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SUBFACTOR OF THE HYPERFINITE II_1 FACTOR WITH COXETER GRAPH E_6 AS INVARIANT

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INTRODUCTION

An important problem in the theory of von Neumann algebras is the classification of the subfactors of a type II_1 factor.

A crucial step was made by V. Jones [2] by the introduction of the notion of index $[M.N]$ of a subfactor N of a type II_1 factor M. Subsequently he defined a finer invariant, namely the derived tower $M_i \cap N' \subset M_{i+1} \cap N'$ of $N \subset M$, where $N \subset M = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_i \subset \ldots$ is the Jones' tower. He proved then that in case $[M : N] < 4$, the graph canonically associated to this derived tower is a Coxeter graph of type A_n, D_n or E_6, E_7, E_8 whose norm is $[M : N]^{\frac{1}{2}}$ ([1], paragraph 4).

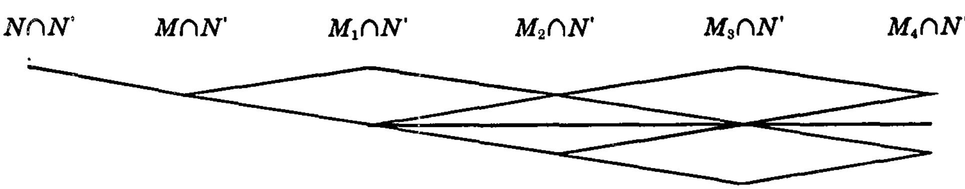
The A_n are realized by the von Neumann algebra M generated by the $(e_i)_{i \geqslant 1}$ and the subfactor N generated by the $(e_i)_{i \geqslant 2}$ ([2], paragraph 4). The question of the realization of the others was left open. A. Ocneanu has announced at conferences in 1987 and 1988 that the graphs which can be realized are exactly the A_n, D_{2n} and E_6 and E_8; but he has not yet presented the details of his proof of his result.

The purpose of this paper is to present an explicit calculation of an example of a subfactor of index < 4 with the (Jones) Coxeter invariant E_6. This not only proves that E_6 is realized as the principal graph associated to a pair of hyperfinite subfactors with index < 4, but also shows how to calculate the derived tower or principal graph in a concrete case. Here the von Neumann algebras N and M ($N \subset \subset M$), are obtained in a natural way from finite dimensional von Neumann algebras whose inclusion diagram is already E_6. The method of the construction and of the computation of the derived tower can also be applied to other Coxeter graphs to decide whether they are realized, but the computation might become lenghty. I am presently working on the generalization.

In the same time, Shingo Okamoto has also computed the derived tower for other examples of Jones' subfactors but in the case of index > 4. His algorithm of computation cannot be applied to the case of index < 4.

After recalling some results, in the first part of this paper we describe a general construction of a factor M and a subfactor N starting with any given Coxeter graph of type A, D or E. Then we prove that we can use unitaries of the Hecke algebra generated by the e_i to compute in general the tower of relative commutants: $M \cap N' = \mathbb{C} \subset M_1 \cap N' \subset M_2 \cap N' \subset \ldots$.

In the second part of this paper we complete the computation in the case of the Coxeter graph E_6. We prove that in that case $M_1 \cap N' \simeq \mathbb{C} \oplus \mathbb{C}$, $M_2 \cap N' \simeq M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$; so the derived tower is:



Thus the principal graph associated to $N \subset M$ is the graph E_6.

I would like to thank V. Jones for encouragement and fruitful discussions.
I also thank the referee for usefull comments.

0. PRELIMINARIES

We will recall first some definitions and results.

1. The basic construction of V. Jones ([2], paragraph 3).

Let $N \subset M$ be factors of type II_1. Let tr be the normalized trace on M. We denote by $e_1 \in \mathcal{L}(L^2(M, \mathrm{tr}))$ the projection associated to the conditional expectation from M to N and by $M_1 = \langle M, e_1 \rangle$ the factor generated by M and e_1. This is the basic construction defined by Jones. Now this basic construction can be iterated to obtain the Jones' tower $N \subset M \subset M_1 \subset M_2 = \langle M_1, e_2 \rangle \subset M_3 = \langle M_2, e_3 \rangle \subset \ldots$.

The projections e_i satisfy the relations: $e_i e_{i+1} e_i = \tau e_i$, $e_i e_{i-1} e_i = \tau e_i$, $e_i e_j = e_j e_i$ if $|i-j| \geqslant 2$ with $\dfrac{1}{\tau} = [M : N]$.

V. Jones proved also that the same results are true when one starts with M and N finite sums of finite dimensional factors, provided one chooses on M and N the Markov trace.

2. The derived tower ([1] chapter 4 paragraph 6).

Let $N \subset M$ be factors of type II_1. The derived tower $\partial M/\partial N$ defined in [1] is the chain of algebras: $\mathbb{C} = N' \cap N \subset N' \cap M \subset \cdots \subset N' \cap M_k \subset \cdots$.

The derived tower has the following properties:

— the inclusion $N' \cap M_k \subset N' \cap M_{k+1}$ is connected.

— if we note Λ_k the inclusion matrix for $N' \cap M_k \subset N' \cap M_{k+1}$ then the matrix Λ_{k+1} has the form $(\Lambda_k^t\ A)$ where A has no column of zeros.

The principal graph Γ of $N \subset M$ is defined as follows ([3] paragraph 2). Let $k \geqslant 0$ denote by p_{k+1} the central support of e_k in $N' \cap M_{k+1}$. Then $N' \cap M_{k+1}$ consists of the "old" part $p_{k+1}(N' \cap M_{k+1})$ which is isomorphic to the canonical extension corresponding to $N' \cap M_{k-1} \subset N' \cap M_k$ and a "new" part $(1-p_{k+1})(N' \cap \cap M_{k+1})$. The subgraph corresponding to the new parts:

$$N' \cap N = \mathbb{C} \to N' \cap M \to (1-p_1)(N' \cap M_1) \to (1-p_2)(N' \cap M_2) \to \ldots$$

is called the principal graph of the inclusion $N \subset M$.

The principal graph Γ is connected, has a distinguished vertex * the unique vertex on floor 0 and the pair $N \subset M$ has finite depth iff Γ is finite. Moreover if $[M : N] < 4$, then the principal graph is a Coxeter graph of type A_n, D_n or E_6, E_7, E_8.

3. Commuting square [6] and [7]

The notion of commuting square is due to Popa.

A diagram

$$\begin{array}{ccc} C_1 & \subset & B_1 \\ \cup & & \cup \\ C_0 & \subset & B_0 \end{array}$$

of finite von Neumann algebras with a finite faithful normal trace tr on B_1 is a commuting square if the diagram

$$\begin{array}{ccc} C_1 & \xleftarrow{E_{C_1}} & B_1 \\ \cup & & \cup \\ C_0 & \xleftarrow{E_{C_0}} & B_0 \end{array}$$

commutes.

1. CONSTRUCTION OF THE FACTORS N AND M

1. Construction of $N \subset M$

Consider a connected pair $A \subset B$ of finite dimensional von Neumann algebras with inclusion matrix Λ. Let $\beta = \|\Lambda\|^2$ and assume that $\beta < 4$. Let $B_1 = \langle B, e_1 \rangle$ be the basic construction obtained from $A \subset B$.

Write $\beta = 2 + t + \frac{1}{t}$ and define $u_1 = te_1 - (1 - e_1)$.

LEMMA 1.1 :

(i) u_1 *is a unitary,* $u_1 \in \langle B, e_1 \rangle \cap A'$.

(ii) *the square*

$$\begin{array}{ccc} u_1 B u_1^{-1} & \subset & \langle B, e_1 \rangle \\ \cup & & \cup \\ A & \subset & B \end{array}$$

is commuting.

Proof. i) $\beta < 4$ so from ([2] paragraph 4) $\beta = 4\cos^2\left(\frac{\pi}{n}\right)$ for some integer $n \geqslant 3$. Then $|t| = 1$ and u_1 is unitary.

(ii) Denote by E_B the conditional expectation: $\langle B, e_1 \rangle \to B$ and E_A the conditional expectation: $u_1 B u_1^{-1} \to A$.

$$\begin{aligned} \forall b \in B,\ E_B(u_1 b u_1^{-1}) &= E_B((t+1)e_1 - 1)b((t^{-1}+1)e_1 - 1)) = \\ &= E_B((t+1)(t^{-1}+1)e_1 b e_1 - (t^{-1}+1)be_1 - (t+1)e_1 b + b) = \\ &= E_B\left(\frac{1}{\tau}E_A(b)e_1 - (t^{-1}+1)be_1 - (t+1)e_1 b + b\right) = \\ &= E_A(b) - (t^{-1}+1)b\tau - (t+1)b\tau + b = E_A(b). \end{aligned}$$

But u_1 commutes with A so $E_A(u_1 b u_1^{-1}) = E_A(b)$.

This proves that the restriction of E_B to $u_1 B u_1^{-1}$ is equal to E_A. So the square is commuting.

