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LATTICE ABSOLUTELY SUMMING OPERATORS

DENNY H. LEUNG

1. INTRODUCTION

Let (e_i) be a normalized unconditional basic sequence in a Banach space. An operator $T : E \to F$ between Banach spaces is called (e_i)-absolutely summing if there is a constant $K < \infty$ such that

$$\Big\| \sum_{i=1}^{n} \|Tx_i\| e_i \Big\| \leqslant K \sup_{\|x'\| \leqslant 1} \Big\| \sum_{i=1}^{n} \langle x_i, x' \rangle e_i \Big\|$$

for every finite set of elements $(x_i)_{i=1}^n \subset E$. We were led to consider such operators while working on questions on Banach lattices of spaces of operators (cf. Corollary 2.2 below). Also, if (e_i) is equivalent to the canonical ℓ^p basis for some $1 \leqslant p < \infty$, then the (e_i)-absolutely summing operators are precisely the p-absolutely summing operators. Thus we may ask which of the known properties of p-absolutely summing operators are preserved by the more general (e_i)-absolutely summing operators.

Our terminology is standard. Let E be a Banach space. Then E' and U_E denote the dual space and the unit ball of E respectively. A (bounded linear) operator $T : E \to F$ between Banach spaces is *strictly singular* if its restriction to any infinite dimensional closed subspace of E is not an isomorphism. T is a *Dunford-Pettis* operator if it maps weakly compact sets onto norm compact sets. We will have occasion to consider Orlicz sequence spaces and Tsirelson's space; references for these spaces may be found in [5] and [3]. Finally, the cardinality of a set A is denoted by $|A|$. Other terms used but not defined may be found in [5].

2. STRICT SINGULARITY OF (e_i)-ABSOLUTLY SUMMING OPERATORS

It is well-known that p-absolutely summing operators are strictly singular. In this section, we show that (e_i)-absolutely summing operators are strictly singular provided (e_i) is not equivalent to the c_0 basis. Observe that since every bounded linear operator is (e_i)-absolutely summing if (e_i) is equivalent to the c_0 basis, the restriction is essential but trivial.

PROPOSITION 2.1. *Every (e_i)-absolutely summing operator is strictly singular provided (e_i) is not equivalent to the c_0 basis.*

Proof. If there is an (e_i)-absolutely summing operator which is not strictly singular, then there is an infinite dimensional Banach space E such that the identity map on E is (e_i)-absolutely summing. By Dvoretzky's theorem [4], for any n, there exists a sequence $(x_i)_{i=1}^n \subset E$ which is 2-equivalent to the $\ell^2(n)$ basis. Observe that

$$\{(\langle x_i, x'\rangle)_{i=1}^n : x' \in U_{E'}\} \subset 2U_{\ell^2(n)}.$$

For any sequence $(a_i)_{i=1}^n \subset \mathbf{R}$, applying the definition of (e_i)-absolutely summing operators with $(a_i x_i)_{i=1}^n \subset E$ yields

$$\left\|\sum_{i=1}^n a_i e_i\right\| \leqslant K \sup\left\{\left\|\sum_{i=1}^n a_i b_i e_i\right\| : (b_i)_{i=1}^n \in U_{\ell^2(n)}\right\},$$

where K is a constant independent of n. Using this inequality twice, we obtain

$$\left\|\sum_{i=1}^n a_i e_i\right\| \leqslant K^2 \sup\left\{\left\|\sum_{i=1}^n a_i b_i c_i e_i\right\| : (b_i)_{i=1}^n, (c_i)_{i=1}^n \in U_{\ell^2(n)}\right\} =$$

$$= K^2 \sup\left\{\left\|\sum_{i=1}^n a_i b_i e_i\right\| : (b_i)_{i=1}^n \in U_{\ell^1(n)}\right\} \leqslant K^2 \sup|a_i|.$$

Hence (e_i) is equivalent to the c_0 basis. ∎

As a corollary of Proposition 2.1, we prove the following results of Cartwright and Lotz [1, Corollary 2] which gives some indication of the connection between (e_i)-absolutely summing operators and Banach lattices of operators.

