J. OPERATOR THEORY
27(1992), 193–203

ON L^p-DOMAINS OF FRACTIONAL POWERS OF SINGULAR ELLIPTIC OPERATORS AND KATO'S CONJECTURE

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Let $A = \sum_{|\alpha|\leqslant m} a_\alpha(x)D^\alpha, (D = \frac{1}{i}\partial)$ be a m-th order elliptic operator on $\mathbf{R}^n$ with the principal symbol $a(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ satisfying

$$\operatorname{Re} a(x,\xi) \geqslant c_0|\xi|^m$$

and possibly singular coefficients $a_\alpha \varepsilon L^{p_\alpha}$.

One can shown such A to be bounded from below: $\operatorname{Re}\langle Au; u\rangle \geqslant c\|u\|^2$, for u in the L^2-domain of A, and m-accretive, i.e.

$$(1) \qquad \|(A-\lambda)^{-1}u\| \leqslant (c - \operatorname{Re}\lambda)^{-1}\|u\|$$

for all λ in the half plane $\operatorname{Re}\lambda < c$ (see [4],[5]).

For any m-accretive operator A with spectrum in the right half plane (i.e. $c = 0$) one can construct the square root $A^{1/2}$ or any fractional power $A^s, s \geqslant 0$ (see [9], Ch. 5). Kato conjectured that for second order elliptic operator $A = \sum_j \partial_i a_{ij}\partial_j$ with $a_{ij} \in L^\infty$, the L^2-domain of $A^{1/2}$ is equal to the Sobolev space H_1. Recently this conjecture was proved in [1], [2], [3] for small (in L^∞-norm) perturbations of the Laplace operator $-\Delta + \sum b_{ij}\partial_{ij}^2$, with $\|b_{ij}\|_\infty < \varepsilon$.

In this paper we shall prove Kato's conjecture for singular perturbations of m-th order elliptic operators and their fractional powers $A^{s/m}(0 \leqslant s \leqslant m)$ in L^p-spaces. Precisely we shall show that $\operatorname{Dom}(A^{s/m}|L^p) = \mathcal{L}_s^p$, the L^p-Sobolev space of order s. For operator A itself ($s = m$) it was done in [4–7].

The essential part of our proof is based upon the recent results of G. David and J. L. Journe [2], developed in connection with generalized Calderon-Zygmond operators. Those will be combined with some of our earlier results on the resolvent and other related kernels for elliptic operators on $\mathbf{R}^n$ ([4–7]). At the end we shall discuss different extensions and ramifications of the main result.

We consider operators in the form $A = A_0 + B$, where A_0 *is a constant coefficient* (or uniformly elliptic) operator $\sum_{|\alpha|=m} a_\alpha(x)D^\alpha$, whose symbol satisfies

$$c_1|\xi|^m \leqslant \operatorname{Re} a(x,\xi) \leqslant c_2|\xi|^m \quad \text{uniformly in } x$$

and coefficients $a_\alpha(x)$ are sufficiently smooth.

The standard (left) convention will be used for differential (pseudodifferential) operators on $\mathbf{R}^n$: to each symbol $\sigma(x,\xi)$ we assign a (distributional) kernel via the Fourier transform

$$\mathcal{F} : \sigma(x,\xi) \to K_\sigma(x,z) = (2\pi)^{-n} \int e^{i\xi\cdot z} a(x;\xi)d\xi, \quad z = x - y.$$