Now we go on with the basic construction for the second column. $\begin{smallmatrix}\langle B, e_1\rangle \\ \cup \\ B\end{smallmatrix}$. Note $B_0 = B$ and $B_1 = \langle B, e_1 \rangle$.

For $j \geqslant 1$ we get $B_{j+1} = \langle B_j, e_{j+1} \rangle$ which we endow with the Markov extension of tr from B_j to B_{j+1}. Setting $C_0 = A$ and $C_1 = u_1 B u_1^{-1}$ for $j \geqslant 1$ we denote by C_{j+1} the subalgebra of B_{j+1} generated by C_j and e_{j+1}.

At each step we obtain a commuting square ([1] chapter 4, paragraph 2). As B_{j+1} is obtained by the basic construction (applied to $B_{j-1} \subset B_j$), the inclusion matrix of $B_j \subset B_{j+1}$ is alternately Λ^t and Λ ([2] paragraph 1.3). Furthermore by [1] paragraph 4.4, C_{j+1} is the result of the basic construction for the inclusion $C_{j-1} \subset C_j$, and the inclusion matrices for $C_j \subset C_{j+1}$ (and also those for $C_j \subset B_j$) are alternately Λ and Λ^t.

Now let $M = (UB_j)''$ be the von Neumann algebra generated by the B_j and $N = (UC_j)''$ the von Neumann algebra generated by the C_j.

This construction was first introduced by V. Jones.

Necessarily the trace vector $\vec{t_j}$ on B_{2j} (respectively $\vec{s_j}$ on B_{2j+1}) and on C_{2j+1} (respectively on C_{2j+2}) is a Perron Frobenius vector for the matrix $\Lambda^t\Lambda$ (respectively

$\Lambda\Lambda^t$). So there is a unique normalized trace on M and on N. It follows that M and N are factors of type II_1.

We now want to compute the derived tower $\partial M/\partial N$.

2. Method of computation of the derived tower.

To do this we proceed with the basic construction for the inclusion $N \subset M$. We note $f_1, \ldots, f_n$ the projections giving the successive basic constructions. We obtain the following diagram T

$$\begin{array}{ccccccccc}
C_{j+1} = \langle C_j, e_{j+1}\rangle & \subset & B_{j+1} = \langle B_j, e_{j+1}\rangle & \subset & \langle B_{j+1}, f_1\rangle & \subset \cdots \subset & \langle B_{j+1}, f_1, \ldots, f_n\rangle & \subset \\
\cup & & \cup & & \cup & & \cup & \\
\vdots & & \vdots & & \vdots & & \vdots & \\
\cup & & \cup & & \cup & & \cup & \\
C_2 = \langle C_1, e_2\rangle & \subset & B_2 = \langle B_1, e_2\rangle & \subset & \langle B_2, f_1\rangle & \subset \cdots \subset & \langle B_2, f_1, \ldots, f_n\rangle & \subset \\
\cup & & \cup & & \cup & & \cup & \\
C_1 = u_1 B u_1^{-1} & \subset & B_1 = \langle B, e_1\rangle & \subset & \langle B_1, f_1\rangle & \subset \cdots \subset & \langle B_1, f_1, \ldots, f_n\rangle & \subset \\
\cup & & \cup & & \cup & & \cup & \\
A & \subset & B & \subset & \langle B, f_1\rangle & \subset \cdots \subset & \langle B, f_1, \ldots, f_n\rangle & \subset
\end{array}$$

It is then easy to see that the restriction to each line j that is $C_j \subset B_j \subset \subset \langle B_j, f_1\rangle \subset \ldots \subset \langle B_j, f_1, \ldots, f_n\rangle \subset \ldots$ is at each step the basic construction ([9]). So the inclusion diagrams are alternately given by Λ and Λ^t ([2] paragraph 1.3.). And also it follows from ([1] proposition 4.1.2) that we always get commuting squares. The computation of the first terms of the derived tower is simple. Indeed we prove the following lemma:

LEMMA 1.2.
(i) $[M : N] = [B : A] = \beta$.
(ii) $M \cap N' = \mathbf{C}$ *and* $M_1 \cap N' \simeq \mathbf{C} \oplus \mathbf{C}$ *or* $M_1 \cap N' \simeq \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$

Proof. (i) From a result of Wenzl ([9] theorem 1.5), it follows that $[M : N] = = \|t_j\|^2/\|s_j\|^2 = \|\Lambda\|^2$. So $[M : N] = [B : A] = \beta$.
(ii) $N' \cap M = \mathbf{C}$ by [2] Corollary 2.2.4. Pimsner and Popa have proved ([5] Proposition 1.9) that the trace of each minimal projection in $M_1 \cap N'$ is greater or equal to τ. So if n is the maximal number of orthogonal minimal projections in $M_1 \cap N'$, $n\tau \leqslant 1$ i.e. $n \leqslant \frac{1}{\tau} = [M : N] < 4$. So $n = 2$ or 3 ($n \neq 1$ because $M \neq N$).
If $n = 2$, as f_1 is a central projection in $M_1 \cap N'$, necessarily $M_1 \cap N' = \mathbf{C}f_1 \oplus \mathbf{C}(1 - f_1)$.
If $n = 3$, the trace of each minimal projection is $< 2\tau$. Again from a result of Pimsner and Popa ([5] Proposition 1.9) it follows that each minimal projection is central. So in that case $M_1 \cap N' \simeq \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$. ∎

To compute the higher terms of the derived tower $\partial M/\partial N$ we will use the following lemma of Skau as it is formulated by Goodman, de la Harpe, Jones ([1] Theorem 4.4.3).

LEMMA OF SKAU. *Let C and D be finite direct sums of finite factors. Assume $[D:C]<4$. Let $C \subset D \subset \langle D,e_1\rangle \subset \ldots \subset \langle D,e_1,\ldots,e_n\rangle \subset \ldots$ be the iterated basic construction. Let D_∞ be the von Neumann algebra generated by D and the $(e_i)_{i\in\mathbb{N}^*}$ then*

$$D_\infty \cap \{e_1,e_2,\ldots\}' = C$$

COROLLARY 1.3. *Let N and M be the factors of type II_1 defined before.*

$$M_i \cap N' = \langle B,f_1,\ldots,f_i\rangle \cap (u_1Bu_1^{-1})'.$$

Proof. $M_i\cap N' = \langle B,f_1,\ldots,f_i,e_1,\ldots,e_n,\ldots\rangle''\cap\langle u_1Bu_1^{-1},e_2,\ldots,e_n\rangle'$. We apply the lemma of Skau with $D=\langle B,f_1,\ldots,f_i,e_1\rangle$ and $C=\langle B,f_1,\ldots,f_i\rangle$. The first basic construction is then given by e_2.
So $\langle B,f_1,\ldots,f_i,e_1,\ldots,e_n,\ldots\rangle''\cap\{e_2,\ldots,e_n,\ldots\}' = \langle B,f_1,\ldots,f_i\rangle$ and then

$$M_i \cap N' = \langle B,f_1,\ldots,f_i\rangle \cap (u_1Bu_1^{-1})'.$$

So now the problem is a problem of finite dimensional algebras. But the difficulty to make the computation is that we don't know a priori the relative positions of the algebras $u_1Bu_1^{-1}$ and $\langle B,f_1,\ldots,f_i\rangle$ inside $\langle B,e_1,f_1,\ldots,f_i\rangle$ because these algebras appear on different lines of the diagram T.
We will now prove that the computation of the derived tower can be made by use of algebras which are on the same line in the diagram T and unitaries which can be easily computed. ∎

DEFINITION. For all i define the unitary v_i by $v_i = tf_i-(1-f_i)$ (with $\beta = 2+t+\frac{1}{t} = [B:A]$).