COROLLARY 2.2. *For a Banach lattice E, the following are equivalent:*

(1) *E is lattice isomorphic to an AM-space;*

(2) *$\sum |x_i|$ converges for every unconditionally convergent series $\sum x_i$ in E;*

(3) *Every compact $T : c_0 \to E$ has a modulus $|T|$.*

Proof. The equivalence of (2) and (3) and the implication (1) $\Rightarrow$ (3) are well--known. We will show that (2) $\Rightarrow$ (1). Let (e_i) be a normalized disjoint sequence in E. By Satz (2b$'$) of [2], it suffices to show that (e_i) is equivalent to the c_0 basis. We will show that the identity operator on ℓ^1 is (e_i)-absolutely summing. An application of Proposition 2.1 then concludes the proof. By a standard argument, we obtain from (2) a fixed constant K such that

$$(*)\qquad \left\|\sum_{i=1}^{n}|x_i|\right\| \leqslant K \sup_{\varepsilon_i=\pm1}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|$$

for any sequence $(x_i)_{i=1}^n \subset E$. Fix k and let $y_i = (a_{ij})_{j=1}^n \in \ell^1, 1 \leqslant i \leqslant k$. Define $x_j = \sum_{i=1}^{k} a_{ij}e_i \in E$ for $1 \leqslant j \leqslant n$. Then it is easy to see that $\left\|\sum_{i=1}^{n}|x_i|\right\| = \left\|\sum_{i=1}^{k}\|y_i\|_1 e_i\right\|$, where $\|\cdot\|_1$ denotes the ℓ^1 norm. On the other hand,

$$\sup_{\varepsilon_j=\pm1}\left\|\sum_{j=1}^{n}\varepsilon_j x_j\right\| = \sup\left\{\left\|\sum_{j=1}^{n}\sum_{i=1}^{k} b_j a_{ij} e_i\right\| : \sup|b_j| \leqslant 1\right\} =$$

$$= \sup\left\{\left\|\sum_{i=1}^{k}\langle y_i, (b_i)\rangle e_i\right\| : \sup|b_j| \leqslant 1\right\} = \sup_{\|y'\|\leqslant 1}\left\|\sum_{i=1}^{k}\langle y_i, y'\rangle e_i\right\|.$$

Thus inequality $(*)$ says that the identity on ℓ^1 is (e_i)-absolutely summing, as desired. ∎

3. THE DUNFORD-PETTIS PROPERTY

It follows immediately from the Grothendieck-Pietsch factorization theorem that p-absolutely summing operators are Dunford-Pettis operators. However, the factorization theorem has no counterpart in general (e_i)-absolutely summing operators. Thus, most of what is done in this (as well as the next) section can be viewed as finding ways to get around the use of the factorization theorem.

Recall that a basic sequence is *subsymmetric* if it is unconditional and equivalent to all its subsequences [5, Definition 3.a.2].

PROPOSITION 3.1. *Let (e_i) be a normalized unconditional basic sequence which is not equivalent to a c_0 basis. Then every (e_i)-absolutely summing operator maps weakly null subsymetric basic sequences onto norm null sequences.*

Proof. Without loss of generality, assume that the unconditional constant of (e_i) is 1. Suppose the Proposition fails. Then there are an (e_i)-absolutely summing operator T, an $\varepsilon > 0$, and a weakly null subsymmetric basic sequence (x_n) such that $\|Tx_n\| > \varepsilon$ for all n. By using a subsequence if necessary, we may furhter assume that (Tx_n) is a semi-normalized basic sequence. Let $S : [Tx_n] \to c_0$ denote the inclusion map $S\left(\sum a_n Tx_n\right) = (a_n)$. Then $S \circ T|_{[x_n]}$ is the inclusion map from $[x_n]$ into c_0, and it is (e_i)-absolutely summing. Hence there is a finite constant K such that

$$\left\|\sum_{i=1}^{n} \|S(Tx_n)\| e_i\right\| \leqslant K \sup_{\|x'\|\leqslant 1}\left\|\sum_{i=1}^{n}\langle x_i, x'\rangle e_i\right\|$$

for every n. Thus, for every n, there exists $\|x_n'\| \leqslant 1$ such that

$$\left\|\sum_{i=1}^{n} e_i\right\| \leqslant K\left\|\sum_{i=1}^{n}\langle x_i, x_n'\rangle e_i\right\|.$$