The coefficients of perturbation $B = \sum_{|\alpha|\leqslant m} b_\alpha(x)D^\alpha$ may be singular, $b_\alpha \in L^{p_\alpha}$. For each term $b_\alpha D^\alpha$ we introduce its "fractional order" ([4;5])

$$d = d_\alpha = \frac{n}{p_\alpha} + |\alpha| \leqslant m.$$

Then B can be written as the sum of "top order part" $B_0 = \sum_{d_\alpha=m} b_\alpha D^\alpha$, and the lower order part $B_1 = \sum_{d_\alpha<m} b_\alpha D^\alpha$. As in [1;2;3] the top order coefficients $\{b_\alpha\}_{|\alpha|=m}$ are assument sufficiently small in norm,

$$\sum \|b_\alpha\|_\infty = \varepsilon \ll 1.$$

In [4], [5] we showed that operators of this typed are m-accretive in the scale,

$$\left\{ L^p : p_0 < p \leqslant \min\left(p_\alpha; \frac{p_0}{p_0-1}\right)\right\}. \tag{2}$$

where $p_0 > 1$ depends on $\varepsilon =$"norm of top order perturbations". Assuming that constant $c = c(A)$ in the definition of accretivity (1) is zero we shall prove the following result.

THEOREM 1. *The L^p-domain of the fractional power $A^{s/m}, \mathcal{D}_p(A^{s/m})$, is equal to the L^p-Sobolev space of order $s, \mathcal{L}^p_s = \Lambda^{-s}(L^p)$ for all p in the accretivity scale (2), and all $0 \leqslant s \leqslant m$.*

Proof. We denote Λ^s the fractional Laplacian $(1-\Delta)^{s/2}$. To show the equality $\mathcal{D}_p(A^{s/m}) = \mathcal{L}_s^p$ it suffices to prove that operators $\Lambda^s(\lambda+A)^{-s/m}$, and $(\lambda+A)^{2/m}\Lambda^{-s}$ are bounded in L^p for some $\lambda > 0$.

We construct $(\lambda + A)^{-s/m}$ by the Cauchy integration of the resolvent kernel $R_\zeta = (\zeta - A)^{-1}$,

$$(\lambda + A)^{-s} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (\lambda+\zeta)^{-s/m} R_\zeta \mathrm{d}\zeta \tag{3}$$

Here contour Γ consist of two rays $\{\, te^{\pm i\theta} : t > 0 \text{ and fixed } \theta > 0 \,\}$.

The resolvent R_ζ is constructed from the "free resolvent" $R_\zeta^0 = (\zeta - A_0)^{-1}$ via the perturbation series expansion

$$R_\zeta = \sum_0^\infty R_\zeta^0 (BR_\zeta^0)^k. \tag{4}$$

In [4], [5] we established convergence of (4) and estimated $\|R_\zeta\|$ in different L^p--spaces. These results will be used here. In fact, we shall first prove the result for the unperturbed operator A_0, namely $\mathcal{D}_p(A_0^{s/m}) = \mathcal{L}_s^p$ for all $s > 0$ and $1 < p < \infty$.

In the constant coefficient case operator $P_0 = \Lambda^s(A_0 + \lambda)^{-s/m}$ is a convolution kernel whose multiplier $(1+\xi^2)^{s/2}(\lambda + a(\xi))^{-2/m}$ is of the classical $S_{1,0}^0$-type. So the Hörmander-Michlin Multiplier theorem applies to show boundedness and invertibility of P_0 (see [11], Ch.4.).

In the variable coefficient (uniformly elliptic) case the free resolvent was shown in [5] to factor into the product $R_\zeta^0 = K_\zeta(I - L_\zeta)^{-1}$, with ψ DO's $K_\zeta(x,z)$ =parametrix of $A_0 + \lambda$ and $L_\zeta(x,z)$ — the remainder, whose symbols

$$\sigma_K = \frac{1}{\zeta - a(x,\xi)}; \quad \sigma_L = \sum_{1 \leqslant |\beta| \leqslant m} \frac{1}{\beta!} D_\xi^\beta a(x,\xi) \partial_x^\beta \left(\frac{1}{\eta - a}\right)$$

have orders $-m$ and -1, respectively. Then Cauchy integration (3) of the series $K_\zeta \sum_0^\infty L_\zeta^k$ yields

$$P_0 = Q + \sum_1^\infty \frac{1}{2\pi i} \int_\Gamma (-\Delta)^{-s/2}(\lambda+\zeta)^{-s/m} K_\zeta L_\zeta^k \mathrm{d}\zeta, \tag{5}$$

where ψ DO Q has symbol $(1 + |\xi|^2)^{s/2}(\lambda + a(x,\xi))^{-s/m}$. So operator Q is bounded and invertible.