LEMMA 1.4. *$f_1e_1f_1=\tau f_1$ and $e_1f_1e_1=\tau e_1$.*

Proof. Let E be the conditional expectation from $\langle B,e_1\rangle$ onto $u_1Bu_1^{-1}$. $\forall x\in B$, $\mathrm{tr}(u_1xu_1^{-1}e_1) = \mathrm{tr}(xu_1^{-1}e_1u_1) = \mathrm{tr}(xe_1)$ because e_1 commutes with u_1, $=\tau\mathrm{tr}(x)=\tau\mathrm{tr}(u_1xu_1^{-1})$. This proves that $E(e_1)=\tau$. $\forall y\in\langle B,e_1\rangle, f_1yf_1=E(y)f_1$ so $f_1e_1f_1=\tau f_1$.
Moreover $u_1Bu_1^{-1}\cap\{e_1\}' = u_1(B\cap\{e_1\}')u_1^{-1} = u_1Au_1^{-1} = A$ because $u_1e_1u_1^{-1}=e_1$. It follows that e_1 is the projection canonically associated to the conditional expectation: $u_1Bu_1^{-1}\to A$. So from a result of Jones ([2] proposition 3.4.1) we also have $e_1f_1e_1 = \tau e_1$. ∎

COROLLARY 1.5. *$e_1, e_2, \dots, e_n, \dots$ are the projections defining the tunnel associated to $N \subset M$ and we can complete the diagram T as follows.*

We denote $u_i = te_i - (1 - e_i)$.

$$\begin{array}{ccccccc}
A \subset u_3u_2(u_1Bu_1^{-1})u_2^{-1}u_3^{-1} & \subset & \langle u_2(u_1Bu_1^{-1})u_2^{-1}, e_3\rangle & \subset & \langle u_1Bu_1^{-1}, e_2, e_3\rangle & \subset & \langle B, e_1, e_2, e_3\rangle \\
\cup & & \cup & & \cup & & \cup \\
A & \subset & u_2(u_1Bu_1^{-1})u_2^{-1} & \subset & \langle u_1Bu_1^{-1}, e_2\rangle & \subset & \langle B, e_1, e_2\rangle \\
 & & \cup & & \cup & & \cup \\
 & & A & \subset & \langle u_1Bu_1^{-1}\rangle & \subset & \langle B, e_1\rangle \\
 & & & & \cup & & \cup \\
 & & & & A & & B
\end{array}$$

At each step we have basic construction, and commuting square.

LEMMA 1.6. *Let e and f be two projections such that $fef = \tau f$ and $efe = \tau e$. Write $\beta = \dfrac{1}{\tau} = 2 + t + t^{-1}$. Define $u = te - (1 - e)$ and $v = tf - (1 - f)$. Then $v^{-1}ev = ufu^{-1}$, and also $vev^{-1} = u^{-1}fu$.*

Proof.

$$\begin{aligned} v^{-1}ev &= (t^{-1}f - (1-f))e(tf - (1-f)) = ((t^{-1}+1)f - 1)e((t+1)f - 1) = \\ &= -(t^{-1}+1)fe - (t+1)ef + (t^{-1}+1)(t+1)\tau f + e. \end{aligned}$$

But $(t^{-1}+1)(t+1) = \beta = \dfrac{1}{\tau}$. So $v^{-1}ev = -(t^{-1}+1)fe - (t+1)ef + f + e$. A similar computation gives: $ufu^{-1} = -(t^{-1}+1)fe - (t+1)ef + f + e$. So $v^{-1}ev = ufu^{-1}$. We also have $vev^{-1} = u^{-1}fu$ (same proof). ∎

PROPOSITION 1.7.

$$\langle B, f_1, \dots, f_n\rangle = u_1^{-1}v_1^{-1}\cdots v_n^{-1}\langle B, e_1, f_1, \dots, f_{n-1}\rangle(u_1^{-1}v_1^{-1}\cdots v_n^{-1})^{-1}.$$

where $v_i = tf_i - (1 - f_i)$.

Proof.

(i) Note first that $\langle B, e_1, f_1, \dots, f_{n-1}\rangle = \langle u_1Bu_1^{-1}, e_1, f_1, \dots, f_{n-1}\rangle$. Now $u_1Bu_1^{-1}$ commutes with $v_1, v_2, \dots, v_n$; so

$$u_1^{-1}v_1^{-1}\cdots v_n^{-1}(u_1Bu_1^{-1})v_nv_{n-1}\cdots v_1u_1 = B$$

(ii) e_1 commutes with $v_2, \dots, v_n$ so

$$u_1^{-1}v_1^{-1}v_2^{-1}\cdots v_n^{-1}e_1v_nv_{n-1}\cdots v_1u_1 = u_1^{-1}v_1^{-1}e_1v_1u_1 = f_1$$

from Lemma 1.5 and Lemma 1.6.

(iii) for all i f_i commutes with $v_{i+2},\ldots,v_n$ so

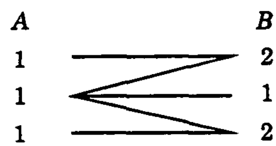
$$u_1^{-1}v_1^{-1}\cdots v_n^{-1}f_iv_n\cdots v_1u_1 = u_1^{-1}v_1^{-1}\cdots v_{i+1}^{-1}f_iv_{i+1}\cdots v_1u_1$$

f_i and f_{i+1} satisfy the hypothesis of Lemma 1.2. So: $v_i^{-1}v_{i+1}^{-1}f_iv_{i+1}v_i = f_{i+1}$ which commutes with $u_1, v_1, \ldots, v_{i-1}$.
We then get $u_1^{-1}v_1^{-1}\cdots v_n^{-1}f_iv_n\cdots v_1u_1 = f_{i+1}$. The proposition results from (i), (ii) and (iii). ■

COROLLARY 1.8. *$\langle B,f_1,\ldots,f_n\rangle\cap(u_1Bu_1^{-1})' = (\langle B,e_1,f_1,\ldots,f_n\rangle\cap(u_1Bu_1^{-1})'\cap$ $\cap U_n\langle B,e_1,f_1,\ldots,f_{n-1}\rangle U_n^{-1}$ with $U_n = u_1^{-1}v_1^{-1}\cdots v_n^{-1}$.*

2. REALIZATION OF THE COXETER GRAPH E_6

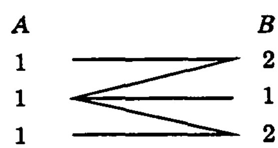
In all what follows $N \subset M$ are the II_1 factors constructed from the inclusion



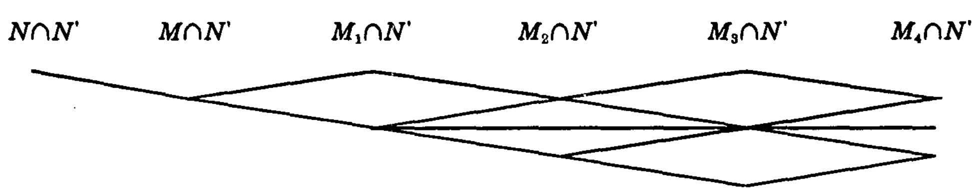
as in I.1.

We will prove the following theorem.

THEOREM 2.1. *Let $N \subset M$ be the II_1 factors constructed from the inclusion*



as in I. The derived tower associated to $N \subset M$ is



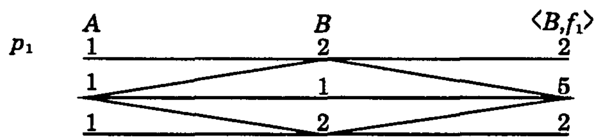
So the principal graph associated to $N \subset M$ is the Coxeter graph E_6.

LEMMA 2.2. *$M \cap N' = \mathbf{C}$ and $M_1 \cap N' = \mathbf{C}f_1 \oplus \mathbf{C}(1-f_1)$.*

Proof. We already know that $M \cap N' = \mathbf{C}$ (Lemma 1.2).
Let p be a non zero projection in $M_1 \cap N'$; $x \to \mathrm{tr}(px)$ is a trace on N; so $px \neq 0$ for all $x \neq 0$ in N.

From Corollary 1.3, $M_1 \cap N' = \langle B, f_1\rangle \cap (u_1 B u_1^{-1})' \subset \langle B, f_1\rangle \cap A'$. Denote by p_1 a minimal projection in A corresponding to a vertex of valence 1 in the Bratteli diagram.

Assume p, q, r are three non zero orthogonal projections in $M_1 \cap N'$.
Then $p_1 p$, $p_1 q$, $p_1 r$ are three non zero orthogonal projections in $\langle B, f_1\rangle_{p_1}$.
But from the diagram



we get $\langle B, f_1\rangle_{p_1} \simeq \mathbf{C} \oplus \mathbf{C}$; which gives a contradiction.
So there are strictly less than three non zero orthogonal projections in $M_1 \cap N'$.
This proves that $M_1 \cap N' = \mathbf{C}f_1 \oplus \mathbf{C}(1-f_1)$. ∎

SOME BASIC COMPUTATIONS

LEMMA 2.3. *Let $\vec{s}$ be the trace vector on A. Let $\vec{t}$ be the trace vector on B. Then $\vec{s} = \frac{1}{\tau+1}\begin{pmatrix}\tau \\ 1-\tau \\ \tau\end{pmatrix}$ and $\vec{t} = \frac{1}{\tau+1}\begin{pmatrix}\tau \\ \tau(1-\tau) \\ \tau\end{pmatrix}$ where $\tau = 2-\sqrt{3}$ and $[B:A] = \frac{1}{\tau} = 2+\sqrt{3}$. $t = \frac{\sqrt{3}+\mathrm{i}}{2}$ (where t is defined as in Lemma 1.6: $\frac{1}{\tau} = 2+t+$ $+t^{-1}$ and t is choosen such that $\mathrm{Im}\, t > 0$).*

