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DIAGONALIZATION IN INDUCTIVE LIMITS OF CIRCLE ALGEBRAS

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0. INTRODUCTION

Recently there has been renewed interest for the C^*-algebras that can build as an inductive limit of C^*-algebras that are (more or less) of the form $C(X) \otimes M_n(\mathbb{C})$ for certain compact spaces X. In particular the preprint [4] of Elliott provides new tools and results that can be used for the study of such algebras. In fact Elliott managed to extend the classification of AF-algebras in terms of K-theory to certain of these C^*-algebras by taking the K_1 group into account. In this paper we study inductive limits of circle algebras, i.e. of C^*-algebras of the form $C(\mathbb{T}) \otimes M_n(\mathbb{C})$. Some of these C^*-algebras, namely those of real rank zero, are subsumed under Elliott's classification result; however the main theme here is not the classification of these inductive limits but rather their internal structure. We shall show that the process of "diagonalization" which is known from the theory of AF-algebras can be carried out in a C^*-algebra B which is the inductive limit of a sequence of circle algebras with unital connecting $*$-homomorphisms.

To be specific, we prove that B contains a unital abelian C^*-subalgebra D which admits a free action $\alpha : G \to \operatorname{Aut} D$ of a countable abelian torsion group G and a unitary 2-cocycle $u : G \times G \to D$ such that B is the (twisted) crossed product $D \underset{\alpha,u}{\times} G$. This is a generalization of a result of Takesaki [5] concerning UHF algebras. By virtue of the fact that the action α is free, many properties of B are equivalent to specific properties of this action. Since the property of having real rank zero has been in focus in the last few years, let us mention that when B is simple, it has this property if and only if the action on D only has one invariant state, i.e. is uniquely ergodic.

We also show that B has $K_1(B) = 0$ if and only if B is the inductive limit C^*-algebra of a sequence of interval algebras $C[0,1] \otimes M_n(\mathbb{C})$ with unital connecting $*$-homomorphisms. In this case the group G can be taken to be an infinite direct sum of cyclic groups and the free action α can be taken to an ordinary (non-twisted) action. Finally, we give examples of simple C^*-algebras B of the above form with $K_1(B) = 0$ that are not subsumed under the classification result of Elliott [4], i.e. do not have real rank 0.

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1. PRELIMINARY RESULTS ON CIRCLE ALGEBRAS

We shall write M_n for the n by n complex matrices and consider $C(\mathbb{T}) \otimes M_n$ as a subalgebra of $C[0,1] \otimes M_n$ in the obvious way. Occasionally we will also identify $C(\mathbb{T}) \otimes M_n$ with the 1-periodic continuous M_n-valued functions on $\mathbb{R}$.

DEFINITION 1.1. An abelian C^*-subalgebra A of $C(\mathbb{T}) \otimes M_n$ is called *maximally homogeneous* when A contains the center $C(\mathbb{T})$ of $C(\mathbb{T}) \otimes M_n$ and $A(t) = \{ a(t) \in \in M_n : a \in A \}$ has dimension n for all $t \in \mathbb{T}$.

By [8, Lemma 1.4] an abelian C^*-subalgebra A of $C(\mathbb{T}) \otimes M_n$ is maximally homogeneous if and only if it has the property that pure states of A extends uniquely to a (pure) state of $C(\mathbb{T}) \otimes M_n$. In particular, a maximally homogeneous C^*-subalgebra must be maximal abelian.

LEMMA 1.2. *Let A be a maximally homogeneous abelian C^*-subalgebra of $C(\mathbb{T}) \otimes M_n$. There is then a set of matrix units $\{ e_{ij} \} \subseteq C[0,1] \otimes M_n$ such that*

$$A = \{ f \in C(\mathbb{T}) \otimes M_n \ : \ f(t) \in \operatorname{span}\{e_{11}(t), e_{22}(t), \ldots, e_{nn}(t) \}, \ t \in [0,1] \}$$

and there is a permutation $\sigma \in \Sigma_n$ so that

$$e_{\sigma(i)\sigma(j)}(0) = e_{ij}(1), \ i,j = 1,2,3,\ldots,n.$$

Furthermore, the functions $e_{ii} : [0,1] \to M_n$ can be extended as continuous projection valued maps $e_{ii} : \mathbb{R} \to M_n$ such that $e_{\sigma(i)\sigma(j)}(t) = e_{ii}(t+1)$, $t \in \mathbb{R}$, $i = = 1,2,3,\ldots,n$.

Proof. Consider $C(\mathbb{T}) \otimes M_n$ as the subalgebra of the C^*-algebra $C_b(\mathbb{R}, M_n)$ of continuous bounded M_n-valued functions on $\mathbb{R}$ consisting of 1-periodic functions. Using that $\dim A(t) = n$ for all $t \in \mathbb{R}$ as in the proof of [7], Lemma 2, or by using that lemma on compact intervals and then patch unitaries together by using that $H^1(\mathbf{R}, U(d)) = 0$ for all d, it follows that there is a unitary $v \in C_b(\mathbf{R}, M_n)$ such that $vAv^*(t) = D_n$ for all $t \in \mathbf{R}$. Here D_n denotes the standard diagonal in M_n. It follows that we can find matrix units $\{ f_{ij} \} \in C_b(\mathbf{R}, M_n)$ such that

$$A = \{ f \in C_b(\mathbf{R}, M_n) \ : \ f(t+1) = f(t) \in \operatorname{span}\{f_{11}(t), f_{22}(t), \dots, f_{nn}(t)\}, t \in \mathbf{R}\}.$$

Since $\{ f_{11}(t), f_{22}(t), \dots, f_{nn}(t) \} = \{ f_{11}(t+1), f_{22}(t+1), \dots, f_{nn}(t+1) \}$ there is a permutation $\sigma_t \in \Sigma_n$ such that $f_{\sigma_t(i)\sigma_t(i)}(t+1) = f_{ii}(t)$, $t \in \mathbf{R}$. By continuity and connectedness of $\mathbf{R}$ the map $t \to \sigma_t$ must be constant. Let σ be this constant value. Since $f_{\sigma(i)\sigma(1)}(0)$ and $f_{i1}(1)$ have the same minimal projections in M_n as range and domain projections, there are scalars $c_i \in \mathbf{T}$ such that

$$f_{\sigma(i)\sigma(1)}(0) = c_i f_{i1}(1), \ i = 1, 2, 3, \dots, n.$$

Let $w_i \ : \ [0,1] \to \mathbf{T}$ be continuous functions with $w_i(0) = 1$ and $w_i(1) = c_i$ and define

$$e_{ij}(t) = w_i(t)\overline{w_j(t)} f_{ij}(t), \ t \in [0,1], \ i, j = 1, 2, 3, \dots, n.$$