For each term of the remaining sum we observv that

$$\|\Lambda^s K_\zeta\| \leqslant c|\zeta|^{s/m-1}, \tag{6}$$

whereas operator L_ζ was shown in [5] to be bounded by

(7) $$\|L_\zeta\| \leqslant c(\theta)R^{-\nu}; \text{ where } \zeta = re^{i\theta}, \text{ and } \nu = \frac{1}{m}.$$

Inserting these estimates in (3) we get

(8) $$\left\|\sum_{k=1}^{\infty}\cdots\right\| \leqslant \frac{1}{\pi}\int^{+\infty} |\lambda + re^{i\theta}|^{-s/m} r^{s/m-1}\left(\frac{c(\theta)r^{-\nu}}{1-c(\theta)r^{-\nu}}\right) dr$$

an absolutely convergent integral.

Thus P_0 is bounded in all $L^p, 1 < p < \infty$.

Invertibility of P_0 does not follow immediately as the right hand side of (8) may not be small. Instead we consider the operator $T = (\lambda + A_0)^{s/m}\Lambda^{-s}$. The left factor is represented by the Cauchy integral of $(\lambda+\zeta)^{m/s}K_\zeta \sum_0^\infty L_\xi^k$. Conjugating each L_ζ with the Bessel potential $\Lambda^s = (1-\Delta)^{s/2}, \tilde{L}_\zeta = \Lambda^s L_\zeta \Lambda^{-s}$, we can "pull" Λ^{-s} from the right to the left side of the product $K_\zeta L_\zeta^k$ and write operator T as

$$(\lambda+\zeta)^{m/s}K_\zeta\Lambda^{-s}\sum_0^\infty \tilde{L}_\zeta^k.$$

Now the standard pseudodifferential calculus techniques (product formulas, remainder estimates, etc.) applies to show that symbol of $\tilde{L}_\zeta$ (consequently, operator $\tilde{L}_\zeta$ itself) behaves like symbol of L_ζ, in particular estimate (7) holds for $\tilde{L}_\zeta$. Noticing that $\|K_\zeta\Lambda^{-s}\| \leqslant c|\zeta|^{-1-s/m}$ we complete the rest of the proof as above for P_0.

Having shown boundedness and invertibility of $P_0 = \Lambda^s(\lambda + A_0)^{-s/m}$ we proceed now to perturbations $A = A_0 + B$.

Let $P = \Lambda^s(\lambda+A)^{-s/m}$. We divide perturbation B into two parts B_0+B_1, where B_0 consists of leading order terms (in the sense of fractional order $d_\alpha = \dfrac{n}{p_\alpha} + |\alpha| = m$) with sufficiently small norms $\sum \|b_\alpha\| = \varepsilon \ll 1$, whereas all terms of B_1 have lower order $d_\alpha = \text{order}\,(b_\alpha D^\alpha) < m$. We shall call the maximal number $\{d_\alpha\}$ over all terms $b_\alpha D^\alpha$ of B_1, the order of B_1, $d = d(B_1) < m$.

We shall introduce an intermediate operator $A_1 = A_0+B_0$ along with its resolvent R_ζ^1, and first show that

(a) $\Lambda^s(\lambda + A_1)^{-s/m}$ is bounded and invertible in the L^p-scale of accretivity (2).

(b) $\|\Lambda^s R_\zeta^1\| \leqslant C|\zeta|^{-1+s/m}$.