For every n, let $A_n = \{1 \leqslant i \leqslant n : |\langle x_i, x_n'\rangle| > (2K)^{-1}\}$ and $B_n = \{1,\ldots,n\} \setminus A_n$. Then

$$\left\|\sum_{i=1}^{n} e_i\right\| \leqslant K\left\|\left(\sum_{i\in A_n} + \sum_{i\in B_n}\right)(\langle x_i, x_n'\rangle e_i)\right\| \leqslant$$

$$\leqslant K\left\{(2K)^{-1}\left\|\sum_{i=1}^{n} e_i\right\| + \left\|\sum_{i\in A_n}\langle x_i, x_n'\rangle e_i\right\|\right\}.$$

Therefore,

$$\frac{1}{2}\left\|\sum_{i=1}^{n} e_i\right\| \leqslant K\left\|\sum_{i\in A_n}\langle x_i, x_n'\rangle e_i\right\| \leqslant K\sup\|x_i\|\left\|\sum_{i\in A_n} e_i\right\|.$$

Since (x_i) is bounded and $\left\|\sum_{i=1}^{n} e_i\right\| \to \infty$ as $n \to \infty$, we must have $|A_n| \to \infty$. For any k, choose n_0 with $|A_{n_0}| \geqslant k$. Enumerate $A_{n_0} = \{i_1 < \ldots < i_j\}$, $j \geqslant k$. Then for any sequence of scalars (a_m),

$$\left\|\sum_{m=1}^{k} a_m x_m\right\| \geqslant M^{-1}\left\|\sum_{m=1}^{k} \operatorname{sgn}(a_m\langle x_{i_m}, x_{n_0}'\rangle) a_m x_{i_m}\right\| \geqslant$$

$$\geqslant M^{-1}\sum_{m=1}^{k} |a_m\langle x_{i_m}, x_{n_0}'\rangle| \geqslant (2KM)^{-1}\sum_{m=1}^{k}|a_m|,$$

where M is the subsymmetric constant of the sequence (x_n). Hence (x_n) is equivalent to the ℓ^1 basis and fails to be weakly null, contrary to the assumption. ∎

As will be shown below, even if (e_i) is not equivalent to the c_0 basis, it is not true that every (e_i)-absolutely summing operator is Dunford-Pettis. We do, however, have the following positive result in a special case.

THEOREM 3.2. *Let (e_i) be the unit vector basis of an Orlicz sequence space ℓ_M which does not contain a copy of c_0. Then every (e_i)-absolutely summing operator is Dunford-Pettis.*

Proof. Suppose the Theorem fails. Then, arguing as in the proof of Proposition 3.1, there is weakly null normalized basic sequence (x_n) such that the inclusion map $T:[x_n]\to c_0$, $T\left(\sum a_n x_n\right)=(a_n)$, is (e_i)-absolutely summing. In particular, there is a finite constant K such that for every sequence of scalars (a_n) and every $i\in\mathbb{N}$,

$$\|(a_1,a_2,\dots,a_i)\|_M\leqslant K\sup\left\{\|(a_1b_1,\dots,a_ib_i)\|_M:\left\|\sum b_n x_n'\right\|\leqslant 1\right\},$$

where $\|\cdot\|_M$ denotes the norm in ℓ_M and (x_n') is the biorthogonal sequence to (x_n). Choose N such that $\left\|\sum b_n x_n'\right\|\leqslant 1\Rightarrow\sup|b_n|\leqslant N$. Since $c_0\not\hookrightarrow\ell_M$, Proposition 4.a.4 in [5] yields a constant C such that $M(KNt)\leqslant CM(t)$ for $0\leqslant t\leqslant 1$. Now, for every sequence of positive numbers $(c_n)_{n=1}^i$ which sums to 1, let $(a_n)_{n=1}^i$ be chosen so that $M(a_n)=c_n$, $1\leqslant n\leqslant i$. Then $\|(a_1,a_2,\dots,a_i)\|_M=1$; hence there exists (b_n), $\left\|\sum b_n x_n'\right\|\leqslant 1$, such that $1\leqslant K\|(a_1b_1,\dots,a_ib_i)\|_M$. Therefore,

$$\begin{aligned}
&1\leqslant\sum_{n=1}^{i}M(Ka_n|b_n|)\leqslant C\sum_{n=1}^{i}M(a_n|b_n|/N)\leqslant\\
(**)\quad&\leqslant C\sum_{n=1}^{i}\frac{|b_n|}{N}M(a_n)= &&\text{since } \frac{|b_n|}{N}\leqslant 1\\
&=CN^{-1}\sum_{n=1}^{i}|b_n|c_n.
\end{aligned}$$