Then $\{ e_{ij} \}$ will be a set of matrix units in $C[0,1] \otimes M_n$ with the stated properties.■

When A is as in Lemma 1.2, we call $\sigma \in \Sigma_n$ the *characteristic permutation of* A. Although σ is not itself uniquely determined by A, it's conjugation class in Σ_n is and this is all that matters to us. A choice of matrix units $\{ e_{ij} \} \subseteq C[0,1] \otimes M_n$ with the properties described in Lemma 1.2 will be called a *matrix system for the position of* A *in* $C(\mathbf{T}) \otimes M_n$.

LEMMA 1.3. *Let* $A \subseteq C(\mathbf{T}) \otimes M_n$ *be a unital abelian* C^**-subalgebra. Then the following conditions are equivalent*

(i) *A is maximally homogeneous and the centralizer $\{\sigma\}'$ of it's characteristic permutation σ acts transitively on $\{ 1, 2, 3, \dots, n \}$,*

(ii) *A is maximally homogeneous and there is an abelian subgroup in the centralizer $\{\sigma\}'$ of it's characteristic permutation which acts transitively on $\{1, 2, \dots, n\}$,*

(iii) *There are natural numbers, k, m, with $km = n$ and a free action $\alpha \ : \ \mathbf{Z}_k \times \mathbf{Z}_m \to \operatorname{Aut} A$ such that the pair $A \subseteq C(\mathbf{T}) \otimes M_n$ is isomorphic to the pair $A \subseteq A \times_\alpha (\mathbf{Z}_k \times \mathbf{Z}_m)$,*

(iv) *There is (discrete) group G and a free action $\alpha \ : \ G \to \operatorname{Aut} A$ such that the pair $A \subseteq C(\mathbf{T}) \otimes M_n$ is isomorphic to the pair $A \subseteq A \times_\alpha G$.*

Proof. (iii)⇒(iv) is trivial.
(iv)⇒(i): If a discrete group acts freely on a unital abelian C^*-algebra A, then the image of A in the crossed product has the extension property of pure states by [1].

So by [8], Lemma 1.4, A must be maximally homogeneous. Let $\sigma \in \Sigma_n$ be the characteristic permutation of A and let $\{ e_{ij} \} \subseteq C[0,1] \otimes M_n$ be a matrix system for the position of A in $C(\mathbb{T}) \otimes M_n$. For every element $\mu \in \{\sigma\}'$ we can define $U_\mu \in C(\mathbb{T}) \otimes M_n$ by

$$U_\mu(t) = \sum_{i=1}^{n} e_{\mu(i)i}(t),\ t \in [0,1].$$

(This is an element of $C(\mathbb{T}) \otimes M_n$ only because μ commutes with σ.) In this way we obtain a unitary representation of $\{\sigma\}'$ such that $U_\mu e_{ii} U_\mu^* = e_{\mu(i)\mu(i)}$, $i = 1,2,3,\ldots,n$. It is straightforward to describe the structure of the group of unitary normalizers of A in $C(\mathbf{T}) \otimes M_n$ as a semi-direct product of the unitary group of A and $\{\sigma\}'$. In particular, this group of unitary normalizers generate $C(\mathbf{T}) \otimes M_n$ if and only if $\{\sigma\}'$ acts transitively on $\{1,2,3,\ldots,n\}$. But $A \times_\alpha G$ is certainly generated by the unitary A normalizers, so we conclude that $\{\sigma\}'$ must act transitively on $\{1,2,\ldots,n\}$.

(i)⇒(ii): Because $\{\sigma\}'$ acts transitively on $\{1,2,\ldots,n\}$ it follows that all σ's orbits are of the same lenght. Thus σ is the product of a number, m say, of disjoint cycles of the same length, k say. Then $km = n$ and it is clear that $\{\sigma\}'$ contains a subgroup isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_m$ which acts transitively on $\{1,2,3,\ldots,n\}$.

(ii)⇒(iii): From the preceding we know that there is a subgroup $G \subseteq \{\sigma\}'$, isomorphic to $\mathbb{Z}_k \times \mathbb{Z}_m$, which acts transitively on $\{1,2,3,\ldots,n\}$. Then $\mu \to \mathrm{Ad}U_\mu$, $\mu \in G$, defines an action of G on A. Because G acts transitively on $\{1,2,3,\ldots,n\}$, a simple partition of unity argument shows that $C(\mathbf{T}) \otimes M_n$ is generated by A and $\{ U_\mu : \mu \in G \}$. Let us show that $\beta = \mathrm{Ad}U$ gives a free action of G on A. So let ω be a pure state of A. We must show that $\omega \circ \mathrm{Ad}U_\mu \neq \omega$ when $\mu \neq 0$. Let $\hat{\omega}$ be the state extension of ω to $C(\mathbf{T}) \otimes M_n$. Since A has the extension property in $C(\mathbf{T}) \otimes M_n$, $\hat{\omega}$ is pure, and we must show that $\hat{\omega} \circ \mathrm{Ad}U_\mu \neq \hat{\omega}$ as states on $C(\mathbf{T}) \otimes M_n$. Since $\hat{\omega}$ is pure there is a $t \in [0,1]$ and a pure state λ of M_n such that $\hat{\omega}(f) = \lambda(f(t))$, $f \in C(\mathbb{T}) \otimes M_n$. Thus $\hat{\omega} \circ \mathrm{Ad}U_\mu(f) = \lambda(U_\mu(t) f(t) U_\mu(t)^*)$, $f \in C(\mathbb{T}) \otimes M_n$. Note that $\lambda(f(t)g(t)) = \hat{\omega}(fg) = \omega(f)\omega(g) = \lambda(f(t))\lambda(g(t))$ for all $f, g \in A$, showing that λ is a pure state on $A(t) \subseteq M_n$. Since $A(t) = \mathrm{span}\{e_{ii}(t)\}$ and $\mathrm{Ad}U_\mu(t)(e_{ii}(t)) = e_{\mu(i)\mu(i)}(t)$, we see that $\mathrm{Ad}U_\mu(t)$ is a free automorphism of $A(t)$ because $\mu \neq 0$. It follows that $\lambda \circ \mathrm{Ad}U_\mu(t) \neq \lambda$ on $A(t)$, and consequently also on M_n. Thus $\hat{\omega} \circ \mathrm{Ad}U(\mu) \neq \hat{\omega}$. It is now easy to see that $C(\mathbb{T}) \otimes M_n \simeq A \times_\beta G$, and in fact this follows from a much more general result like Corollary 15 of [6]. Since G is isomorphic to $\mathbb{Z}_{\frac{n}{k}} \times \mathbb{Z}_k$, this completes the proof. ∎

