}}{=} (\partial_t - G_1)(\varphi\Gamma_1)(D_x)U(t) =$$

$$= ((\varphi\Gamma_1)(D_x)b^{-1}G_0 - b^{-1}(ia\varphi\Gamma_1)(D_x))U(t).$$

Clearly, $(\mathrm{i}a\varphi\Gamma_1)(D_x) = (\mathrm{i}\varphi\lambda_1\Gamma_1)(D_x) = (\varphi\Gamma_1)(D_x)G_0$. Hence we can deduce that $Z(t) = Z_1 G U(t)$ where $Z_1 = \left[(\varphi\Gamma_1)(D_x), b^{-1}\right] b$. Since by the assumption (1.2), $(\varphi\Gamma_1)(\xi) \in \mathbb{C}^\infty(\mathbb{R}^n)$, we have $Z_1 \in \mathcal{L}(H, H^1)$. Now by Duhamel's formula,

$$(\varphi\Gamma_1)(D_x)U(t) = U_1(t)(\varphi\Gamma_1)(D_x) + \int_0^t U_1(t-s)Z(s)\,\mathrm{d}s =$$

$$= U_1(t)(\varphi\Gamma_1)(D_x) + \int_0^t U_1(t-s)Z_1\partial_s U(s)\,\mathrm{d}s =$$

$$= U_1(t)(\varphi\Gamma_1)(D_x) - U_1(t)Z_1 + \partial_t\left(\int_0^t U_1(t-s)Z_1U(s)\,\mathrm{d}s\right).$$

Multiplying this equality by $\eta_1(t,x)$ on the left and by χ on the right we obtain

$$\begin{gathered}\eta_1(t,x)(\varphi\Gamma_1)(D_x)U(t)\chi = \eta_1(t,x)U_1(t)\chi(\varphi\Gamma_1)(D_x)+\\ +\eta_1(t,x)U_1(t)\left[(\varphi\Gamma_1)(D_x),\chi\right] - \eta_1(t,x)U_1(t)Z_1\chi-\\ -(\partial_t\eta_1)\int_0^t U_1(t-s)Z_1U(s)\,\mathrm{d}s\chi + \partial_t\left(\eta_1\int_0^t U_1(t-s)Z_1U(s)\,\mathrm{d}s\chi\right) =\\ = Z_2(t) + \partial_t Z_3(t),\end{gathered}$$

where $Z_2(t)$ is the sum of the first four terms in the right-hand side of this equality; the definition of $Z_3(t)$ is clear. We claim that if T is large enough, then

$$\eta_1(t,x)U_1(t)\chi \in \mathcal{L}(H, H^1), \quad \forall t. \tag{4.8}$$

Assume for a moment that (4.8) is fulfilled. Now, taking into account that by the theory of hyperbolic equations $U_1(t) \in \mathcal{L}(H^1, H^1)$, $\forall t$, we conclude that $Z_j(t) \in \mathcal{L}(H, H^1)$, $\forall t$, $j = 2,3$. Next, in view of above equality, we have

$$S_3'(z)\chi = \int_0^{T_1} \mathrm{e}^{-tz}\int_0^t U_0(t-s)\eta_1(s,x)(\varphi\Gamma_1)(D_x)U(s)\chi\,\mathrm{d}s\,\mathrm{d}t =$$

$$= \int_0^{T_1} \mathrm{e}^{-tz}\int_0^t U_0(t-s)(Z_2(s) + \partial_s Z_3(s))\,\mathrm{d}s\,\mathrm{d}t =$$

$$= \int_0^{T_1} \mathrm{e}^{-tz}\int_0^t U_0(t-s)Z_2(s)\,\mathrm{d}s\,\mathrm{d}t - \int_0^{T_1} \mathrm{e}^{-tz}U_0(t)Z_3(0)\,\mathrm{d}t +$$