Since (x_n) is weakly null, there is a positive sequence $(c_n)_{n=1}^i$ which sums to 1 such that

$$\left\|\sum_{n=1}^{i}\varepsilon_n c_n x_n\right\|\leqslant N/(2C)$$

for every choice of signs $\varepsilon_n=\pm1$. Then $\left\|\sum b_n x_n'\right\|\leqslant 1$ implies

$$\sum_{n=1}^{i}|b_n|c_n=\sum_{n=1}^{i}(\operatorname{sgn}b_n)b_nc_n=$$

$$=\left\langle\sum_{n=1}^{i}(\operatorname{sgn}b_n)c_nx_n,\sum b_nx_n'\right\rangle\leqslant\left\|\sum_{n=1}^{i}(\operatorname{sgn}b_n)c_nx_n\right\|\leqslant N/(2C).$$

This contradicts inequality (**) above and finishes the proof. ∎

Example 3.4 below shows that the restriction on ℓ_M cannot be removed. We will need the following observation in the course of verifying the example.

PROPOSITION 3.3. *Let T be a 1-absolutely summing operator with constant K. Then for every normalized unconditional basic sequence (e_i), T is (e_i)-absolutely summing with constant $\leqslant\alpha K$, where α is the unconditional basis constant of (e_i).*

Proof. Suppose $T : E \to F$ is 1-absolutely summing with constant K. Then by the Grothendieck-Pietsch factorization theorem [5, Theorem 2.b.2], there is a regular probability measure μ on $U_{E'}$ endowed with the weak* topology so that

$$\|Tx\| \leqslant K \int_{U_{E'}} |\langle x, x'\rangle| \mathrm{d}\mu(x').$$

Let (e_i') be the biorthogonal sequence to (e_i). Then

$$\Big\|\sum_{i=1}^{n} \|Tx_i\| e_i\Big\| \leqslant K \Big\|\sum_{i=1}^{n} \int_{U_{E'}} |\langle x_i, x'\rangle| \mathrm{d}\mu(x') e_i\Big\| =$$

$$= K\Big\|\sum_{i=1}^{n} \|x_i\|_{L^1(\mu)} e_i\Big\| = K\sup\left\{\sum_{i=1}^{n} b_i \|x_i\|_{L^1} : \Big\|\sum b_i e_i'\Big\| \leqslant 1\right\} \leqslant$$

$$\leqslant K\sup\left\{\Big\|\sum_{i=1}^{n} |b_i x_i|\Big\|_{L^1} : \Big\|\sum b_i e_i'\Big\| \leqslant 1\right\} \leqslant$$

$$\leqslant K\sup\left\{\sum_{i=1}^{n} |b_i \langle x_i, x'\rangle| : \Big\|\sum b_i e_i'\Big\| \leqslant 1,\ x' \in U_{E'}\right\} \leqslant$$

$$\leqslant \alpha K\sup\left\{\sum_{i=1}^{n} b_i \langle x_i, x'\rangle : \Big\|\sum b_i e_i'\Big\| \leqslant 1,\ x' \in U_{E'}\right\} =$$

$$= \alpha K\sup\left\{\Big\|\sum_{i=1}^{n} \langle x_i, x'\rangle e_i\Big\| : x' \in U_{E'}\right\}.$$

Thus T is (e_i)-absolutely summing with constant αK. ■

EXAMPLE 3.4. *There is a non-degenerate Orlicz function M so that if (e_i) denotes the coordinate unit vectors in ℓ_M, then there is a non-Dunford-Petis operator which is (e_i)-absolutely summing.*

We divide the proof into a sequence of lemmas.