A unital abelian C^*-subalgebra A of $C(\mathbf{T}) \otimes M_n$ will be called a *normal diagonal* when it satisfies the equivalent conditions of Lemma 1.3. The justification for this name comes from the fact that $C(\mathbb{T}) \otimes M_n$ can be considered as an extension of

$\mathbf{Z}_{\frac{n}{k}} \times \mathbf{Z}_k$ by A. In this analogy with group extensions A plays the role of the normal subgroup.

Now we turn the attention to $*$-homomorphisms between circle algebras.

DEFINITION 1.4. A unital $*$-homomorphism $\phi : C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$ is called *maximally homogeneous* when (the vector space dimension) $\dim \phi(C(\mathbf{T}) \otimes \otimes M_n)(t) = mn$ for all $t \in \mathbf{T}$.

Note that the $\dim \phi(C(\mathbf{T}) \otimes M_n)(t) \leqslant mn$ for all t whenever ϕ is a unital $*$-homomorphism. Thus the requirement on a maximally homogeneous one is that the dimension of the range is maximal in each simple quotient of $C(\mathbf{T}) \otimes M_m$. Such $*$-homomorphisms, for more general spaces in place of $\mathbf{T}$, have been studied to some extend before, cf. [7], [3].

The importance of the maximally homogeneous $*$-homomorphisms, in particular for the study of inductive limit C^*-algebras build from C^*-algebras of the form $C(\mathbf{T}) \otimes \otimes M_n$, comes from the following result of Elliott, cf. [4], Theorem 4.4.

THEOREM 1.5. (Elliott) *The maximally homogeneous $*$-homomorphisms are dense, in the topology of pointwise normconvergence, among the unital $*$-homomorphisms $C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$.*

As it follows from Elliott's investigations in [4], one very important feature of a maximally homogeneous $*$-homomorphism is the variation over $\mathbf{T}$ of the eigenvalues of the image of the canonical unitary generator of the center of the domain algebra. We introduce this in a way which is convenient for our present purposes.

DEFINITION 1.6. Let $\phi : C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$ be a maximally homogeneous $*$-homomorphism. By [7] there are then $\frac{m}{n}$ continuous functions $g_i : [0,1] \to \mathbf{T}$ and a unitary $U \in C[0,1] \otimes M_m$ such that

$$\phi(f) = U \begin{bmatrix} f \circ g_1 & 0 & \dots & 0 \\ 0 & f \circ g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ g_{\frac{m}{n}} \end{bmatrix} U^*.$$

The functions $g_1, g_2, \dots, g_{\frac{m}{n}}$ will be called the *characteristic functions of* ϕ.

Let $\phi : C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$ be a maximally homogeneous $*$-homomorphism with characteristic functions $g_1, g_2, \dots, g_{\frac{m}{n}}$. There are then continuous functions $G_1, G_2, \dots, G_{\frac{m}{n}} : [0,1] \to \mathbf{R}$ such that

$$g_j = \exp(2\pi \mathrm{i} G_j),\ j = 1, 2, 3, \dots, \frac{m}{n}.$$

Since the G_j's are only determined mod $\mathbb{Z}$, we can assume that

$$\{G_1(0), G_2(0), \ldots, G_{\frac{m}{n}}(0)\} \subseteq [0,1[. \tag{1.1}$$

After a renumbering we can furthermore assume that $G_1(0) < G_2(0) < \cdots < G_{\frac{m}{n}}(0)$. Since $g_j(t) \neq g_i(t)$ for $t \in [0,1]$, $i \neq j$, it follows that

$$G_1(t) < G_2(t) < \cdots < G_{\frac{m}{n}}(t) \text{ for all } t \in [0,1]. \tag{1.2}$$

We call $G_1, G_2, \ldots, G_{\frac{m}{n}}$, subject to both (1.1) and (1.2), for the *characteristic real valued functions for* ϕ. Since we have that

$$\{g_j \,:\, j = 1,2,3,\ldots,\frac{m}{n}\} = \{\exp(2\pi i G_j) \,:\, j = 1,2,3,\ldots,\frac{m}{n}\},$$

there is a permutation $\sigma \in \Sigma_{\frac{m}{n}}$ such that $G_{\sigma(i)}(0) - G_i(1) \in \mathbb{Z}$. We call σ for the *characteristic permutation of* ϕ.

For the formulation of the next lemma, which index the realvalued characteristic functions of a maximally homogeneous $*$-homomorphism in a suitable way, we consider $\mathbb{Z}_n$ as the subgroup of Σ_n generated by the cyclic permutation in $\{1,2,3,\ldots,n\}$.

LEMMA 1.7. *Let* $G_1, G_2, \ldots, G_n : [0,1] \to \mathbb{R}$ *be continuous functions and* $\sigma \in \Sigma_n$ *a permutation such that*

(i) $G_1(t) < G_2(t) < \cdots < G_n(t)$, $t \in [0,1]$,
(ii) $G_1(0), G_2(0), \ldots, G_n(0) \in [0,1[$,
(iii) $\exp(2\pi i G_j(t)) \neq \exp(2\pi i G_k(t))$, $t \in [0,1]$, $j \neq k$,
(iv) $\exp(2\pi i G_{\sigma(j)}(0)) = \exp(2\pi i G_j(1))$, $j = 1,2,3,\ldots,n$.
Then
(a) $\sigma \in \mathbb{Z}_n$.
(b) *Let k be the order of σ and let $\mu \in \Sigma_k$ be cyclic permutation. There is a bijection* $\phi : \{1,2,3,\ldots,k\} \times \{1,2,3,\ldots,\frac{n}{k}\} \to \{1,2,\ldots,n\}$ *and a function* $f : \{1,2,3,\ldots,k\} \to \mathbb{Z}$ *such that*

$$H_{(\mu(i),j)}(0) = H_{(i,j)}(1) - f(i),$$

where $H_{(i,j)} = G_{\phi(i,j)}$, $(i,j) \in \{1,2,3,\ldots,k\} \times \{1,2,\ldots,\frac{n}{k}\}$.

