J. OPERATOR THEORY
28(1992), 93–103

HYPERCYCLIC OPERATORS AND CHAOS

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1. INTRODUCTION

A Hilbert space operator T is called *hypercyclic* if there is a vector y such that the orbit

$$\mathrm{Orb}(T;y) = \{y, Ty, T^2y, \ldots\}$$

is (norm) dense in the space.

In [4, p. 50], R. L. Devaney has proposed that a continuous mapping of a topological space be called *chaotic* if it is topologically transitive (some element has a dense orbit), has a dense set of periodic points, and possesses a certain "sensitivity to initial conditions." Topological transitivity is, in our setting, hypercyclicity. G. Godefroy and J. H. Shapiro have indicated that a hypercyclic operator T on a Banach space $\mathcal{H}$ is sensitive to initial conditions in a rather dramatic form: the set $HC(T)$ of all hypercyclic vectors of T is a G_δ-dense subset of $\mathcal{H}$; thus, for each $x \in \mathcal{H}$ the set $S(x) = x + HC(T)$ is G_δ-dense, and

$$\liminf_{n\to\infty} \|T^n x - T^n w\| = 0, \quad \limsup_{n\to\infty} \|T^n x - T^n w\| = \infty$$

for every $w \in S(x)$. Therefore, a hypercyclic operator is chaotic if and only if it has a dense set of periodic points (see[7, section 6]). This observation allows the authors of the above article to exhibit a large number of chaotic operators.

Indeed, it is not difficult to check that the set $\mathrm{Per}(T)$ of periodic points of a (not necessarily hypercyclic) operator T coincides with the linear span of $\{\ker(\lambda - T) : \lambda \in \exp(2\pi i\mathbb{Q})\}$, where $\mathbb{Q}$ denotes the set of rational numbers.

The main result of this article says that if $\mathcal{H}$ is a (complex separable, infinite dimensional) Hilbert space, and $\mathcal{L}(\mathcal{H})$ is the algebra of all (bounded linear) operators acting on $\mathcal{H}$, then $\mathcal{L}(\mathcal{H})$ includes a large supply of chaotic operators:

If $T \in \mathcal{L}(\mathcal{H})$, T^* has no eigenvalues, and $\mathcal{H} = \vee\{\ker(\lambda - T) : \lambda \in \partial\mathbb{D}, \lambda - T$ is semi-Fredholm$\}$, then T is chaotic.

(Here $\mathbb{D}$ denotes the open unit disk, with boundary $\partial\mathbb{D}$.) Actually, it will be shown that these operators have several other interesting properties. The main result of [11] indicates that every hypercyclic operator on $\mathcal{H}$ is the norm-limit of a sequence of chaotic operators of the above described type. The results also include new information about limits of hypercyclic operators.

2. CHAOTIC OPERATORS

THEOREM 1. *Let $T \in \mathcal{L}(\mathcal{H})$, and assume that T^* has no eigenvalues and*

$$\mathcal{H} = \bigvee\{\ker(\lambda - T) : \lambda \in \partial\mathbb{D}, \lambda - T \text{ is semi-Fredholm}\};$$

then:

(i) $\operatorname{Per}(T)$ *is dense in* $\mathcal{H}$;

(ii) *there exists a dense linear manifold $\mathcal{X}$ invariant under T such that $\|T^n x\| \to 0$ $(n \to \infty)$ for each x in $\mathcal{X}$; moreover, each $x \in \mathcal{X} \setminus \{0\}$ is a cyclic vector of T;*

(iii) *there exists a dense linear manifold $\mathcal{Y}$ and a (not necessarily continuous) linear mapping S on $\mathcal{Y}$ such that $S\mathcal{Y} \subset \mathcal{Y}$, $\|S^n y\| \to 0$ $(n \to \infty)$ for each y in $\mathcal{Y}$, and $TS = 1_{\mathcal{Y}}$; moreover, each $y \in \mathcal{Y} \setminus \{0\}$ is a cyclic vector of T satisfying $\|T^n y\| \to \infty$ $(n \to \infty)$.*