LEMMA 3.5. *There is a non-degenerate Orlicz function M such that $M(4t) \geqslant \sqrt{M(t)}$ for every $t \geqslant 0$.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive reals such that $a_{n-1} \geqslant (2^{n-1}a_n)^{\frac{1}{2}}$ for $n \geqslant 3$ and $\sum_{n=1}^{\infty} 2^{-n} a_n = 1$. Let M be the piecewise linear continuous function such that $M(0) = 0$,

$$M'(t) = a_n,\ \ 2^{-n} < t < 2^{-(n-1)},\ \ 1 \leqslant n,$$

and $M'(t) = a_1$, $1 < t$. Then M is non-degenerate Orlicz function. For $1/4 \leqslant t \leqslant 1$, $M(4t) \geqslant M(1) = 1 \geqslant \sqrt{M(t)}$. For $t > 1$, $M(4t) \geqslant M(t) \geqslant \sqrt{M(t)}$. For $t < 1/4$, choose $n \geqslant 3$ such that $2^{-n} \leqslant t < 2^{-(n-1)}$. Then

$$M(4t) \geqslant \int_{2^{-(n-1)}}^{2^{-(n-2)}} M'(s)\mathrm{d}s = 2^{-(n-1)}a_{n-1} \geqslant (2^{-(n-1)}a_n)^{\frac{1}{2}} \geqslant$$

$$\geqslant \left\{ \int_0^{2^{-(n-1)}} M'(s)\mathrm{d}s \right\}^{\frac{1}{2}} \geqslant \sqrt{M(t)}.$$

■

For the remainder of the section, fix M as given by Lemma 3.5. For $y = (a_n) \in \ell_M$, let $\tau(y) = \sum M(|a_n|)$, where the sum is taken to be ∞ if it diverges. Note that τ is additive on disjoint sequences in ℓ_M and $\|y\|_M \leqslant 1$ if and only if $\tau(y) \leqslant 1$.

LEMMA 3.6. *Let (y_i) be a sequence of pairwise disjoint elements in ℓ_M such that $\left\|\sum y_i\right\|_M = 1$. Then there exist $A \subset \mathbb{N}$ and $j \in \mathbb{N}$ such that $|A| = 2^j$, $\min A \geqslant j$, and $\left\|\sum_{i\in A} y_i\right\| > \frac{1}{32}$.*

Proof. We first observe that $\tau(4y) \geqslant \sqrt{\tau(y)}$ for all $y \in \ell_M$. Indeed, let $y = (a_n)$ and let $\|y\|_0$ denote the ℓ^∞ norm of y. Then

$$\tau(4y) = \sum M(4|a_n|) \geqslant \sum \sqrt{M(|a_n|)} \geqslant$$

$$\geqslant \left(M(\|y\|_0)\right)^{-\frac{1}{2}} \sum M(|a_n|) \geqslant \left(\tau(y)\right)^{-\frac{1}{2}} \sum M(|a_n|) = \sqrt{\tau(y)}.$$

Now let (y_i) be as given and let $b_i = \tau(y_i)$ for all i. Then $\sum b_i = \tau\left(\sum y_i\right)$ by the disjointness. Hence $\sum b_i = 1$ since $\left\|\sum y_i\right\|_M = 1$. Now suppose for all $A \subset \mathbb{N}$ and $j \in \mathbb{N}$ such that $|A| = 2^j$, $\min A \geqslant j$, we have $\left\|\sum_{i\in A} y_i\right\| \leqslant \frac{1}{32}$. Then

$$(\dagger) \qquad \begin{aligned} \sum_{i\in A} \sqrt{b_i} &= \sum_{i\in A} \sqrt{\tau(y_i)} \leqslant \sum_{i\in A} \tau(4y_i) \leqslant \\ &\leqslant \frac{1}{8} \sum_{i\in A} \tau(32y_i) \leqslant \frac{1}{8}. \end{aligned}$$

It is not hard to see the decreasing rearrangement of the sequence (b_i) retains the property $(\dagger)$. Without loss of generality, we may thus assume additionally that (b_i)

is decreasing. Define a sequence of integers $(f(j))$ inductively by $f(0) = 1$, $f(j) = 2^{f(j-1)}$ for $j \geqslant 1$ and let $c_j = \sum_{i=f(j-1)}^{f(j)-1} b_i$. Then $\sum c_j = \sum b_i = 1$. Also

$$[f(j) - f(j-1)]b_{f(j)-1} \leqslant c_j \text{ since } (b_i) \text{ is decreasing } \Rightarrow$$
$$\Rightarrow b_{f(j)} \leqslant b_{f(j)-1} \leqslant c_j[f(j) - f(j-1)]^{-1} \leqslant [f(j) - f(j-1)]^{-1}.$$