Proof. Since $G_n(0) - 1 < G_1(0) < G_2(0) < \cdots < G_n(0)$ by (ii), conditions (iii) and (i) imply that $G_n(t) - 1 < G_1(t) < G_2(t) < \cdots < G_n(t)$ for all $t \in [0,1]$. Thus the permutation σ which satisfies (iv) must be some power of the cyclic permutatiion. This proves (a). Furthermore it shows that if $\sigma = \mu^j$, where $j \in \{0,1,\ldots,n-1\}$, then there is an integer x such that

$$G_i(1) - G_{\sigma(i)}(0) = G_i(1) - G_{i+j}(0) = x, \; i = 1,2,\ldots,n-j,$$

and

$$G_i(1) - G_{\sigma(i)}(0) = G_i(1) - G_{i+j-n}(0) = x + 1, \ i = n - j + 1, \dots, n.$$

Set $g(i) = G_i(1) - G_{\sigma(i)}(0)$, $i = 1, 2, 3, \dots, n$. The crucial observation is that g is constant on the sets $\{1, 2, 3, \dots, \frac{n}{k}\}$, $\{\frac{n}{k} + 1, \frac{n}{k} + 2, \dots, 2\frac{n}{k}\}, \dots, \{(k-1)\frac{n}{k} + 1, (k-1)\frac{n}{k} + 2, \dots, n\}$. This follows by noting that $j = z\frac{n}{k}$ for some $z \in \{1, 2, \dots, k-1\}$ and that $n - j = (k - z)\frac{n}{k}$.

Define $\psi : \{1, 2, 3, \dots, k\} \times \{1, 2, 3, \dots, \frac{n}{k}\} \to \{1, 2, 3, \dots, n\}$ by

$$\psi(i, j) = (i - 1)\frac{n}{k} + j.$$

Then $\psi^{-1} \circ \mu^{\frac{n}{k}} \circ \psi(i, j) = (\mu(i), j)$ and the value of $g \circ \psi(i, j)$ depends only on i. Let $g'(i) = g \circ \psi(i, 1)$ and let $H'_{(i,j)} = G_{\psi(i,j)}$, $(i, j) \in \{1, 2, \dots, k\} \times \{1, 2, 3, \dots, \frac{n}{k}\}$. Then $H'_{(\mu^z(i),j)}(0) = G_{\psi(\mu^z(i),j)}(0) = G_{\sigma \circ \psi(i,j)}(0) = G_{\psi(i,j)}(1) - g \circ \psi(i, j) = H'_{(i,j)}(1) - g'(i)$ for all i, j. Since the order of μ^z is the order of σ which is k, we conclude that μ^z is conjugate in Σ_k to μ. Let $\beta \in \Sigma_k$ such that $\beta^{-1} \circ \mu^z \circ \beta = \mu$. Define ϕ : $\{1, 2, \dots, k\} \times \{1, 2, 3, \dots, \frac{n}{k}\} \to \{1, 2, \dots, n\}$ by $\phi(i, j) = \psi(\beta(i), j)$ and set $H_{(i,j)} = G_{\phi(i,j)} = H'_{(\beta(i),j)}$. Then

$$\begin{aligned} H_{(\mu(i),j)}(0) &= H'_{(\beta(\mu(i)),j)}(0) = H'_{(\mu^z(\beta(i)),j)}(0) = \\ &= H'_{(\beta(i),j)}(1) - g'(\beta(i)) = H_{(i,j)}(1) - f(i) \end{aligned}$$

for all i, j when f is defined by $f = g' \circ \beta$. ∎

The following proposition shows how maximally homogeneous abelian C^*-subalgebras of circle algebras are related by maximally homogeneous $*$-homomorphisms. It constitutes the crucial step in the proof of our main result.

PROPOSITION 1.8. *Let $\phi : C(\mathbb{T}) \otimes M_n \to C(\mathbb{T}) \otimes M_m$ be a maximally homogeneous $*$-homomorphism and $A \subseteq C(\mathbb{T}) \otimes M_n$ a maximally homogeneous abelian C^*-subalgebra. Let $\mu \in \Sigma_n$ and $\sigma \in \Sigma_{\frac{m}{n}}$ be the characteristic permutations of A and ϕ, respectively.*

(i) *There is a unique maximally homogeneous abelian C^*-subalgebra $B \subseteq C(\mathbb{T}) \otimes M_m$ such that $\phi(A) \subseteq B$. B is generated by $\phi(A)$ and the center of $C(\mathbb{T}) \otimes M_m$.*

(ii) *If A is a normal diagonal, then so is B.*

(iii) *Let p be the order of σ. The conjugacy class of the characteristic permutation of B is represented by the bijection Φ of $\{1, 2, \dots, n\} \times \{1, 2, \dots, p\} \times \{1, 2, \dots, \frac{m}{pn}\}$ given by*

$$\Phi(i, j, k) = (\mu^{f(j)}(i), c(j), k), \ (i, j, k) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, p\} \times \{1, 2, \dots, \frac{m}{pn}\},$$

where c is cyclic permutation and f is an integer valued map.