Notice that (b) easily follows from the perturbation series expansion

$$R_\zeta^1 = R_\zeta^0[(I - B_0R_\zeta^0])^{-1} = \sum_0^\infty R_\zeta^0(B_0R_\zeta^0)^k.$$

Indeed, estimate (b) have already been proven for $\Lambda^s R^0_\zeta$. On the other hand one can show that the norm, $\|B_0 R^0_\zeta\|$, is sufficiently small, (cf. [5]):

$$\|B_0 R^0_\zeta\| \leqslant \text{const} \sum \|b_\alpha\|.$$

Once (a)–(b) are established we proceed to the "full" resolvent R_ζ, by perturbing R^1_ζ with lower order terms B_1 and utilizing estimates of [4]–[5]. Namely,

$$\Lambda^s R_\zeta = \Lambda^s R^1_\zeta \sum_0^\infty (B_1 R^1_\zeta)^k,$$

where $\Lambda^s R^1_\zeta$ is estimated by (b), while

$$B_1 R^1_\zeta = B_1 R^0_\zeta (I - B_0 R^0_\zeta)^{-1}$$

was shown in [4]–[5] to admit an estimate similar to (7). Precisely

$$\|B_1 R^0_\zeta\| \leqslant c(\theta) r^{-\nu}, \quad \text{with}\, \nu = 1 - d/m,$$

The rest of the argument then proceeds as in the uniformly elliptic case above.

Now we come to the most difficult part of the proof, statement (a).

We need to estimate the difference between two operators

$$P = \Lambda^s (\lambda + A_1)^{-s/m} \quad \text{and} \quad P_0 = \Lambda^s (\lambda + A_0)^{-s/m},$$

$$P - P_0 = \sum_{k=1}^\infty \frac{1}{2\pi \mathrm{i}} \int (\lambda + \zeta)^{-s/m} \Lambda^s R^0_\zeta (B_0 R^0_\zeta)^k \mathrm{d}\zeta = \sum_1^\infty T_k; \tag{9}$$

and to show that each term T_k is bounded by $(C_p \sum \|b_\alpha\|)^k$. This would ensure the convergence of series (9) provided $C_p \sum \|b_\alpha\| < 1$.

Notice, however, that unlike the preceding cases, norm estimate can not be carried now inside the integral, since $\|B_0 R^0_\zeta\|$ does not decay along any ray $\{ re^{\mathrm{i}\theta}; r > 0 \}$. Instead, a careful analysis of each term is required to show cancellation of singulariries. This analysis was carried out by G. David and J. -L. Journe in [2], whose results and method we shall adopt here.

First let us observ that

$$(B_0 R^0_\zeta)^k = \sum b_1 M^{(1)}_\zeta \dots b_k M^{(k)}_\zeta, \tag{10}$$

is a linear combination of products of multiplication operators with b_j (one of coefficients $\{b_\alpha\}$ of B) and ψ DO's $M^{(j)}_\zeta$ of order 0, whose symbols

$$\sigma^{(j)} = \frac{\xi^\alpha}{\zeta - a(x, \xi)} \in S^0_{1,0}.$$

Using homogenuity of $a(x;\xi)$ in ξ we pull out $r = |\zeta|$ and write $M_\zeta^{(j)}(x,z)$ as an L^1-dilation $r^{n/m}M^{(j)}(r^{1/m}z)$ of the kernel

$$M^{(j)}(x,z) = \mathcal{F}[\xi^\alpha/\mathrm{e}^{\mathrm{i}\theta} - a(x,\xi)].$$

Let us observ that $M^{(j)}(x,z)$ is a singular integral kernel of the Calderon--Zygmund type (CZ) in z near $\{0\}$ and has "nice" radial bound at $\{\infty\}$

$$M^j(x;z) = \begin{cases} CZ; & |z| \leqslant 1 \\ |z|^{-n-1}; & |z| > 1. \end{cases} \tag{11}$$