Moreover,

$$c_{j+1} = \sum_{i=f(j)}^{f(j+1)-1} b_i \leqslant \sqrt{b_{f(j)}} \sum_{i=f(j)}^{f(j+1)-1} \sqrt{b_i} \leqslant \frac{\sqrt{b_{f(j)}}}{8}$$

by (†). Note that $f(j-1) \leqslant f(j) - f(j-1)$ for $j \geqslant 2$. Hence the two preceding inequalities yields that $c_{j+1} \leqslant \left(8\sqrt{f(j-1)}\right)^{-1}$ for $j \geqslant 2$. A straightforward computation show that this implies $\sum_{j\geqslant 3} c_j \leqslant \frac{1}{5}$. Hence we must have $c_1 + c_2 \geqslant \frac{4}{5}$. Finally, it is not hard to see that this implies $\sum_{i\in A} \sqrt{b_i} > \frac{1}{8}$ for $A = \{1\}$ or $A = \{2,3\}$. This contradicts (†) and proves the lemma. ■

LEMMA 3.7. *Let (e_i) be the coordinate unit vectors in ℓ_M and let I be the formal identity from the Tsirelson space T into c_0. Then I is an (e_i)-absolutely summing operator which is not Dunford-Pettis.*

Proof. It is clear that I is not Dunford-Pettis since the Tsirelson space is reflexive and I is not compact. For $B \subset \mathbb{N}$, define the projection $\mathcal{X}_B$ on T by $\mathcal{X}_B(a_n) = (b_n)$, where $b_n = a_n$ if $n \in B$, and 0 otherwise. We will need the following property of T: there is a finite constant λ such that for every $B \subset \mathbb{N}$ with $|B| = 2^j$, $\min B \geqslant j$ for some $j \in \mathbb{N}$, the formal identity map $\mathcal{X}_B T \to \ell^1(|B|)$ has norm$\leqslant \lambda$. This follows from Theorems IV.c.1 and V.3 of [3]. Now let $x_1, \ldots, x_n \in T$ with $\|(\|Ix_1\|, \ldots, \|Ix_n\|)\|_M = 1$. For $1 \leqslant i \leqslant n$, choose $p_i \in \mathbb{N}$ such that $\|Ix_i\| = |x_i(p_i)|$, where we write $x_i = (x_i(j))$. For all $k \in \mathbb{N}$, define $A_k = \{1 \leqslant i \leqslant n : p_i = k\}$ and let $y_k = (y_k(i)) \in \ell_M$ be given by $y_k(i) = |x_i(p_i)|$ if $i \in A_k$ and 0 otherwise. Then (y_k) is a pairwise disjoint sequence in ℓ_M such that $\left\|\sum y_k\right\| = \|(\|Ix_1\|, \ldots, \|Ix_n\|)\|_M = 1$. Applying Lemma 3.6, there exists $B \subset \mathbb{N}$, $j \in \mathbb{N}$ such that $|B| = 2^j$, $\min B \geqslant j$, and $\left\|\sum_{k\in B} y_k\right\| > \frac{1}{32}$. Note that the last inequalty implies

$$\left\|(\|I\mathcal{X}_B x_1\|, \ldots, \|I\mathcal{X}_b x_n\|)\right\|_M \geqslant \left\|\sum_{k\in B} y_k\right\|_M > \frac{1}{32}.$$

Let I_B be the restriction of I to $\mathcal{X}_B T$. Then I_B may be factorized by

$$\mathcal{X}_B T \xrightarrow{J_1} \ell^1(|B|) \xrightarrow{J_2} \ell^2(|B|) \xrightarrow{J_3} c_0,$$

where each J_i, $1 \leqslant i \leqslant 3$, is the formal identity. Now J_2 is 1-absolutely summing with constant$\leqslant K_G$ by Theorem 2.b.6 of [5]. Hence it is (e_i)-absolutely summing with constant$\leqslant K_G$ by Proposition 3.3. Also $\|J_1\| \leqslant \lambda$ and $\|J_3\| = 1$. Thus I_B is (e_i)-absolutely summing with constant$\leqslant \lambda K_G$. Hence

$$\frac{1}{32} < \left\| \left(\|I\mathcal{X}_B x_1\|, \dots, \|I\mathcal{X}_B x_n\| \right) \right\|_M \leqslant$$

$$\leqslant \lambda K_G \sup \left\{ \left\| \sum_{i=1}^{n} \langle x_i, x' \rangle e_i \right\| : \|x'\| \leqslant 1 \right\}.$$