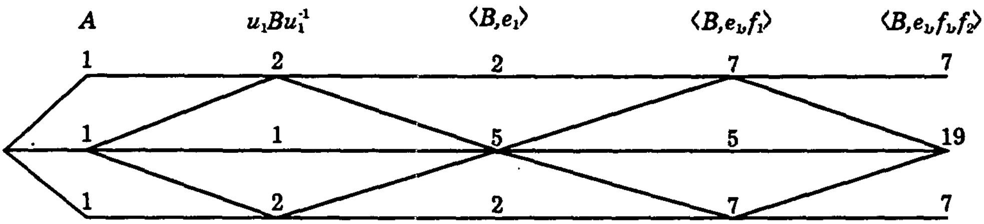
Proof. Let

$$\Lambda = \begin{pmatrix}1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1\end{pmatrix}$$

be the inclusion matrix of A in B.
V. Jones has proved ([2] paragraph 3.2) that $\vec{t}$ and $\vec{s}$ satisfy the conditions $\Lambda^t \Lambda \vec{t} = \frac{1}{\tau}\vec{t}$ and $\vec{s} = \Lambda \vec{t}$. So we get:

$$\begin{cases} t_1 = t_3 \\ t_2 = t_1(1-\tau) \\ \tau^2 - 4\tau + 1 = 0 \end{cases} \quad \text{and } \tau < 1 \text{ so } \tau = 2-\sqrt{3};$$

also $2t_1 + t_2 + 2t_3 = 1$ so $t_3 = t_1 = \dfrac{\tau}{\tau+1}, t_2 = \dfrac{\tau(1-\tau)}{\tau+1}$. The expression of $\vec{s}$ follows from $\vec{s} = \Lambda\vec{t}$. $\sqrt{3} = t + t^{-1} = 2\mathrm{Re}(t)$, so $t = \dfrac{\sqrt{3}+\mathrm{i}}{2}$ (because $|t| = 1$ and $\mathrm{Im}\, t > 0$). Now consider the Bratteli diagrams of the following inclusions: (D)



We make a choice of basis in these algebras well adapted to the successive inclusions (the elements of the basis are f_{ij} where i and j are paths from $*$ to the same point v at the level of the algebra). ■

We want to compute the matrices of e_1, f_1 and f_2.

LEMMA 2.4.
(i) *the matrices of* e_1 *in* $\langle B, e_1\rangle$ *are*

$$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & p & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \text{ with } p = \begin{pmatrix} \frac{\tau}{1-\tau} & \frac{\tau}{\sqrt{1-\tau}} & \frac{\tau}{1-\tau} \\ \frac{\tau}{\sqrt{1-\tau}} & \tau & \frac{\tau}{\sqrt{1-\tau}} \\ \frac{\tau}{1-\tau} & \frac{\tau}{\sqrt{1-\tau}} & \frac{\tau}{1-\tau} \end{pmatrix}$$

(ii) *the matrices of* f_1 *in* $\langle B, e_1, f_1\rangle$ *are:*

$$\begin{pmatrix} q & \\ & 0_3 \end{pmatrix} \begin{pmatrix} 0_2 & & \\ & 1 & \\ & & 0_2 \end{pmatrix} \begin{pmatrix} 0_3 & \\ & q' \end{pmatrix} \text{ with}$$

$$q = \begin{pmatrix} \tau & 0 & \sqrt{\tau(1-\tau)} & 0 \\ 0 & \tau & 0 & \sqrt{\tau(1-\tau)} \\ \sqrt{\tau(1-\tau)} & 0 & 1-\tau & 0 \\ 0 & \sqrt{\tau(1-\tau)} & 0 & 1-\tau \end{pmatrix} \text{ and } q_{ij} = q_{5-i,5-j}$$

(iii) *the matrices of* f_2 *in* $\langle B, e_1, f_1, f_2\rangle$ *are:*

$$\begin{pmatrix} I_2 & \\ & 0_5 \end{pmatrix} \begin{pmatrix} 0_2 & & & & \\ & p_{11}I_5 & p_{12}I_5 & p_{13}I_5 & \\ & p_{21}I_5 & p_{22}I_5 & p_{23}I_5 & \\ & p_{31}I_5 & p_{32}I_5 & p_{33}I_5 & \\ & & & & 0_2 \end{pmatrix} \begin{pmatrix} 0_5 & \\ & I_2 \end{pmatrix}$$

where $p = (p_{ij})_{1 \leqslant i,j \leqslant 3}$ *is the matrix of* (i).

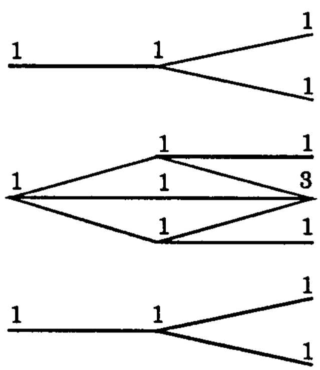
Proof. Sunder [8] and Ocneanu [3] have found a formula giving the projection of the basic construction for $A \subset u_1 B u_1^{-1} \subset \langle B, e_1 \rangle$

$$e_1 = \sum \frac{t(w)^{1/2} t(w')^{1/2}}{s(v)} f_{i,i'},$$

where i (resp. i') is a path in the Bratteli diagram of $A \subset B \subset \langle B, e_1 \rangle$ with vertices v, w, v (resp. v, w', v).

(i) From the diagram (D) we get

$\langle B,e_1 \rangle \cap A' =$



Applying the above formula of e_1 we get the following expression for e_1

$$\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \\ \begin{pmatrix} \frac{\tau}{1-\tau} & \frac{\tau}{\sqrt{1-\tau}} & \frac{\tau}{1-\tau} \\ \frac{\tau}{\sqrt{1-\tau}} & \tau & \frac{\tau}{\sqrt{1-\tau}} \\ \frac{\tau}{1-\tau} & \frac{\tau}{\sqrt{1-\tau}} & \frac{\tau}{1-\tau} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix} = p \text{ in } \langle B, e_1 \rangle \cap A'.$$

From (D) it follows also that an element $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ of A is written

$$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$$

$$\begin{pmatrix}\alpha &&&& \\ & \beta && 0 & \\ && \beta && \\ & 0 && \beta & \\ &&&& \gamma\end{pmatrix}$$

$$\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}$$

in $\langle B, e_1\rangle$.

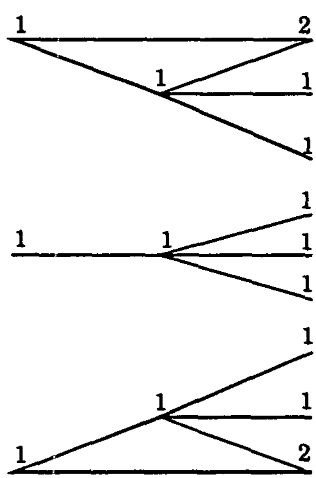
Now e_1 belongs to the commutant of A, so we get the announced expression for e_1 in $\langle B, e_1\rangle$.

(ii) The same formula applied to f_1 for the basic construction $u_1Bu_1^{-1} \subset \langle B, e_1\rangle \subset$ $\subset \langle B, e_1, f_1\rangle$ gives

$$f_1 = \sum \frac{s_1(v)^{1/2}s_1(v')^{1/2}}{t(w)} f_{j,j'},$$

where j (respectively j') is a path in the Bratteli diagram of $u_1Bu_1^{-1} \subset \langle B, e_1\rangle \subset$ $\subset \langle B, e_1, f_1\rangle$ with vertices wvw (respectively $wv'w$), and $\vec{s}_1$ is the trace vector on $\langle B, e_1\rangle$: $\vec{s}_1 = \tau\vec{s}$ ([1] paragraph 3.2).

From (D) again we get the following expression for $\langle B, e_1, f_1\rangle \cap (u_1Bu_1^{-1})'$:



And then the expression of f_1 in $\langle B, e_1, f_1\rangle \cap (u_1 B u_1^{-1})'$ is

$$\begin{array}{c} \begin{pmatrix} \tau & \sqrt{\tau(1-\tau)} \\ \sqrt{\tau(1-\tau)} & 1-\tau \end{pmatrix} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \begin{pmatrix} 1-\tau & \sqrt{\tau(1-\tau)} \\ \sqrt{\tau(1-\tau)} & \tau \end{pmatrix} \end{array}$$

Using (D) again we get the announced expression for f_1 in $\langle B, e_1, f_1\rangle$.

(iii) Note that the expression of f_2 in $\langle B, e_1, f_1, f_2\rangle \cap \langle B, e_1\rangle'$ is the same as the expression of e_1 in $\langle B, e_1\rangle \cap A'$.