In particular, $M^{(j)} = \rho + \omega$ in the notation of [2], with distribution $\rho(x;z)$ supported in the ball $|z| < 1$, and an L^1-radially bounded function $\omega(x;z)$ estimated by

$$|\omega(x;z)| + \nabla_z\omega(x;z)| \leqslant C|z|^{-n-1}. \tag{12}$$

Operator $(B_0R^0)^k$ is mutiplied on the left by $M_\zeta^{(0)} = \Lambda^s R_\zeta^0 = r^{-1+s/m}\times$"$L^1$--dilation of kernel $M^{(0)}(x,z)$ in z", whose symbol is $\dfrac{(1+\xi^2)^{s/2}}{(\mathrm{e}^{\mathrm{i}\theta} - a)}$. Since $s < m$, kernel $M^{(0)}(z)$ has L^1-radial bounds at $\{0\}$ and $\{\infty\}$,

$$|M^{(0)}(x,z)| \leqslant c \begin{cases} |z|^{-\mu}; & |z| < 1; \mu = n - (m-s) \\ |z|^{-n-1}; & |z| > 1 \end{cases} \tag{13}$$

Finally, following [2] we replace the variable of integration $r = |\zeta|$ by $t = r^{-1/m}$, and rewrite the Cauchy integral of the product $M_\zeta^{(0)} b_1 M_\zeta^{(1)} \ldots b_k M_\zeta^{(k)}$ as

$$K = -\frac{m}{\pi}\int_0^\infty M_t^{(0)} b_1 \ldots b_k M_t^{(k)} \frac{\mathrm{d}t}{t} \tag{14}$$

Here M_t means the usual L^1-dilation: $t^{-n}M\left(x; \dfrac{x-y}{t}\right)$; the product inside the integral represents a composition of convolution-type operators $M_t^{(j)}$ and multiplication operators with coefficients b_j.

Two basic results of [2], Propositions 1 and 2, will be adopted in our proof. The 2nd Proposition states that the kernel $K(x,y)$ of operator (14) is of the Calderon--Zygmund type, i.e.

$$\int_{|x-y| \geqslant 2|x-x'|} |K(x,y) - K(x',y)|\mathrm{d}y \leqslant C \tag{15}$$

with constants $C \leqslant C_0 \prod_1^k \|b_j\|$. Moreover, if the utmost right (convolution-type) factor $M^{(k)}(x,z)$ in (14), itself satisfied (12), then $K(x,y)$ would obey the basic estimates of the Calderon-Zygmund theory (see [2] p. 372)

(a) $|K(x;y)| \leqslant c|x-y|^{-n}$

(b) $|K(x';y) - K(x;y)| + |K(y;x') - K(y;x)| \leqslant c\dfrac{|x-x'|^\delta}{|x-y|^{n+\delta}}$.

The argument of [2] does not apply directly in our case, since $M^{(k)}$ is now a ψDO-kernel of order 0 (for ($|\alpha| = m$) with symbol $\sigma(x;\xi) = \dfrac{\xi^\alpha}{e^{i\theta} - a(x;\xi)}$, rather then a simple CZ-convolution kernel.

To proceed we split σ into the sum of the 0-th order homogeneous in ξ symbol $\sigma' = \dfrac{\xi^\alpha}{a(x;\xi)}$, and a negative order symbol $\sigma'' = \dfrac{e^{i\theta}\xi^\alpha}{a(e^{i\theta} - a)} \in S^{-m}$ (cf. [2]; [6]).

Correspondingly, kernel $M^{(k)}$ (the last term of (14)), splits into the sum, $M'(x;z) + M''(x;z)$. The former kernel, M', is homogeneous in z of degree $-n$, while the latter (of negative-order!) admits an L^1-radial bound (13).

Both are treated separately.