Thus I is (e_i)-absolutely summing, as claimed. ■

4. WEAK COMPACTNESS OF (e_i)-ABSOLUTELY SUMMING OPERATORS

Suppose that (e_i) is not equivalent to the c_0 basis. Then every (e_i)-absolutely summing operator $T : E \to F$ is strictly singular by Proposition 2.1. In particular, T cannot fix a copy of ℓ^1. Hence TU_E is weakly sequentially precompact. Again, more can be said in the special case of Orlicz sequence spaces.

THEOREM 4.1. *Let (e_i) be the unit vector basis of an Orlicz sequence space ℓ_M which does not contain a copy of c_0. Then every (e_i)-absolutely summing operator is weakly compact.*

We begin by considering the following lemma.

LEMMA 4.2. *Let (u_n) and (v_n) be the standard bases for ℓ^1 and c_0 respectively, and let $R : \ell^1 \to c_0$ be given by $Ru_n = \sum_{i=1}^{n} v_i$. For every $\varepsilon > 0$, there exists $(x_i)_{i=1}^n \subset \subset \ell^1$ such that $\sum_{i=1}^{n} \|Rx_i\| = 1$, $\sup \left\{ \sum_{i=1}^{n} |\langle x_i, x' \rangle| : \|x'\| \leqslant 1 \right\} \leqslant \varepsilon$, and $\|Rx_i\| \geqslant \frac{\|x_i\|}{2}$ for $1 \leqslant i \leqslant n$.*

REMARK. It follows immediately from the fact that R is not 1-absolutely summing that there is a sequence in ℓ^1 having the first two properties stated in the Lemma. What we want is an example satisfying in addition the last inequality.

Proof. Let (h_k) be the L^∞-normalized Haar functions on [0,1] and let $g_1 = h_1$, $g_k = \frac{h_k}{n}$ for $2^{n-2} < k \leqslant 2^{n-1}$, $1 \leqslant n$. Let $\|\cdot\|_2$ denote the norm in L^2. By the orthogonality of the sequence (g_k) and the fact that $\sum \|g_k\|_2^2 < \infty$, $\sum g_k$ is unconditonally convergent in L^2 and consequently L^1. Let E_n be the subspace of L^1 generated by the characteristic functions of the intervals $I_i \equiv [2^{-(n-1)}(i-1), 2^{-(n-1)}i)$, $1 \leqslant i \leqslant 2^{n-1}$, and let $j_n : E_n \to \ell^1$ be the obvious (into) isometry $j_n(\mathcal{X}_{I_i}) = \frac{u_i}{2^{n-1}}$. Define $x_k = j_n g_k$, $1 \leqslant k \leqslant 2^{n-1}$. Then it is easy to see that $\sum \|Rx_k\| = 2^{-1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$, $\|Rx_k\| = \frac{\|x_k\|}{2}$, and $\sup\left\{\sum |\langle x_k, x'\rangle| : \|x'\| \leqslant 1\right\} = \sup_{\varepsilon_k=\pm 1}\left\|\sum \varepsilon_k x_k\right\| \leqslant K$, where $K = \sup_{\varepsilon_k=\pm 1}\left\|\sum_{k=1}^{\infty}\varepsilon_k g_k\right\|_{L^1} < \infty$. The Lemma follows immediately by scaling (x_k) by a constant factor. ■