$$+ \int_0^{T_1} \mathrm{e}^{-tz}\partial_t\left(\int_0^t U_0(t-s)Z_3(s)\,\mathrm{d}s\right)\,\mathrm{d}t,$$

and since $Z_3(0)=0$, integrating by parts, we get

$$S_3'(z)\chi=\int_0^{T_1}\mathrm{e}^{-tz}\int_0^t U_0(t-s)Z_2(s)\,\mathrm{d}s\,\mathrm{d}t+\mathrm{e}^{-zT_1}\int_0^{T_1}U_0(T_1-s)Z_3(s)\,\mathrm{d}s+$$
$$+z\int_0^{T_1}\mathrm{e}^{-tz}\int_0^t U_0(t-s)Z_3(s)\,\mathrm{d}s\,\mathrm{d}t.$$

This representation show that $S_3'(z)\chi$ has an analytic continuation to the whole $\mathbf{C}$ with values in $\mathcal{L}(H,H^1)$, which is the desired result.

It remains to establish (4.8). In doing so we shall use some well known results on the propagation of the singularities of the solutions to the Cauchy problem for hyperbolic equations. We refer the reader to [1], Chapter 23, Section 1, fo. the details. Consider the Cauchy problem

$$\text{(4.9)}\qquad \begin{cases}(\partial_t-\mathrm{i}b(x)^{-1}a(D_x))u(t,x)=0 & \text{on } \mathbf{R}^{n+1},\\ u(0,x)=f(x), & x\in\mathbf{R}^n,\end{cases}$$

where $u(t,x)$ and $f(x)$ are scalar functions. It is clear by the definition of $U_1(t)$ that given any $f=(f_1,\dots,f_d)\in L^2(\mathbf{R}^n;\mathbf{C}^d)$ we have $U_1(t)f=(u_1(t,x),\dots,u_d(t,x))$, where $u_j(t,x)$ is the solution to (4.9) with initial data f_j. Hence, to study the singularities of $U_1(t)\chi f$, $f\in L^2(\mathbf{R}^n;\mathbf{C}^d)$, it suffices to study the singularities of the solution to (4.9) with initial data χf, $f\in L^2(\mathbf{R}^n)$. Choose a function $\chi_2\in C_0^\infty(\mathbf{R}^n)$ so that $\chi_2=1$ for $|x|\leqslant\rho+5$. We would like to show that there exists a $T_2>0$ so that for any $f\in L^2(\mathbf{R}^n)$,

$$\text{(4.10)}\qquad \chi_2 u(t,x)\in C^\infty(\mathbf{R}^n)\quad\text{for}\quad t\geqslant T_2,$$

where $u(t,x)$ is the solution to (4.9) with initial data χf. Then, if $T>T_2$, (4.8) follows from (4.10) and the above remarks. On the other hand, according to the results in [1], to prove (4.10) it sufficies to show that the bicharacteristic curves of the operator $\mathrm{i}b(x)^{-1}a(D_x)$ are nontrapping, i.e. if $(x(t),\xi(t))$ is such a curve, then $|x(t)|\to+\infty$ as $t\to+\infty$. To see this we shall exploit the fact that $x(t)$ satisfies the Hamilton equation

$$\mathrm{d}x(t)/\mathrm{d}t=b(x(t))^{-1}\nabla_\xi a(\xi(t)).$$

Now, in view of (4.7) and the assumption (1.1), we have

$$x_1(t)=x_1(0)+\int_0^t b(x(s))^{-1}\partial_{\xi_1}a(\xi(s))\,\mathrm{d}s\geqslant$$
$$\geqslant x_1(0)+c_2t$$

for any $t > 0$, with a constant $c_2 > 0$. Here x_1 is the first component of x. Hence $x_1(t) \to +\infty$ as $t \to +\infty$, which implies $|x(t)| \to +\infty$ as $t \to +\infty$. This completes the proof of (4.8), and hence the proof of Lemma 4.

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