Using again (D) we obtain the expression of f_2 in $\langle B, e_1, f_1, f_2\rangle$. ■

COROLLARY 2.5.
(i) *the matrices of* e_1 *in* $\langle B, e_1, f_1, f_2\rangle$ *are:*

$$\begin{pmatrix} 1 &&&& \\ & 0 &&& \\ && 0 && \\ &&& p & \\ &&&& 0 \end{pmatrix} \begin{pmatrix} 1 &&&&&&&&&&&& \\ & 0 &&&&&&&&&&& \\ && 0 &&&&&&&&&& \\ &&& p &&&&&&&&& \\ &&&& 0 &&&&&&&& \\ &&&&& 0 &&&&&&& \\ &&&&&& p &&&&&& \\ &&&&&&& 0 &&&&& \\ &&&&&&&& 0 &&&& \\ &&&&&&&&& p &&& \\ &&&&&&&&&& 0 && \\ &&&&&&&&&&& 0 & \\ &&&&&&&&&&&& 1 \end{pmatrix} \begin{pmatrix} 0 &&&& \\ & p &&& \\ && 0 && \\ &&& 0 & \\ &&&& 1 \end{pmatrix}$$

(ii) *the matrices of* f_1 *in* $\langle B, e_1, f_1, f_2\rangle$ *are:*

$$\begin{pmatrix} q & \\ & 0_3 \end{pmatrix} \begin{pmatrix} q &&&& \\ & 0_5 &&& \\ && 1 && \\ &&& 0_5 & \\ &&&& q' \end{pmatrix} \begin{pmatrix} 0_3 & \\ & q' \end{pmatrix}$$

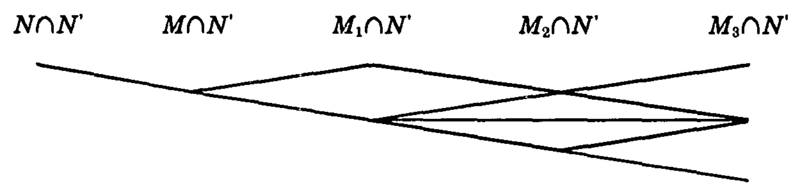
with p, q *and* q' *as in Lemma 2.4.*

This is immediate from Lemma 2.4 and (D).

COMPUTATION OF $M_2 \cap N'$

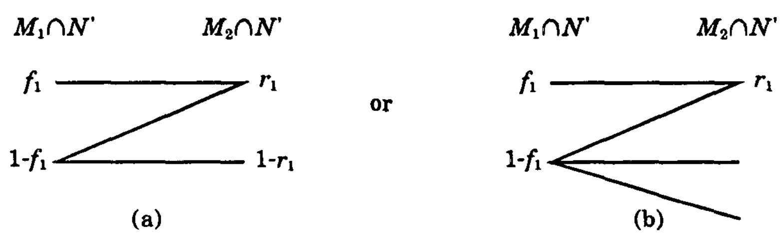
PROPOSITION 2.6. *Let r_1 be the minimal central projection in $M_2 \cap N'$ such that $f_1 r_1 = f_1$ (r_1 is the central support of f_1 in $M_2 \cap N'$).*

Assume that there is a projection P in $M_2 \cap N'$ such that $P \neq 1 - r_1$ and P is orthogonal to f_1 and f_2. Then necessarily the derived tower associated to $N \subset M$ is:



So the principal graph associated to $N \subset M$ is E_6. 

Proof.: Goodman, de la Harpe and Jones have proved ([1] Corollary 4.6.6) that the principal graph of $N \subset M$ is a Coxeter graph A D or E with norm $[M : N]^{1/2}$. So $M_2 \cap N'$ and the inclusion $M_1 \cap N' \subset M_2 \cap N'$ are necessarily



In case (a) the only projection in $M_2 \cap N'$ orthogonal to f_1 and f_2 is $1 - r_1$. So the existence of P satisfying the hypothesis of the lemma implies that $M_2 \cap N' \simeq$
$\simeq M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$.
Now the index $[M : N] = [B : A] = 2 + \sqrt{3}$ (Lemma 1.2 and Lemma 2.3).
And the only Coxeter graphs with norm $(2 + \sqrt{3})^{1/2}$, i.e. with Coxeter number 12 are E_6, D_7 and A_{11} (cf. table 1.4.5 in [1]).
Among those graphs the only graph starting by  is E_6. This proves that the principal graph associated to $N \subset M$ is E_6, and the derived tower is as announced.■

Now we will prove the existence of such a projection P.
We note U the unitary $U = v_2 v_1 u_1$.
For each element X in $\langle B, e_1, f_1, f_2 \rangle$ we note $X^{(1)}, X^{(2)}, X^{(3)}$ its matrices in the three

components of $\langle B, e_1, f_1, f_2\rangle$.
Now we define the projection P.

DEFINITION 2.7.

$$\text{Let } X = \begin{pmatrix} & & & & 0 & & \\ & 0_4 & & & 0 & & \\ & & & & 0 & & \\ & & & & 0 & & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & 0 & 0_2 & \\ & & & & 0 & & \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0_2 & & 0 & & & & \\ & & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \\ & & 0 & & 0_4 & & \\ & & 0 & & & & \\ & & 0 & & & & \end{pmatrix}.$$

Let

$$M^{(1)} = XU^{(1)^*} \; M^{(3)} = YU^{(3)^*} \; M^{(2)} = \begin{pmatrix} XU^{(1)^*} & & & & & \\ & 1 & & & & \\ & & 0_3 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & YU^{(3)^*} \end{pmatrix}.$$

Define $P = (MU)^* MU$.

LEMMA 2.8.
(i) *P is a projection in $\langle B, e_1, f_1, f_2\rangle$ $P^{(1)} = X$ and $P^{(3)} = Y$* .
(ii) *P belongs to $\langle B, f_1, f_2\rangle$*
(iii) *P is orthogonal to f_1 and f_2.*
(iv) *$p \neq 1 - r_1$*

Proof. (i)

$$P^{(1)} = (M^{(1)}U^{(1)})^* M^{(1)}U^{(1)} = X^* X = X$$
$$P^{(3)} = (M^{(3)}U^{(3)})^* M^{(3)}U^{(3)} = Y$$

$X^2 = X = X^*, Y^2 = Y = Y^*$. We note $B = \begin{pmatrix} 1 & & \\ & 0_3 & \\ & & 1 \end{pmatrix}$

then $P^{(2)} + U^{(2)^*} \begin{pmatrix} U^{(1)}XU^{(1)^*} & & \\ & B & \\ & & U^{(3)}YU^{(3)^*} \end{pmatrix} U^{(2)}$ and $B^2 = B = B^*$

So $P^2 = P = P^*$ i.e. P is a projection in $\langle B, e_1, f_1, f_2\rangle$.

(ii) By definition $M \in \langle B, e_1, f_1\rangle$ so $P = U^*M^*MU \in U^*\langle B, e_1, f_1\rangle U = \langle B, f_1, f_2\rangle$ (Proposition 1.7.)

From the expressions of X, Y and f_1 (Corollary 2.5.ii) and f_2 (Lemma 2.4.iii) we obviously get : $Xf_1^{(1)} = f_1^{(1)}X = Xf_2^{(1)} = f_2^{(1)}X = 0$ and $Yf_1^{(3)} = f_1^{(3)}Y = Yf_2^{(3)} = f_2^{(3)}Y = 0$. So to prove that P is orthogonal to f_1 and f_2 from (i) it only remains to prove that $P^{(2)}f_1^{(2)} = P^{(2)}f_2^{(2)} = 0$. But $P = (MU)^*MU$ and $f_1 = U^*e_1U, f_2 = U^*f_1U$. So it is enough to prove that $M^{(2)}e_1^{(2)} = M^{(2)}f_1^{(2)} = 0$. Using the expressions of e_1 and f_1 given in Corollary 2.5, we obtain:

$$M^{(2)}e_1^{(2)} = \begin{pmatrix} XU^{(1)^*}e_1^{(1)} & & \\ & \begin{pmatrix} 1 & & \\ & 0_3 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 0 & & \\ & p & \\ & & 0 \end{pmatrix} & \\ & & YU^{(3)^*}e_1^{(3)} \end{pmatrix}$$

But $XU^{(1)^*}e_1^{(1)} = Xf_1^{(1)}U^{(1)^*} = 0$ and $YU^{(3)}e_1^{(3)} = Yf_1^{(3)}U^{(3)^*} = 0$. So $M^{(2)}e_1^{(2)} = 0$.

$$M^{(2)}f_1^{(2)} = \begin{pmatrix} XU^{(1)^*}f_1^{(1)} & & \\ & \begin{pmatrix} 1 & & \\ & 0_3 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 0_2 & & \\ & 1 & \\ & & 0_2 \end{pmatrix} & \\ & & YU^{(3)^*}f_1^{(3)} \end{pmatrix}$$

And $XU^{(1)^*}f_1^{(1)} = Xf_2^{(1)}U^{(1)^*} = 0$ and $YU^{(3)^*}f_1^{(3)} = Yf_2^{(3)}U^{(3)^*} = 0$. So $M^{(2)}f_1^{(2)} = 0$. We have then proved that P is a projection orthogonal to f_1 and f_2.

(iv) It is easy to compute $\operatorname{tr}(P)$. The trace vector on $\langle B, e_1, f_1, f_2\rangle$ is equal to $\tau^2\vec{s} = \frac{\tau^2}{1+\tau}\begin{pmatrix} \tau \\ 1-\tau \\ \tau \end{pmatrix}$ (this is immediate from the results of V. Jones ([1] paragraph 3.2) applied to the diagram D).