Proof. Since ϕ is maximally homogeneous there are orthogonal projections $p_1, p_2, \ldots, p_{\frac{m}{n}} \in C[0,1] \otimes M_{\frac{m}{n}}$ of sum 1 such that $p_{\sigma(i)}(0) = p_i(1)$, $i = 1, 2, 3, \ldots, \frac{m}{n}$, and

$$\phi(h)(t) = \sum_{i=1}^{\frac{m}{n}} h \circ g_i(t) \otimes p_i(t), \; h \in C(\mathbb{T}) \otimes M_n, \; t \in [0,1],$$

where $g_j = \exp(2\pi i G_j)$ and $G_1, G_2, \ldots, G_{\frac{m}{n}}$ are the characteristic realvalued functions for ϕ. In fact we can be a little more specific. By Lemma 1.7 we can find continuous functions $H_{(j,k)} : [0,1] \to \mathbb{R}$ and a function $f : \{1, 2, \ldots, p\} \to \mathbb{Z}$ such that

$$H_{(c(j),k)}(0) = H_{(j,k)}(1) - f(j),$$

$(j,k) \in \{1, 2, 3, \ldots, p\} \times \{1, 2, 3, \ldots, \frac{m}{pn}\}$, and $\{\exp(2\pi i H_{(j,k)}) : j = 1, 2, 3, \ldots, p$, $k = 1, 2, 3, \ldots, \frac{m}{kn}\}$ are the characteristic functions for ϕ. Let $p_{j,k}$, $j = 1, 2, 3, \ldots, p$, $k = 1, 2, \ldots, \frac{m}{pn}$, be an orthogonal set of projections in $C[0,1] \otimes M_{\frac{m}{n}}$ of sum 1 such that $p_{c(j),k}(0) = p_{j,k}(1)$ for all j, k. Set

$$P_{(i,j,k)}(t) = e_{ii} \circ H_{(j,k)}(t) \otimes p_{j,k}(t), \; t \in [0,1],$$

$(i,j,k) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, p\} \times \{1, 2, \ldots, \frac{m}{pn}\}$, where $\{e_{ij}\}$ is a matrix system for the position of A in $C(\mathbb{T}) \otimes M_n$. By viewing $C(\mathbb{T})$ as consisting of 1-periodic continuous functions on $\mathbb{R}$, we can assume that ϕ is given by

$$\phi(f)(t) = \sum_{j,k} f \circ H_{(j,k)}(t) \otimes p_{j,k}(t).$$

In this description it is clear that ϕ maps A into

$$B = \{ f \in C(\mathbb{T}) \otimes M_m : f(t) \in \operatorname{span}\{ P_{(i,j,k)}(t) \}, \; t \in [0,1] \}.$$

By using the last statement of Lemma 1.2 we find that

$$P_{(i,j,k)}(1) = P_{(\mu^{f(j)}(i), c(j), k)}(0), \; \text{ for all } i, j, k,$$

so that we can prove (i) and (iii) by showing that B is generated by $\phi(A)$ and $C(\mathbb{T}) \otimes 1$. By a partition of unity argument it suffices to fix $t \in [0,1]$ and prove that $\phi(A)(t) = \operatorname{span}\{ P_{(i,j,k)}(t) \}$. Fix $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, 3, \ldots, p\}$ and $k \in \{1, 2, 3, \ldots, \frac{m}{pn}\}$. Since $\exp(2\pi i H_{(j,k)}(t)) \neq \exp(2\pi i H_{(j_1,k_1)}(t))$, $(j,k) \neq (j_1,k_1)$, we can find h in the center of $C(\mathbb{T}) \otimes M_n$ such that $h(\exp(2\pi i H_{(j,k)}(t))) = 1$ and $h(\exp(2\pi i H_{(j_1,k_1)}(t))) = 0$, $(j_1,k_1) \neq (j,k)$. Furthermore since $A(\exp(2\pi i H_{(j,k)}(t))) =$

span$\{e_{ii}(H_{(j,k)}(t))\}$, we can find $h_1 \in A$ such that $h_1(\exp(2\pi i H_{(j,k)}(t))) = = e_{ii}(H_{(j,k)}(t))$. Then $hh_1 \in A$ and $\phi(hh_1)(t) = P_{(i,j,k)}(t)$. Thus $\phi(A)$ and $c(\mathbf{T}) \otimes 1$ generate B and (i) and (iii) are proved.

To prove (ii) assume that $\{\mu\}'$ contains a subgroup G which acts transitively on $\{1, 2, 3, \ldots, n\}$. For $g \in G$, define $g \times \mathrm{id} \in \Sigma_m$ by

$$g \times \mathrm{id}(i, j, k) = (g(i), j, k),$$

$(i, j, k) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, p\} \times \{1, 2, \ldots, \frac{m}{pn}\}$. For $h \in \mathbb{Z}_{\frac{m}{pn}}$, define $\mathrm{id} \times h \in \in \Sigma_m$ in a similar way:

$$\mathrm{id} \times h(i, j, k) = (i, j, h(k)).$$

Let $h \subseteq \Sigma_m$ be the subgroup generated by $\{g \times \mathrm{id} : g \in G\}$, $\{id \times h : h \in \mathbb{Z}_{\frac{m}{pn}}\}$ and Φ. Then H is an abelian subgroup of the centralizer of Φ in Σ_m which acts transitively on $\{1, 2, 3, \ldots, m\}$ (identified with $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, p\} \times \{1, 2, \ldots, \frac{m}{pn}\}$). Consequently B is a normal diagonal and (ii) is proved. ∎

2. INDUCTIVE LIMIT C^*-ALGEBRAS OF SEQUENCES OF CIRCLE ALGEBRAS WITH UNITAL CONNECTING $*$-HOMOMORPHISMS.

In this section we study inductive limits of sequences

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \xrightarrow{\phi_4} \cdots$$

consisting of circle algebras A_n and unital connecting $*$-homomorphisms.

LEMMA 2.1. (Elliott) *Let A be the inductive limit of a sequence of circle algebras with unital connecting $*$-homomorphisms. The A is $*$-isomorphic to the inductive limit C^*-algebra of a sequence*

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \xrightarrow{\phi_4} \cdots$$

of circle algebras $A_i = C(\mathbf{T}) \otimes M_{n_i}$ with maximally homogeneous connecting $*$-homomorphisms ψ_n.

Proof. By Theorem 2.1 of [4] we can exchange the given connecting $*$-homomorphisms with others that approximate the original ones sufficiently closely without changing the isomorphism class of the direct limit C^*-algebra. By Theorem 1.5 we can therefore exchange them with maximally homogeneous ones. ∎

THEOREM 2.2. *Let B be the inductive limit C^*-algebra of a sequence of circle algebras with unital connecting $*$-homomorphisms.*

There is then a unital abelian C^*-subalgebra $D \subseteq B$, a countable abelian torsion group G and a free twisted action (α, u) of G on D such that $B \simeq D \underset{\alpha,u}{\times} G$.