For M' we observe that t-dilations in z do not change M', so it can be pulled out of the integral (14), which leaves a composition of $(k-1)$ terms inside the integral. Using the cancellation property,

$$M'(x) = \int M'(x;z)\mathrm{d}z \in L^\infty, \quad \text{(uniform ellipticity!)}$$

one can show that M' is L^p-bounded, which reduces the estimate to $(k-1)$ remaining terms of the integral. Then the inductive step in k applies.

As for M'' we use the radial bound (13) and the second half of Proposition 2 of David-Journe.

The remaining steps of the argument are based on "BMO-type" cancellation conditions of David-Journe, $\hat{K}(x) = \int K(x;z)\mathrm{d}z \in \mathrm{BMO}$, and are carried over, as in [2].

Thus we get, L^p-boundedness of the integral kernel $K, \|K\| \leqslant C_p \prod_j \|b_j\|$. Consequently, each term $\{T_k\}$ of the series expansion (9) is estimated by

$$\|T_k\| \leqslant C_p(\sum_\alpha \|b_\alpha\|)^k.$$

Summing up all estimates we complete the proof of part (a) and Theorem 1 altogether.

REMARK 1. Theorem 1 extends to operators A, which are not uniformly elliptic, but whose coefficients may increase unboundedly or degenerate. One such class of operators was studied in [6]. It is defined in terms of the function $\rho(x) > 0$, which satisfies

$$(\partial^\alpha \rho)(x) = O(\rho^{1-|\alpha|/m})$$

for all partial derivatives $\partial^\alpha \rho, |\alpha| \leqslant m$ (for m-th order operators). In particular, $\rho(x) = O(|x|^m)$.

Precisely, operator $A = \rho A_0 + B$, where A_0 is uniformly elliptic and coefficients of B belong to weighted L^p-spaces,

$$L^{p_\alpha}_{w_\alpha} = \left\{ \int |bw_\alpha|^{p_\alpha} dx < \infty \right\}.$$

Here weight $w_\alpha = \rho^{-d/m}$ depends on ρ and the fractional order $d = \frac{n}{p_\alpha} + |\alpha|$ of the α-th term. In other words coefficients b_α are allowed certain growth or decay relative to ρ, which depends on their fractional order d.

In [6] we showed that such operators are m-accretive in L^p-spaces and the L^p--domain of A coinsides with the weighted Sobolev space

$$\mathcal{L}^p_m(\rho) = \left\{ u : \|\frac{1}{\rho}(1-\Delta)^{m/2} u\|_{L^p} < \infty \right\}.$$

This result can be extended to fractional powers of operators A. We shall state it without proof, which is somewhat involved, but whose ideas follows Theorem 1 and the techniques of [2] and [6–7].

THEOREM 2. *If the top order terms of perturbation B are sufficiently small, then the L^p-domain of $A^{s/m}$ is the weighted Sobolev space $\mathcal{L}^p_s(\rho^{s/m})$. Here $0 \leqslant s \leqslant m$, and $p_0 < p \leqslant \min\left\{ p_\alpha; \frac{p_0}{p_0 - 1} \right\}$.*

REMARK 2. The limitation on the scale on L^p-spaces ($p \leqslant p_0$) in Theorems 1, 2 is due to a nonsymmetric form of operators under consideration, i.e. $A = \sum a_\alpha(x) D^\alpha$, where differentiation is followed by multiplication. A symmetric (and more general) form of A is $\sum_{|\alpha+\beta| \leqslant 2m} D^\alpha a_{\alpha\beta} D^\beta$. The order of A has now been changed to $2m$. As above coefficients $\{a_{\alpha\beta}\}$ are taken in L^p classes ($p = p_{\alpha\beta}$), so that $d = \frac{n}{p} + |\alpha + \beta| \leqslant$ $\leqslant 2m$.