Now from definition of P we get $\operatorname{tr}(P) = \frac{\tau^2}{1+\tau}(\tau + 4(1-\tau) + \tau) = \frac{\tau^2(4-2\tau)}{1+\tau}$ but $\tau(4-2\tau) = 4\tau - 2(4\tau - 1) = 2 - 4\tau$.

$\frac{2-4\tau}{1+\tau} = \frac{4\sqrt{3}-6}{3-\sqrt{3}} = \frac{(4\sqrt{3}-6)(3+\sqrt{3})}{6} = \sqrt{3} - 1 = 1 - \tau$. So $\operatorname{tr}(P) = \tau(1-\tau)$.

And $\operatorname{tr}(1 - r_1) = 1 - 2\tau = \tau + \operatorname{tr}(P)$. So $P \neq 1 - r_1$. ∎

Now it remains to prove that P commutes with $u_1 B u_1^{-1}$.

For this we will compute $M^{(2)}U^{(2)}$ and then $P^{(2)} = (M^{(2)}U^{(2)})^*(M^{(2)}U^{(2)})$.

LEMMA 2.9.

$$XU^{(1)^*} = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & & & & & & \\ 2 & & & & & & & \\ 3 & & & & & & & \\ 4 & & & & & & & \\ 5 & 0 & \bar{t}^2\sqrt{\tau} & 0 & -\frac{\tau\bar{t}^2}{\sqrt{1-\tau}} & (\bar{t}+1)\tau-1 & (\bar{t}+1)p_{23} & 0 \\ 6 & & & & & & & \\ 7 & & & & 0 & & & \end{array}$$

$YU^{(3)^*}_{i,j} = XU^{(1)^*}_{8-i,8-j}$.

Proof. The second relation is immediate from the expressions of X, Y and e_1, f_1, f_2. So we just have to compute $XU^{(1)^*} = X(u_1^* v_1^* v_2^*)^{(1)}$.
In X the lines of index 1, 2, 3, 4, 6, 7 are zero. So in $Xu_1^{(1)^*}, X(u_1^* v_1^*)^{(1)}, X(u_1^* v_1^* v_2^*)^{(1)}$ we just have to compute the line of index 5.

$$u_1^* = (\bar{t}+1)e_1 - 1, \quad v_1^* = (\bar{t}+1)f_1 - 1, \quad v_2^* = (\bar{t}+1)f_2 - 1.$$

So using the expressions of e_1, f_1 as in Corollary 2.5 and of f_2 as in Lemma 2.4 we obtain: (we note $p = (p_{ij})_{1 \leqslant i,j \leqslant 3}$)

The line of index 5 in $Xu_1^{(1)^*}$ is: $0, 0, 0, (\bar{t}+1)p_{21}, (\bar{t}+1)p_{22} - 1, (\bar{t}+1)p_{23}, 0$

The line of index 5 in $X(u_1^* v_1^*)^{(1)}$ is: $0, (\bar{t}+1)p_{21}\sqrt{\tau(1-\tau)}, 0, (\bar{t}+1)p_{21}[(\bar{t}+ +1)(1-\tau)-1], -(\bar{t}+1)p_{22} - 1, -(\bar{t}+1)p_{23}, 0$.

The line of index 5 in $X(u_1^* v_1^* v_2^*)^{(1)}$ is: $0, \bar{t}(t+1)^2 p_{21}\sqrt{\tau(1-\tau)}, 0, -(\bar{t}+1)p_{21}[(\bar{t}+ +1)(1-\tau)-1], (\bar{t}+1)p_{22} - 1, (\bar{t}+1)p_{23}, 0$.

Now, we replace the p_{ij} by their values, we use the relation $\tau = \frac{\bar{t}}{(1+\bar{t})^2}$, and we obtain the announced matrix for $XU^{(1)^*}$. ∎

COROLLARY 2.10. *The only lines not equal to zero in $M^{(2)}$, $(Mv_2)^{(2)}$, $(Mv_2v_1)^{(2)}$, $(Mv_2v_1u_1)^{(2)}$ are the lines of index 5,8,12,15. Moreover all these matrices satisfy the relation: $(A_{i,j}) = A_{20-i,20-j}$. So we just have to compute the lines 5 and 8.*

Proof. This is immediate from the preceding lemma, the expression of

$$M^{(2)} = \begin{pmatrix} XU^{(1)^*} & & \\ & \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix} & \\ & & YU^{(3)^*} \end{pmatrix}$$

and the expressions of u_1, v_1, v_2.

LEMMA 2.11. (i) *the line of index* 5 *in* $(Mv_2)^{(2)}$ *is equal to:*

$$0, -\bar{t}^2\sqrt{\tau}, 0, \frac{\tau^2\bar{t}^2}{t(1-\tau)\sqrt{1-\tau}}, \frac{-[(\bar{t}+1)p_{22}-1]\tau}{t(1-\tau)}, \frac{-(\bar{t}+1)p_{23}\tau}{t(1-\tau)}, 0, 0,$$
$$\frac{-\tau\bar{t}^2 p_{12}(\bar{t}+1)}{\sqrt{1-\tau}}, [(\bar{t}+1)-1]p_{12}(t+1), (\bar{t}+1)p_{23}(t+1)p_{12}, 0, 0, \frac{-\tau\bar{t}^2}{\sqrt{1-\tau}}(t+1)P_{13},$$
$$[(\bar{t}+1)\tau-1](t+1)p_{13}, (\bar{t}+1)p_{23}(t+1)p_{13}, 0, 0, 0.$$

(ii) *the line of index* 8 *in* $(Mv_2)^{(2)}$ *is equal to:*

$$0, 0, (t+1)p_{12}, 0, 0, 0, 0, (t+1)p_{22}-1, 0, 0, 0, 0, (t+1)p_{23}, 0, 0, 0, 0, 0, 0.$$

Proof. The product $(Mv_2)^{(2)}$ is very easy to compute because each term non zero in the product is only the product of one term in $M^{(2)}$ and one term in $v_2^{(2)}$ (and never a sum of product).

(i) The only relation used here is the equality $(t+1)p_{11} - 1 = \dfrac{-\tau}{t(1-\tau)}$ (which is used to compute the terms of place 5.4, 5.5, and 5.6.

(ii) trivial computation. ■

LEMMA 2.12. (i) *the line of index* 5 *in* $(Mv_2v_1)^{(2)}$ *is equal to:*

$$0, \frac{\bar{t}^2\sqrt{\tau}}{(1-\tau)(1+t)}, 0, (\bar{t}+1)p_{21}\bar{t}\left(\frac{2\tau-1}{1-\tau}\right), \frac{[(\bar{t}+1)p_{22}-1]\tau}{t(1-\tau)}, \frac{(\bar{t}+1)p_{23}\tau}{t(1-\tau)}, 0, 0,$$
$$\frac{\tau\bar{t}^2 p_{12}(t+1)}{\sqrt{1-\tau}}, [(\bar{t}+1)\tau-1]p_{12}(t+1)t, -(\bar{t}+1)p_{23}(t+1)p_{12}, 0, 0, \frac{\tau\bar{t}^2(1+t)p_{13}}{\sqrt{1-\tau}}, tp_{13},$$
$$t(t+1)p_{13}p_{23}, 0, (t+1)P_{13}\sqrt{\tau}, 0.$$

(ii) *the line of index* 8 *in* $(Mv_2v_1)^{(2)}$ *is equal to:*

$$(t+1)^2p_{12}\sqrt{\tau(1-\tau)}, 0, (t+1)p_{12}[(t+1)(1-\tau)-1], 0, 0, 0, 0, -[(t+1)p_{22}-1], 0, 0, 0, 0,$$
$$-(t+1)p_{23}, 0, 0, 0, 0, 0, 0.$$

Proof. The computation here is also easy because the only terms in $(Mv_2v_1)^{(2)}$ which are sums of two products are the terms of place 5,2 and 5,4; all the others are either trivially equal to zero or just equal to the product of one term in $(Mv_2)^{(2)}$ and one term in $v_1^{(2)}$. To obtain the announced expression we have just used the following relations: $\tau = \dfrac{t}{(t+1)^2}$; $(t+1)\tau - 1 = \dfrac{t}{1+t} - 1 = \dfrac{-1}{1+t}$; $(t+1)(1-\tau) - 1 = \dfrac{t^2}{1+t} = \dfrac{t}{1+\bar{t}} = \tau t(t+1)$ and $|t| = 1$. ■

We note $Z = (Mv_2v_1u_1)^{(2)}$.

LEMMA 2.13. *The terms $Z_{5,5}, Z_{5,10}$ and $Z_{5,15}$ are equal to zero.*

Proof.