Proof. By Lemma 2.1, we can assume that $B = \varinjlim A_n$ is the inductive limit of a sequence of circle algebras with maximally homogeneous connecting $*$-homomorphisms ϕ_n. So by Proposition 1.8 we can find a sequence $D_n \subseteq A_n$ of normal diagonals such that $\phi_n(A_n) \subseteq A_{n+1}$. Let $\rho_i : A_i \to B$ be the canonical $*$-homomorphisms and set $D = \overline{\bigcup_n \rho_n(D_n)}$. Let $\sigma_i \in \Sigma_{n_i}$ be the characteristic permutation of $D_i \subseteq \subseteq C(\mathbb{T}) \otimes M_{n_i}$, $i = 1, 2, 3, \ldots$.

We want to construct by induction an abelian subgroup G_i of the centralizer of σ_i in Σ_{n_i} acting transitively on $\{1, 2, \ldots, n_i\}$, and a map $V^i : G_i \to A_i$ of G_i into the unitary normalizers of D_i such that $V_0^i = 1$ and for some matrix system $\{e_{ij}\} \subseteq C[0,1] \otimes M_{n_i}$ for the position of D_i in $C(\mathbb{T}) \otimes M_{n_i}$ we have

$$V_g^i e_{jj} V_g^{i*} = e_{g(j)g(j)}, \ j = 1, 2, 3, \ldots, n_i, \ g \in G_i. \tag{2.1}$$

Furthermore we want the sequence to be compatible with the ϕ_i's in the sense that there are injective group homomorphisms $\lambda_i : G_i \to G_{i+1}$ such that $V^{i+1} \circ \lambda_i = \phi_i \circ V^i$ for all i.

To start the induction we just repeat the construction in the proof of Lemma 1.3. We proceed to the induction step. So assume that we have constructed everything for $i \leqslant k$.

Let $m = \dfrac{n_{k+1}}{n_k}$ and let p be the order of the characteristic permutation of ϕ_k. By Lemma 1.7, we have continuous functions $H_{(j,k)} : [0,1] \to \mathbb{R}$, $j = 1, 2, \ldots, p$, $k = 1, 2, \ldots, \dfrac{m}{n}$, and a function $f : \{1, 2, 3, \ldots, p\} \to \mathbb{Z}$ such that $H_{(c(j),k)}(0) = = H_{(j,k)}(1) - f(j)$ for all j, k and such that ϕ_k is given by

$$\phi_k(g)(t) = \sum_{j,k} g \circ H_{(j,k)}(t) \otimes p_{(j,k)}(t), \ g \in C(\mathbb{T}) \otimes M_{n_k}, \ t \in [0,1],$$

where $p_{(j,k)}$, $(j,k) \in \{1, 2, 3, \ldots, p\} \times \{1, 2, 3, \ldots, \dfrac{m}{n}\}$, is a set of orthogonal projections in $C[0,1] \otimes M_m$ of sum 1 such that $p_{(c(j),k)}(0) = p_{(j,k)}(1)$ for all j, k. By Proposition 1.8 we can identify $\{1, 2, \ldots, n_{k+1}\}$ with $\{1, 2, 3, \ldots, n_k\} \times \{1, 2, 3, \ldots, p\} \times \times \{1, 2, 3, \ldots, \dfrac{m}{p}\}$, where p is the order of the characteristic permutation of ϕ_k, in such that way that the characteristic permutation σ_{k+1} of A_{k+1} is given by

$$\sigma_{k+1}(i, j, k) = (\sigma_k^{f(j)}(i), c(j), k).$$

Let $V^k : G_k \to A_k$ be the unitary map of D_k normalizers and $\{e_{ij}\}$ a matrix system for the position of D_k in A_k such that (2.1) holds for $i = k$. There is then a matrix system $\{F_{(i,j,k)(i_1,j_1,k_1)}\} \subseteq C[0,1] \otimes M_{n_{k+1}}$ for the position of D_{k+1} in A_{k+1} such that

$$F_{(i,j,k)(i,j,k)}(t) = e_{ii} \circ H_{(j,k)}(t) \otimes p_{(j,k)}(t).$$

Define $\lambda_k : G_k \to \Sigma_{n_{k+1}}$ by

$$\lambda_k(g)(i,j,k) = (g(i),j,k),$$

for $(i,j,k) \in \{1,2,3,\ldots,n_k\} \times \{1,2,3,\ldots,p\} \times \{1,2,\ldots,\frac{m}{p}\}$. As shown in the proof of Proposition 1.8 (ii) there is an abelian subgroup G_{k+1} of the centralizer of σ_{k+1} in $\Sigma_{n_{k+1}}$ which acts transitively on $\{1,2,\ldots,n_k\}$ and contains $\lambda_k(G_k)$.

Define W_g, $g \in G_{k+1}$, by

$$\sum_{i,j,k} F_{g(i,j,k)(i,j,k)}$$

Then $g \to W_g$ is a representation of G_{k+1} as unitary D_{k+1} normalizers in A_{k+1}. Furthermore,

$$W_g F_{(i,j,k)(i,j,k)} W_g^* = F_{g(i,j,k)g(i,j,k)},$$

$g \in G_{k+1}$, $(i,j,k) \in \{1,2,3,\ldots,n_k\} \times \{1,2,3,\ldots,p\} \times \{1,2,\ldots,\frac{m}{p}\}$. Note that

$$\mathrm{Ad}\phi_k(V^k g)(F_{(i,j,k)(i,j,k)})(t) = e_{g(i)g(i)} \circ H_{(j,k)}(t) \otimes p_{(j,k)}(t) =$$
$$= F_{\lambda_k(g)(i,j,k)\lambda_k(g)(i,j,k)}(t) = \mathrm{Ad}W_{\lambda_k(g)}(F_{(i,j,k)(i,j,k)})(t),\ t \in [0,1],\ g \in G_k,$$

for all i,j,k. Now define V_g^{k+1}, $g \in G_{k+1}$, by

$$V_g^{k+1} = \phi_k(V^k_{\lambda_k^{-1}(g)}) \text{ for } g \in \lambda_k(G_k)$$

and

$$V_g^{k+1} = W_g \text{ for } g \notin \lambda_k(G).$$

It is straightforward to check that with these definitions we have completed the induction step.

We need three observations:

(i) For each $i \in \mathbb{N}$ and $g,h \in G_i$, we have $V_g^i V_h^i V_{g+h}^{i*} \in D_i$.