(iv) *T is chaotic.*

Proof. It readily follows from [8], [10, Chapter 3], [12] that if T satisfies the hypotheses of the theorem, then T admits an upper triangular matrix representation (with respect to some orthonormal basis of $\mathcal{H}$), with diagonal entries in

$$\partial\mathbb{D} \cap \sigma(T) \cap \rho_{s-F}(T),$$

where $\sigma(T)$ and $\rho_{s-F}(T)$ denote the spectrum of T and, respectively, the semi-Fredholm domain of T; moreover, $\partial\mathbb{D} \cup \sigma(T)$ is a connected set, and, $\ker(\lambda - T) \neq \{0\}$, $\ker(\lambda - T)^* = \{0\}$ and $\operatorname{ind}(\lambda - T) > 0$ for all $\lambda \in \sigma(T) \cap \rho_{s-F}(T)$. (The reader is referred to [10], [14] for definition and properties of the semi-Fredholm operators.)

The continuity properties of semi-Fredholm operators indicate that

$$\mathcal{H} = \bigvee\{\ker(\lambda - T) : \lambda \in \rho_{s-F}(T) \cap \exp(2\pi i\mathbb{Q})\} = \operatorname{Per}(T)^-,$$

so that T satisfies (i).

Clearly, $\sigma(T) \cap \rho_{s-F}(T)$ has at most denumerably many components $\Omega_1, \Omega_2, \ldots$, that intersect $\partial\mathbb{D}$. Let $\alpha_k, \beta_k, \gamma_k$ be any three points in Ω_k with the same argument

θ_k such that $0 < |\alpha_k| < |\beta_k| = 1 < |\gamma_k|$. Construct a sequence $\{\alpha_k(h)\}$ of points of Ω_k such that $\arg \alpha_k(h) = \theta_k$ and $|\alpha_k(h)|$ strictly increases to $|\alpha_k|$. Let $x_k(h)$ be a unit vector in $\ker(\alpha_k(h) - T)$, $h \geqslant 1$. If the $x_k(h)$'s are carefully chosen, then for each $k \geqslant 1$ we have:

$$\bigvee\{x_k(h) : h \geqslant 1\} \supset \bigvee\{\ker(\alpha_k - T)^m : m \geqslant 1\}$$

and, a fortiori,

$$\bigvee\{x_k(h) : k, h \geqslant 1\} \supset \bigvee\{\ker(\alpha_k - T)^m : k, m \geqslant 1\} = \mathcal{H}$$

(see, e.g., [3], [10], [14]).

Now define

$$x = \sum 2^{-k-h} x_k(h).$$

We see that

$$\|T^n x\| = \left\|\sum 2^{-k-h} \alpha_k(h)^n x_k(h)\right\| \leqslant \sum 2^{-k-h} |\alpha_k|^n \to 0 \ (n \to \infty).$$

By using standard arguments of approximation, we can easily check that

$$x_k(h) \in \bigvee\{T^n x : n \geqslant 0\},$$

for all $k, h \geqslant 1$, whence we infer that x *is a cyclic vector of* T (see, e.g., [9, Section 5.B]). Furthermore, it is easily seen, (for instance, by using the properties of Vandermonde determinants) that, if

$$z = \sum_{n=0}^{N} c_n T^n x = \sum 2^{-k-h} \left[\sum_{n=0}^{N} \alpha_k(h)^n c_n\right] x_k(h),$$

and $\sum_{n=0}^{N} \alpha_k(h)^n c_n = 0$ for more than N pairs (k, h), then $c_0 = c_1 = c_2 = \cdots = c_N = 0$. By using this observation, it is not difficult to check that if z belongs to

$$\mathcal{X} = \text{linear span } \{T^n x : n \geqslant 0\}$$

and $z \neq 0$, then z is a cyclic vector of T and $\|T^n z\| \to 0$ $(n \to \infty)$. Hence, T satisfies (ii).