If an additional smoothness assumption on $\{a_{\alpha\beta}\}$ is imposed, namely, $a_{\alpha\beta} \in$ $\in \mathcal{L}^q_{|\alpha|}$, where $\frac{n}{q} + |\beta| \leqslant 2m$; then the Leibnitz rule:

$$D^\alpha a_{\alpha\beta} = \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma} D^\gamma a_{\alpha\beta} D^{\alpha-\gamma},$$

will reduce A to the above type perturbation.

If, furthermore

$$a_{\alpha\beta} \in \mathcal{L}^{q'}_{|\beta|}, \quad \text{such that } \frac{n}{q'} + |\alpha| \leqslant 2m$$

(in particular, if $A = \sum_{|\alpha|\leqslant m} a_\alpha D^\alpha$ with $a_\alpha \in \mathcal{L}^{\alpha}_{|\alpha|}$ such that $\frac{n}{q_\alpha} \leqslant m$, in the former notation), then the adjoint A^* of A can be shown to belong to the same class of perturbations (see [5], [6], [7]). Therefore Theorem 1 extends to a symmetric scale of L^p-spaces: $p_0 < p < \frac{p_0}{p_0 - 1}$.

With no additional smoothness imposed on coefficients $\{ a_{\alpha\beta} \}$, the "square root version" of Kato's conjecture can be shown for operators in a more special form

$$A = \sum_{|\alpha|,|\beta|\leqslant m} D^\alpha a_{\alpha\beta} D^\beta = A_0 + B. \tag{16}$$

Here each of multi indices $|\alpha|, |\beta| \leqslant m$ rather that $|\alpha + \beta| \leqslant 2m$. As above we take $a_{\alpha\beta} \in L^p$ with $\frac{n}{p} + |\alpha + \beta| \leqslant 2m$. Given any pair of functions $u \in \mathcal{L}^p_m, v \in \mathcal{L}^{p'}_m$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, one can shown as in [4], [5], that

$$|\langle Au; v\rangle| \leqslant C\|u\|_{\mathcal{L}^p_m}\|v\|_{\mathcal{L}^{p'}_m}, \quad \text{with}\, C = \sum \|a_{\alpha\beta}\|.$$

In particular, the square root of A (if it exists) is estimated by

$$\|A^{1/2}u\| \leqslant C\|(1-\Delta)^{m/2}u\|.$$

Therefore, $\mathcal{D}_p(A^{1/2}) \supseteq \mathcal{L}^p_m = \mathcal{D}_p((1-\Delta)^{m/2})$.

To show the converse inclusion we assume, as above, the leading part $A_0 = \sum_{|\alpha|=|\beta|=m} D^\alpha a_{\alpha\beta} D^\beta$ to be sufficiently smooth, and take a "singular" perturbation $B = \sum_{|\alpha|,|\beta|\leqslant m} D^\alpha b_{\alpha\beta} D^\beta$. The leading terms of B are assumed to be small.

Then we write

$$A_0 + B + \lambda = (A_0 + \lambda)^{1/2}[(A_0 + \lambda)^{-1/2} B (A_0 + \lambda)^{-1/2} + I](A_0 + \lambda)^{1/2}.$$

We denote by $P = (A_0 + \lambda)^{-1/2} B (A_0 + \lambda)^{-1/2}$, and observe that the argument of Theorem 1. yields $\|P\| < 1$ for sufficiently large $\lambda > 0$.

Now the L^2-estimate easily follows,

$$\operatorname{Re}\langle (A_0 + B + \lambda)u; u\rangle = \operatorname{Re}\langle (I + P)(A_0 + \lambda)^{1/2}u; (A_0 + \lambda)^{1/2}u\rangle \geqslant C\|(A_0 + \lambda)^{1/2}u\|^2.$$

But the latter was shown in Theorem 1 to be estimated from below by

$$C\|(1-\Delta)^{m-2}u\|^2.$$

Thus we get the following version of Kato's conjecture for operators form $A = \sum_{|\alpha|;|\beta|\leqslant m} D^\alpha a_{\alpha\beta} D^\beta = A_0 + B$ in L^2-spaces.