Proof of Theorem 4.1. We first show that the operator R in Lemma 4.2 cannot be (e_i)-absolutely summing. Since $c_0 \not\hookrightarrow \ell_M$, for every $K < \infty$, there is a finite constant $\varphi(K)$ such that $M(Kt) \leqslant \varphi(K)M(t)$ for $0 \leqslant t \leqslant 2$. Suppose R is (e_i)-absolutely summing with constant C. Let $(x_i)_{i=1}^n \subset \ell^1$ be as given by Lemma 4.2 with $\varepsilon = (2\varphi(c)\varphi(2)\|R\|)^{-1}$. Choose $c_i \geqslant 0$ such that $M(c_i\|Rx_i\|) = \|Rx_i\|$ for $1 \leqslant i \leqslant n$. Then $\left\|\sum_{i=1}^n \|R(c_ix_i)\|e_i\right\| = 1$ since $\sum_{i=1}^n M(\|R(c_ix_i)\|) = 1$. Therefore, there must be a $x' \in U_{\ell^\infty}$ such that $\left\|\sum_{i=1}^n c_i\langle x_i, x'\rangle e_i\right\| \geqslant C^{-1}$. Note that $M(c_i\|Rx_i\|) = \|Rx_i\| \leqslant 1 = M(1)$ implies $c_i\|Rx_i\| \leqslant 1$. Hence $c_i|\langle x_i, x'\rangle| \leqslant c_i\|x_i\| \leqslant 2c_i\|Rx_i\| \leqslant 2$. Thus

$$1 \leqslant \sum_{i=1}^n M(Cc_i|\langle x_i, x'\rangle|) \leqslant \varphi(C)\sum_{i=1}^n M(c_i|\langle x_i, x'\rangle|) \leqslant$$

$$\leqslant \varphi(C)\sum_{i=1}^n M\left(2\|Rx_i\|\cdot\frac{c_i|\langle x_i, x'\rangle|}{\|x_i\|}\right) \leqslant \qquad \text{since } 2\|Rx_i\| \geqslant \|x_i\|$$

$$\leqslant \varphi(C)\varphi(2)\sum_{i=1}^n M\left(c_i\|Rx_i\|\cdot\frac{|\langle x_i, x'\rangle|}{\|x_i\|}\right) \leqslant$$

$$\leqslant \varphi(C)\varphi(2)\sum_{i=1}^n M\left(c_i\|Rx_i\|\right)\cdot\frac{|\langle x_i, x'\rangle|}{\|x_i\|} \leqslant \qquad \text{since } \frac{|\langle x_i, x'\rangle|}{\|x_i\|} \leqslant 1$$

$$\leqslant \varphi(C)\varphi(2)\|R\|\sum_{i=1}^n |\langle x_i, x'\rangle| \leqslant \varphi(C)\varphi(2)\|R\|\varepsilon = \frac{1}{2},$$

a contradiction. Now let $T : E \to F$ be a general (e_i)-absolutely summing operator. Let (x_n) be a bounded sequence in E, we wish to show that (Tx_n) has a weakly convergent subsequence. If (x_n) has a weakly Cauchy subsequence, then (Tx_n) has even a norm convergent subsequence as T is Dunford-Pettis by Theorem 3.2. Otherwise, by Rosenthal's Theorem, (x_n) has a subsequence (u_n) equivalent to the ℓ^1 basis. If (Tu_n) is not weakly convergent, then by one of James' characterization of weak compactness and a standard perturbation argument, one obtains a bounded weak* null sequence (u'_n) in the dual of $G \equiv [Tu_n]$ such that $\langle Tu_m, u'_n \rangle = 1$ if $m \geqslant n$ and 0 otherwise. Let $S : G \to c_0$ be given by $y \mapsto ((\langle y, u'_n \rangle))_n$. Then $S \circ T\big|_{[u_n]}$ is precisely the map R of Lemma 4.2. Thus R must be (e_i)-absolutely summing, contrary to what was established above. ∎

We close with a few outstanding questions.

PROBLEM 1. Is every (e_i)-absolutely summing operator weakly compact provided (e_i) is not equivalent to the c_0 basis?

PROBLEM 2. Let (e_i) be a normalized unconditional basic sequences such that $[e_i]$ does not contain a copy of c_0, is every (e_i)-absolutely summing operator Dunford--Pettis?

PROBLEM 3. Let (e_i) and (f_i) be normalized unconditional basic sequences such that (e_i) dominates (f_i). Is every (e_i)-absolutely summing operator (f_i)-absolutely summing?

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DENNY H. LEUNG
Department of Mathematics,
The University of Texas at Austin,
Austin, Tx 78712,
U.S.A.

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