(i) $$Z_{5,5} = (t+1)p_{21}(\bar{t}+1)p_{21}\bar{t}\left(\frac{2\tau-1}{1-\tau}\right) + \frac{[(\bar{t}+1)p_{22}-1][(t+1)p_{22}-1]\tau}{t(1-\tau)} + \\ + \frac{(\bar{t}+1)p_{23}(t+1)p_{23}\tau}{t(1-\tau)}.$$

$(t+1)p - I_3$ is a unitary, so $(t+1)p_{21}(\bar{t}+1)p_{21} + [(\bar{t}+1)p_{22}-1][(t+1)p_{22}-1] + (\bar{t}+ +1)P_{23}(t+1)P_{23} = 1$.
We then get $Z_{5,5} = \dfrac{\tau}{t(1-\tau)} + \dfrac{|t+1|^2p_{21}^2}{t(1-\tau)}(2\tau-1-\tau) = 0$, using $|t+1|^2 = \dfrac{1}{\tau}$.

(ii) $$Z_{5,10} = p_{12}(t+1)\left[\frac{\tau\bar{t}}{\sqrt{1-\tau}}\frac{(\bar{t}+1)\tau}{\sqrt{1-\tau}} + \frac{t}{(1+\bar{t})(1+t)} - \frac{|t+1|^2\tau^2}{1-\tau}\right] =$$
$$= \frac{p_{12}(t+1)\tau}{1-\tau}[\tau\bar{t} + \tau\bar{t}^2 + t(1-\tau) - 1] = \frac{p_{12}(t+1)\tau}{1-\tau}[\tau(\bar{t}^2+\bar{t}-t) + t - 1].$$

From Lemma 2.3 we know that $t = \dfrac{\sqrt{3}+\mathrm{i}}{2}$.

$$\tau(\bar{t}^2+\bar{t}-t) = (2-\sqrt{3})\left(\frac{1}{2} - \left(\frac{\sqrt{3}}{2}+1\right)\mathrm{i}\right) = 1 - \frac{\sqrt{3}}{2} - \frac{\mathrm{i}}{2} = 1 - t.$$

So $Z_{5,10} = 0$.

(iii) $$Z_{5,15} = \frac{\tau\bar{t}^2(1+t)p_{13}}{\sqrt{1-\tau}}(t+1)p_{21} + tp_{13}[(t+1)p_{22}-1] + t(t+1)p_{13}p_{23}(t+1)p_{23} =$$
$$= p_{13}\left(\frac{\bar{t}\tau}{1-\tau} - \frac{t}{t+1} + \frac{t^2}{1-\tau}\right)$$

(replacing the p_{ij} by their values and using the relation $\tau = \dfrac{t}{(t+1)^2}$)

$$= \frac{p_{13}}{(1-\tau)(1+t)}(\tau(\bar{t}+1)(1+\mathrm{i}) + t\tau - t).$$

Now we use $t = \dfrac{\sqrt{3}+\mathrm{i}}{2}$ and $\tau = 2-\sqrt{3}$ and we obtain 0. ∎

PROPOSITION 2.14. *On the line 5 of Z the only terms non zero are $Z_{5,2}$, $Z_{5,4}$, $Z_{5,6}$, $Z_{5,9}$, $Z_{5,11}$, $Z_{5,14}$, $Z_{5,16}$, $Z_{5,18}$; these terms satify the relation:*

$$\begin{pmatrix} Z_{5,2} \\ Z_{5,4} \\ Z_{5,9} \\ Z_{5,14} \end{pmatrix} = -\mathrm{i}\begin{pmatrix} Z_{5,18} \\ Z_{5,16} \\ Z_{5,11} \\ Z_{5,6} \end{pmatrix}$$

Proof. The fact that the other terms of the line 5 are equal to zero is an immediate consequence of Lemma 2.13. Note $A = (Mv_2v_1)^{(2)}$.

$$Z_{5,2} = -a_{5,2} \quad Z_{5,18} = -a_{5,18}.$$

$$Z_{5,2} = \frac{-\bar{t}^2\sqrt{\tau}}{(1-\tau)(1+t)} \quad Z_{5,18} = \frac{-(t+1)\tau\sqrt{\tau}}{(1-\tau)}$$

$$\frac{Z_{5,2}}{Z_{5,18}} = \frac{\bar{t}^2}{\tau(t+1)^2} = \frac{\bar{t}^2}{t} = \bar{t}^3 = -\mathrm{i}$$

Using $Z_{5,5} = Z_{5,15} = 0$ we obtain:

$$Z_{5,4} = -a_{5,4} + \frac{1}{\sqrt{1-\tau}}a_{5,5} \quad Z_{5,16} = -a_{5,16} + \frac{1}{\sqrt{1-\tau}}a_{5,15}.$$

$$Z_{5,14} = -a_{5,14} + \frac{1}{\sqrt{1-\tau}}a_{5,15} \quad Z_{5,6} = -a_{5,6} + \frac{1}{\sqrt{1-\tau}}a_{5,5}.$$

$$Z_{5,4} = \frac{-(\bar{t}+1)\tau}{\sqrt{1-\tau}}\frac{\bar{t}(2\tau-1)}{1-\tau} + \frac{1}{\sqrt{1-\tau}}\frac{((\bar{t}+1)\tau-1)\tau}{t(1-\tau)} =$$

$$= \frac{\tau\bar{t}}{(1-\tau)\sqrt{1-\tau}}[(\bar{t}+1)(1-\tau)-1] = \frac{\tau\bar{t}^3}{(1-\tau)\sqrt{1-\tau}(\bar{t}+1)}$$

$$Z_{5,16} = \frac{\tau t}{(1-\tau)\sqrt{1-\tau}}[1-\tau(t+1)] = \frac{\tau}{(1-\tau)\sqrt{1-\tau}} \times \frac{1}{\bar{t}+1}.$$

So $\dfrac{Z_{5,4}}{Z_{5,16}} = \bar{t}^3 = -\mathrm{i}$.

$$Z_{5,14} = \frac{\tau}{(1-\tau)\sqrt{1-\tau}}(t-\tau\bar{t}^2(1+t)) = \frac{\tau}{(1-\tau)\sqrt{1-\tau}}\left(\frac{t+t^2-\bar{t}}{1+t}\right)$$

but $t+t^2-\bar{t} = t+\mathrm{i}\bar{t}-\bar{t} = \mathrm{i}+\mathrm{i}\bar{t} = \mathrm{i}(1+\bar{t})$. So $Z_{5,14} = \dfrac{\mathrm{i}\tau}{t(1-\tau)\sqrt{1-\tau}}$.

$$Z_{5,6} = \frac{-(\bar{t}+1)\tau^2}{t(1-\tau)\sqrt{1-\tau}} + \frac{1}{\sqrt{1-\tau}}\frac{((\bar{t}+1)\tau-1)\tau}{t(1-\tau)} = \frac{-\tau}{t(1-\tau)\sqrt{1-\tau}}.$$

So $Z_{5,14} = -\mathrm{i}(Z_{5,6})$.

Now from $Z_{5,10} = 0$, we obtain $Z_{5,9} = -a_{5,9} + \dfrac{1}{\sqrt{1-\tau}}a_{5,10}$ and $Z_{5,11} = -a_{5,11} +$ $+\dfrac{1}{\sqrt{1-\tau}}a_{5,10}$.

$$Z_{5,9} = \frac{-\tau\bar{t}^2 p_{12}(t+1)}{\sqrt{1-\tau}} + \frac{[(\bar{t}+1)\tau-1]p_{12}(t+1)t}{\sqrt{1-\tau}} \quad (\bar{t}+1)\tau-1 = -\frac{\tau}{\bar{t}}(1+\bar{t}) = -\tau(1+t).$$

So $Z_{5,9} = \dfrac{-\tau p_{12}(t+1)}{\sqrt{1-\tau}}(\bar{t}^2 + t + t^2)$.

But $\bar{t}^2 + t + t^2 = t(1 + t + \bar{t}^3) = t(1 + t - \mathrm{i}) = t(1 + \bar{t})$ and $\tau = \dfrac{1}{|t+1|^2}$.