THEOREM 3. *If an operator A of (16) has coefficients $a_{\alpha\beta} \in L^p$, of order $d = \frac{n}{p} + |\alpha+\beta| \leqslant 2m$, and the top order terms of perturbation B are small in the appropiate L^∞-norm, then the L^2-domain of $A^{1/2}$ is equal to $\mathcal{L}^2_m$.*

REMARK 3. One can naturally ask, if above results extend for higher fractional powers of singular elliptic operators, i.e. if $\mathcal{D}_p(A^{s/m}) = \mathcal{L}^p_s$ also for $s > m$. In general the answer is negative: for $s > m$ the scale $\{ (\lambda + A)^{s/m} L^p \}$ differs from $\mathcal{L}^p_s$. Indeed, one can construct a Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$ which has the eigenfunction $\psi \in \mathcal{L}^p_2$, but not $\mathcal{L}^p_s$ with $s > 2$ (see [10]).

It turns out that the question of extension is closely related to smoothing properties of the semigroup e^{-tA} in the Sobolev scale or, equivalently, to the fact that conjugates

$$\tilde{A} = (1-\Delta)^{s/2} A (1-\Delta)^{-s/2}$$

of A have the resolvent (and semigroup) kernels of the same type as A. The latter in turn depends on smoothness of coefficients of A.

Precisely, we have proved in [8], that operators $A = A_0 + B$, whose coefficients are s-smooth, i.e. $b_\alpha \in \mathcal{L}^{1_\alpha}_s \left(\frac{n}{q_\alpha} - s + |\alpha| \leqslant m \right)$, are well defined (closed) in all $\mathcal{L}^p_\tau (0 \leqslant \tau \leqslant s)$ with the domain $\mathrm{Dom}(A|\mathcal{L}^p_\tau) = \mathcal{L}^p_{\tau+m}$.

This results allows to extend the range of s beyond $s = m$. Namely,

THEOREM 4. *If coefficients of $A = A_0 + B$ are s_0-smooth,* i.e. *$b_\alpha \in \mathcal{L}^{q_\alpha}_{s_0}$ such that $\frac{n}{q_\alpha} + |\alpha| - s_0 \leqslant m$, then the L^p-domain of $A^{s/m}$ is equal to $\mathcal{L}^p_s$, for all $0 \leqslant s \leqslant m + s_0$ and $p_0 < p \leqslant \min\left\{ q_\alpha; \frac{p_0}{p_0 - 1} \right\}$.*

Indeed, taking $s = m + \tau$, writing $(\lambda + A)^{-s/m} = (\lambda + A)^{-1}(\lambda + A)^{-\tau/m}$, $(1-\Delta)^{s/2} = (1-\Delta)^{m/2}(1-\Delta)^{\tau/2}$ and conjugating $(\lambda + A)^{-1}$ with $(1-\Delta)^{\tau/2}$, we get

$$(1-\Delta)^{s/2}(\lambda + A)^{-s/m} = (1-\Delta)^{m/2}(\lambda + \tilde{A})^{-1}(1-\Delta)^{\tau/2}(\lambda + A)^{-\tau/m}.$$

Here $\tilde{A}$ means the conjugate of A with $(1-\Delta)^{\tau/2}$. By [8] the operator $(1-\Delta)^{m/2}(\lambda + \tilde{A})^{-1}$ is bounded and invertible, while for $(1-\Delta)^{\tau/2}(\lambda + A)^{-\tau/2}$ this follows from Theorem 1, provided $\tau \leqslant m$. If $\tau > m$, we write $\tau = m + t$ and continue the process.

The process will obviously stop after finitely many steps which proves the theorem.

Acknowledgement. We would like to thank Y. Meyer for pointing out some gaps in the original version of the paper.

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Received May 7, 1990; revised August 15, 1990.