(ii) For each $i \in \mathbb{N}$, the action $g \to \mathrm{Ad}V_g^i$ of G_i on D_i is free.

(iii) D_i and $\{ V_g^i : g \in G_i \}$ generate A_i.

(i) follows from (2.1) and the fact that D_i is maximal abelian in A_i. Thus $\mathrm{Ad}V_g^i$ does indeed define an action of G_i on D_i. Note next that every abelian subgroup of Σ_n which acts transitively must also act freely. Thus in particular, G_i acts freely on $\{1,2,3,\ldots,n_i\}$. Then (ii) follows as in the proof of (ii)⇒(iii) in Lemma 1.3. (iii) follows from a partition of unity argument and the fact that G_i acts transitively on $\{1,2,\ldots,n_i\}$ upon using (2.1) again.

Let G be the inductive limit group of the sequence $\lambda_i : G_i \to G_{i+1}$ and let $\mu_i : G_i \to G$ be the canonical embeddings. We have then a welldefined map $U : G \to B$ given by

$$U(\mu_i(g)) = \rho_i(V_g^i),\ g \in G_i,\ i = 1,2,3,\ldots$$

By construction each $U(g)$, $g \in G$, is a unitary D normalizer and by (i) above we have that $u_{g,h} = U(g)U(h)U(g+h)^* \in D$ for all $g\ h \in G$. Thus $\mathrm{Ad}U(g)$ gives us an action of G on D and u is clearly a 2-cocycle with respect to this action. Furthermore, by (iii) above, D and $\{U(g) : g \in G\}$ together generate B. If ω is a pure state of D and $g \in G_i$ satisfies that $\omega \circ \mathrm{Ad}U(\mu_i(g)) = \omega$, then $\omega \circ \varphi_i$ is a pure state of D_i which is fixed under the (dual of the) action of $\mathrm{Ad}V_g^i$. Hence from (ii) above we conclude that $g = 0$. This shows that $\alpha_g = \mathrm{Ad}U(g)$ defines a free action of G on D. It follows in particular from this that D has the extension property (of pure states) in B. (If this is not wellknown, it can be proved as Proposition 4 of [6]). Thus there is a conditional expectation $P : B \to D$ given by requirering $\omega(P(b))$ to equal $\hat{\omega}(b)$, $b \in B$, for any pure state ω of D with pure state extension $\hat{\omega}$. As it follows from the proof of the Proposition 4 in [6], we have $P(aU(g)) = 0$ for all $a \in D, g \in G \setminus \{0\}$. By the universal property of the twisted crossed product, cf.[9], there is a $*$-homomorphism $\pi : D \times_{\alpha,u} G \to B$ which is surjective because D and $\{U(g) : g \in G\}$ generate B and which is obviously injective on the canonical image of D in $D \times_{\alpha,u} G$. Since G is abelian and therefore in particular amenable, we can identify $D \times_{\alpha,u} G$ with the corresponding reduced crossed product so that we have a faithful conditional expectation $P_1 : D \times_{\alpha,u} G \to D$, cf.[9]. From our observations on the conditional expectation $P : B \to D$ it follows immediately that $P \circ \pi = \pi \circ P_1$. Using this identity the faithfulness of π on all of B follows from the faithfulness of P_1 and the faithfulness of π on D. ■

Many of the important properties of B can be read off from the free action $G \to \mathrm{Aut}\,D$ described in Theorem 3.2. For example, by [9], there are bijective correspondences between G-invariant ideals in D and ideals in B and between G-invariant states on D and finite traces on B. So B is simple if and only if the action of G on the pure state space of D is minimal, and B has a unique tracial state if and only if the action only has one invariant state. It is easy to see that any pair of tracial states on an inductive limit C^*-algebra of the type we consider here must agree on projections. Consequently such a C^*-algebra is, provided it is simple, of real rank zero if and only if it only has one tracial state by Theorem 1.3 of [2]. Thus such a B is of real rank zero if and only if the action of α on D only has one invariant state.

It must also be remarked, that the abelian C^*-subalgebra which can serve as D in Theorem 2.2 is not unique. For example, in a given Bunce-Deddens algebra, the process in the proof of Theorem 2.2 can be applied to many non-isomorphic abelian C^*-subalgebras.

We next turn to the case where K_1 is zero.

LEMMA 2.3. *Let $\psi : C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$ be a maximally homogeneous $*$-homomorphism. Then $\psi_* = 0$ on K_1 if and only if the realvalued characteristic functions of ψ are 1-periodic.*

Proof. The generator of $K_1(C(\mathbf{T}) \otimes M_n) \simeq \mathbf{Z}$ is represented by the unitary

$$u(t) = \operatorname{diag}(e^{2\pi i t}, 1, 1, \ldots .1), \ t \in [0,1].$$

So $\psi_* = 0$ on K_1 if and only if $\psi(u)$ represents 0 in $K_1(C(\mathbf{T}) \otimes M_m)$. This happens if and only if the loop $t \to \det(\psi(u)(t))$ in $\mathbf{T}$ is homotopic to the constant loop. Since

$$\det \psi(u)(t) = \exp(2\pi i \sum_{j=1}^{\frac{m}{n}} G_j(t)),$$

where $\{G_1, G_2, \ldots, G_{\frac{m}{n}}\}$ are the realvalued characteristic functions of ψ, this happens if and only if

$$\sum_{j=1}^{\frac{m}{n}} (G_j(1) - G_j(0)) = 0.$$

But from the proof of Lemma 1.7 we know that there is an integer x and a $j \in \{0, 1, 2, \ldots, \frac{m}{n} - 1\}$ such that

$$\sum_{j=1}^{\frac{m}{n}} (G_j(1) - G_j(0)) = (\frac{m}{n} - j)x + j(x+1).$$

This expression is 0 if and only if $x = 0$ and $j = 0$. This happens precisely if $G_i(1) = G_i(0)$ for all $i = 1, 2, \ldots, \frac{m}{n}$, cf. the proof of Lemma 1.7. ∎

LEMMA 2.4. *Let $\psi : C(\mathbf{T}) \otimes M_n \to C(\mathbf{T}) \otimes M_m$ be a maximally homogeneous $*$-homomorphism such that $\psi_* = 0$ on K_1. There is then a unital injection $\phi_1 : C(\mathbf{T}) \otimes$ $\otimes M_n \to C[0,1] \otimes M_n$ and a unital $*$-homomorphism $\phi_2 : C[0,1] \otimes M_n \to C(\mathbf{T}) \otimes M_m$ such that $\psi = \phi_2 \circ \phi_1$.*