So $Z_{5,9} = \dfrac{-tp_{12}}{\sqrt{1-\tau}}$.

$$Z_{5,11} = \frac{p_{12}}{\sqrt{1-\tau}} - \frac{p_{12}t^2}{\sqrt{1-\tau}} = \frac{p_{12}}{\sqrt{1-\tau}}t(\bar{t} - t) = \frac{-\mathrm{i}tp_{12}}{\sqrt{1-\tau}}.$$

So $Z_{5,9} = \dfrac{1}{\mathrm{i}} Z_{5,11} = -\mathrm{i} Z_{5,11}$. ■

PROPOSITION 2.15. *On the line 8 of Z the only terms non zero are $Z_{8,1}$;$Z_{8,3}$; $Z_{8,8}$ and $Z_{8,13}$.*
Moreover:

$$\begin{pmatrix} Z_{8,1} \\ Z_{8,3} \\ Z_{8,8} \\ Z_{8,13} \end{pmatrix} = -\mathrm{i}t(1+t)(1-\tau) \begin{pmatrix} Z_{5,2} \\ Z_{5,4} \\ Z_{5,9} \\ Z_{5,14} \end{pmatrix}$$

Proof. From the expression of e_1 (Corollary 2.5.i) and the line 8 of Mv_2 (Lemma 2.12.ii) we obtain immediately that the only terms non zero in line 8 are $Z_{8,1}$, $Z_{8,3}$, $Z_{8,8}$ and $Z_{8,13}$ and that $Z_{8,1} = t(t+1)^2 p_{12}\sqrt{\tau(1-\tau)}$; $Z_{8,3} = -(t+1)P_{12}\tau t(1+t)$ (using $(t+1)(1-\tau) - 1 = \tau t(1+t)$ as in Lemma 2.12)

$$Z_{8,8} = \frac{-1}{1+t} \quad Z_{8,13} = (t+1)p_{23} = \frac{(t+1)\tau}{\sqrt{1-\tau}}.$$

$$\text{So } Z_{8,1} = t(t+1)^2\tau\sqrt{\tau} = t^2\sqrt{\tau} \quad \frac{Z_{8,1}}{Z_{5,2}} = \frac{t^2(1-\tau)(1+t)}{-\bar{t}^2} = -\mathrm{i}t(1+t)(1-\tau)$$

$$Z_{8,3} = \frac{-\tau t^2}{\sqrt{1-\tau}} \quad \frac{Z_{8,3}}{Z_{5,4}} = \frac{-\tau t^2(1-\tau)(1+\bar{t})}{-\mathrm{i}\tau} = -\mathrm{i}t(1+t)(1-\tau)$$

$$Z_{8,8} = \frac{-1}{1+t} \quad \frac{Z_{8,8}}{Z_{5,9}} = \frac{(1-\tau)}{(1+t)t\tau} = \frac{(1+t)(1-\tau)}{t^2} = -\mathrm{i}t(1+t)(1-\tau) \text{ (cf. } t^3 = i)$$

$$\frac{Z_{8,13}}{Z_{5,14}} = \frac{(t+1)t(1-\tau)}{\mathrm{i}} = -\mathrm{i}t(t+1)(1-\tau)$$ ■

PROPOSITION 2.16. *The second matrix of the projection P in $\langle B, e, f_1, f_2\rangle$ is:*

$$P^{(2)} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19 \end{array}
\begin{array}{c} \begin{array}{ccccccccccccccccccc} 1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19 \end{array} \\
\left(\begin{array}{ccccccccccccccccccc}
a_{11}&0&a_{12}&0&0&&&a_{13}&0&0&&&a_{14}&0&0&&&&\\
0&a_{11}&0&a_{12}&0&&&0&a_{13}&0&&&0&a_{14}&0&&&&\\
a_{21}&0&a_{22}&0&0&&&a_{23}&0&0&&&a_{24}&0&0&&&&\\
0&a_{21}&0&a_{22}&0&&&0&a_{23}&0&&&0&a_{24}&0&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&a_{44}&0&&&0&a_{43}&0&&&0&a_{42}&0&a_{41}&0\\
&&&&0&0&a_{44}&&&0&0&a_{43}&&&0&0&a_{42}&0&a_{41}\\
a_{31}&0&a_{32}&0&0&&&a_{33}&0&0&&&a_{34}&0&0&&&&\\
0&a_{31}&0&a_{32}&0&&&0&a_{33}&0&&&0&a_{34}&0&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&a_{34}&0&&&0&a_{33}&0&&&0&a_{32}&0&a_{31}&0\\
&&&&0&0&a_{34}&&&0&0&a_{33}&&&0&0&a_{32}&0&a_{31}\\
a_{41}&0&a_{42}&0&0&&&a_{43}&0&0&&&a_{44}&0&0&&&&\\
0&a_{41}&0&a_{42}&0&&&0&a_{43}&0&&&0&a_{44}&0&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
&&&&0&a_{24}&0&&&0&a_{23}&0&&&0&a_{22}&0&a_{21}&0\\
&&&&0&0&a_{24}&&&0&0&a_{23}&&&0&0&a_{22}&0&a_{21}\\
&&&&0&a_{14}&0&&&0&a_{13}&0&&&0&a_{12}&0&a_{11}&0\\
&&&&0&0&a_{14}&&&0&0&a_{13}&&&0&0&a_{12}&0&a_{11}
\end{array}\right) \end{array}$$

Proof. $P^{(2)}_{ij} = (Z^*Z)_{ij}$. From Corollary 2.10 and Propositions 2.14 and 2.15, we obtain immediately:

$$P^{(2)}_{ij} = \overline{Z}_{8i}Z_{8j} \text{ if } \quad i,j \in \{1,3,8,13\} = I_1$$

$$P^{(2)}_{ij} = \overline{Z}_{12,i}Z_{12,j} = Z_{8,20-i,20-j} \text{ if } \quad i,j \in \{7,12,17,19\} = I_2$$

$$P^{(2)}_{ij} = \overline{Z}_{5,i}Z_{5,j} + \overline{Z}_{15,i}Z_{15,j} =$$

$$= \overline{Z}_{5,i}Z_{5,j} + \overline{Z}_{5,20-i}Z_{5,20-j} \text{ if } \quad i,j \in \{2,4,9,14,6,11,16,18\} = I_3.$$

$$P^{(2)}_{ij} = 0 \text{ in other cases.}$$

and also $P^{(2)}_{ij} = P^{(2)}_{20-i,20-j}\ \forall i,j$.
Now from Proposition 2.14, if $i \in \{2,4,9,14\}$ and $j \in \{6,11,16,18\}$,

$$P^{(2)}_{i,j} = \overline{Z}_{5,i}Z_{5,j} + \overline{Z}_{5,20-i}Z_{5,20-j} = \qquad (20-j \in \{2,4,9,14\})$$

$$= \mathrm{i}\overline{Z}_{5,20-i}Z_{5,j} + (-\mathrm{i})\overline{Z}_{5,20-i}Z_{5,j} = 0$$

So if $i \in \{2,4,9,14\}$, $P^{(2)}_{ij} = 0$ if $j \notin \{2,4,9,14\}$.
Furthermore if $i,j \in \{1,3,8,13\}$
$P^{(2)}_{ij} = \overline{Z}_{8,i}Z_{8,j} = |-\mathrm{i}t(t+1)(1-\tau)|^2\overline{Z}_{5,i+1}Z_{5,j+1}$ (from Proposition 2.15)
$|-\mathrm{i}t(t+1)(1-\tau)|^2 = \dfrac{(1-\tau)^2}{\tau} = \dfrac{(1-\sqrt{3})^2}{2-\sqrt{3}} = 2.$
And $P^{(2)}_{i+1,j+1} = \overline{Z}_{5,i+1}Z_{5,j+1} + \overline{Z}_{5,20-i-1}Z_{5,20-j-1} = 2\overline{Z}_{5,i+1}Z_{5,j+1}$ (from Proposition 2.14 because $20-i-1$ and $20-j-1 \in \{6,11,16,18\}$.

So $P^{(2)}_{ij} = P^{(2)}_{i+1,j+1} \forall i,j \in \{1,3,8,13\}$.
This proves that $P^{(2)}$ is as announced. ∎

COROLLARY 2.17. *P commutes with $u_1 B u_1^{-1}$.*

Proof. From the diagram (D) giving the inclusion of $u_1 B u_1^{-1}$ in $\langle B, e_1, f_1, f_2\rangle$ we obtain that the matrices of an element B, α, C of $u_1 B u_1^{-1}$ viewed as element of $\langle B, e_1, f_1, f_2\rangle$ are

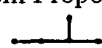
$$\begin{pmatrix} B & & & \\ & B & & \\ & & \alpha & \\ & & & C \end{pmatrix} \begin{pmatrix} B \\ & B \\ & & \alpha \\ & & & C \\ & & & & B \\ & & & & & \alpha \\ & & & & & & C \\ & & & & & & & B \\ & & & & & & & & \alpha \\ & & & & & & & & & C \\ & & & & & & & & & & C \end{pmatrix} \begin{pmatrix} B & & & \\ & \alpha & & \\ & & C & \\ & & & C \end{pmatrix}$$

$(B, C \in M_2(\mathbb{C}), \alpha \in \mathbb{C})$.

These matrices trivially commute respectively with $P^{(1)} = X, P^{(2)}$ (as in Proposition 2.16) and $P^{(3)} = Y$. This proves that P commutes with $u_1 B u_1^{-1}$. ∎

Proof of theorem 2.1. From Lemma 2.8 and Corollary 2.17,

$$P \in \langle B, f_1, f_2\rangle \cap (u_1 B u_1^{-1})'.$$

So $p \in M_2 \cap N'$ (Corollary 1.3) and $P \neq 1 - r_1$ (Lemma 2.8). So now from Proposition 2.6 the principal graph associated to $N \subset M$ is the Coxeter graph E_6.  ∎

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