Proof. By Lemma 2.3, the characteristic realvalued functions of ψ are 1-periodic. Therefore the characteristic permutation of ψ is trivial and ψ is inner equivalent to the $*$-homomorphism

$$f \to \operatorname{diag}(f \circ g_1, f \circ g_2, \ldots, f \circ g_{\frac{m}{n}})$$

where $g_j(t) = \exp(2\pi i G_j(t))$, $j = 1, 2, \ldots, \frac{m}{n}$, are the characteristic functions for ψ. By identifying $C(\mathbf{T})$ with the continuous 1-periodic functions on $\mathbf{R}$, we can thus assume that

$$\psi(f) = \operatorname{diag}(f \circ G_1, f \circ G_2, \ldots, f \circ G_{\frac{m}{n}}),$$

where $G_1, G_2, \ldots, G_{\frac{m}{n}}$ are the realvalued characteristic functions of ψ. Choose $p \in \mathbb{N}$ so large that $G_i([0,1]) \subseteq [-p, p]$ for all i. Under the present identification of $C(\mathbb{T})$ we can define an injection $\phi_1 : C(\mathbb{T}) \otimes M_n \to C[-p,p] \otimes M_n$ by restriction. If we define $\phi_2 : C[-p,p] \otimes M_n \to C(\mathbb{T}) \otimes M_m$ by

$$\phi_2(f) = \operatorname{diag}(f \circ G_1, f \circ G_2, \ldots, f \circ G_{\frac{m}{n}}),$$

then $\psi = \phi_2 \circ \phi_1$. The lemma follows by identifying $C[-p,p] \otimes M_n$ with $C[0,1] \otimes M_n$. ■

THEOREM 2.5. *Let B be the inductive limit of a sequence of circle algebras with unital connecting $*$-homomorphisms. Then $K_1(B) = 0$ if and only if B is $*$-isomorphic to the inductive limit of a sequence of interval algebras with unital connecting $*$-homomorphisms, and in that case there is a unital abelian C^*-subalgebra A of B, a sequence $\{k_i\} \subseteq \mathbb{N}$ and a free action $\alpha : \bigoplus_{i=1}^{\infty} \mathbb{Z}/k_i\mathbb{Z} \to \operatorname{Aut} A$ such that $B \simeq A \times_{\alpha} \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}/k_i\mathbb{Z} \right)$.*

Proof. Since an interval algebra has trivial K_1 the necessity of the condition is clear. To prove that it is also sufficient, assume that

$$B_1 \xrightarrow{\phi_1} B_2 \xrightarrow{\phi_2} B_3 \xrightarrow{\phi_3} B_4 \xrightarrow{\phi_4} \cdots$$

is a sequence of circle algebras with B as inductive limit. Since $K_1(B) = 0$ and $K_1(B_i)$ is singly generated, we can assume that $\phi_{i*} = 0$ on K_1 for all i. By the result of Elliott, stated as Theorem 1.5 above, and using Theorem 2.1 of [4], we can furthermore assume that each ϕ_i is maximally homogeneous. But then an application of Lemma 2.4 to each ϕ_{i*} gives that B is the inductive limit of a sequence of interval algebras with unital connecting $*$-homomorphisms. In fact, by construction the connecting $*$-homomorphisms can be choosen to be standard in the sense of [8], so the statement regarding the crossed product decomposition of B follows from Theorem 2.3 of [8]. ■

By [4] the only C^*-algebras of real rank zero to which Theorem 2.5 applies are the UHF algebras. It is therefore appropiate to show that the result has less trivial applications, also to simple C^*-algebras.

EXAMPLE 2.6 In this example ,we exhibit some simple C^*-algebras to which Theorem 2.5 applies, but which are not subsumed under the classification result of Elliott [4]. Let $f_n : [0,1] \to [0,1]$, $n = 1, 2, 3, \ldots$, be an arbitrary sequence of continuous maps, let $c_1, c_2, c_3, \ldots$ be a sequence in $\mathbb{N}$ such that $\sum_{n=1}^{\infty} c_n^{-1} < \infty$ and let $q_1, q_2, q_3, \ldots$ be a dense sequence in $[0,1]$. Define $m_n \in \mathbb{N}$ inductively by $m_1 = 1, m_{n+1} = n(c_n +$

$+2)m_n$, $n \geqslant 1$. We can then define a $*$-homomorphism $\phi_n : C[0,1] \otimes M_{m_n} \to C[0,1] \otimes$ $\otimes M_{m_{n+1}}$, for each $n \geqslant 1$, as the standard homomorphism, in the sense of [8], Definition 1.11, given by the function f_n repeated nc_n times, the function $t \to t$ repeated n times and the constant function $t \to q_n$ repeated n times. Then the resulting inductive limit C^*-algebra A has the following properties:

(i) A is simple.

(ii) $K_0(A) = \mathbb{Q}$; the scale being $\mathbb{Q} \cap [0,1]$.

(iii) The space of extremal tracial states of A is homeomorphic, in the weak* topology, to the inverse limit space X of the sequence

$$[0,1] \xleftarrow{f_1} [0,1] \xleftarrow{f_2} [0,1] \xleftarrow{f_3} [0,1] \leftarrow \cdots .$$

(i) and (ii) are easily verified. To indicate how to prove (iii), let B be the inductive limit C^*-algebra of the sequence $\psi_n : C[0,1] \otimes M_{m_n} \to C[0,1] \otimes M_{m_{n+1}}$, where ψ_n is the standard homomorphism corresponding to the function f_n repeated $n(c_n+2)$ times. It is then easy to use the assumption $\sum\limits_{n=1}^{\infty} c_n^{-1} < \infty$ to prove that the tracial state spaces of A and B are homeomorphic. But $B \simeq C(X) \otimes D$, where D is the universal UHF-algebra, so it is clear that the space of extremal tracial states on B is as described under (iii).

It follows that the above construction gives us at least as many mutually non-isomorphic simple C^*-algebras with the same K-theory as there are homeomorphism classes of inverse limit spaces of intervals. By [2] A is of real rank zero if and only if X is a point, in which case it is the universal UHF-algebra, [4].

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