The proof of (iii) is very similar: construct $\{\gamma_k(h)\} \subset \Omega_k$, such that $\arg \gamma_k(h) = \vartheta_k$ and $|\gamma_k(h)|$ strictly decreases to $|\gamma_k|$, and define

$$y = \sum 2^{-k-h} y_k(h)$$

for suitably chosen unit vectors $y_k(h) \in \ker(\gamma_k(h) - T)$ $(k, h \geqslant 1)$. A formal repetition of the previous argument shows that if z belongs to

$$\mathcal{Y} = \text{linear span } \left\{ \sum 2^{-k-h} \gamma_k(h)^{-n} y_k(h) : n \geqslant 0 \right\}$$

and $z \neq 0$, then z is a cyclic vector of T. Define S by

$$S\left(\sum 2^{-k-h} \gamma_k(h)^{-n} y_k(h) \right) = \sum 2^{-k-h} \gamma_k(h)^{-n-1} y_k(h),$$

and extend to $\mathcal{Y}$ by linearity. Clearly, $S\mathcal{Y} \subset \mathcal{Y}$ and $\|S^n z\| \to 0$ $(n \to 0)$ for all $z \in \mathcal{Y}$.

Furthermore, $TSz = z$ for all z in $\mathcal{Y}$. Hence T also satisfies (iii).

In order to complete the proof, it only remains to show that T is hypercyclic. But this follows immediately from the Kitai-Gethner-Shapiro criterion [6], [7], [15]: it readily follows from (ii) and (iii) that $\mathcal{H}$ includes the two dense linear manifolds, $\mathcal{X}$ and $\mathcal{Y}$, such that $T\mathcal{X} \subset \mathcal{X}$, $\mathcal{Y}$ is invariant under a linear mapping S such that $TS = 1_{\mathcal{Y}}$, and

$$\|T^n x\| \to 0 \quad \text{and} \quad \|S^n y\| \to 0 \ (n \to \infty),$$

for all $x \in \mathcal{X}$ and, respectively, for all $y \in \mathcal{Y}$.

By the above mentioned criterion, T is hypercyclic. The proof of Theorem 1 is now complete. ∎

3. NOTES AND REMARKS

1. The classes $HC(\mathcal{H})$ and $SC(\mathcal{H})$ of all hypercyclic and, respectively, all supercyclic operators are invariant under similarity. (T is *supercyclic* if $\{\lambda T^n y : \lambda \in \mathbf{C}, n \geqslant \geqslant 0\}$ is dense in $\mathcal{H}$ for some vector y.) Moreover, if $A, B \in \mathcal{L}(\mathcal{H}), A$ is hypercyclic (supercyclic) and $XA = BX$ for some $X \in \mathcal{L}(\mathcal{H})$ with *dense range*, then so is B. Indeed, if $\text{Orb}(A; y)$ is dense in $\mathcal{H}$, then

$$\text{Orb}(B; Xy)^- = \{XA^n y : n \geqslant 0\}^- = [X(\text{Orb}(A; y))]^- = (X\mathcal{H})^- = \mathcal{H},$$

so that Xy is a hypercyclic vector for B. (The case when A is merely supercyclic can be similarly analized.)

2. It was shown in [3, Proposition 2.5] that if

$$A = \begin{pmatrix} A_1 & A_{12} & A_{13} & \cdots \\ & A_2 & A_{23} & \cdots \\ & & A_3 & \cdots \\ & 0 & & \ddots \end{pmatrix}$$

(with respect to a suitable orthogonal direct sum decomposition $\mathcal{H} = \sum \bigoplus_{k=1}^{\infty} \mathcal{H}_k$) and $\sigma(A_j) \cap \sigma(A_k) = \emptyset$ for $j \neq k$, then there is a quasiaffinity X (i.e., X is injective and has dense range) such that

$$AX = X\left(\sum \bigoplus_{k=1}^{\infty} A_k\right).$$

Thus, if $\sum \bigoplus_{k=1}^{\infty} A_k$ is hypercyclic (supercyclic), then so is A. (Observe that $\sigma(A)$ and $\sigma\left(\sum \bigoplus_{k=1}^{\infty} A_k\right)$ can be a lot larger than $\left[\bigcup_{k=1}^{\infty} \sigma(A_k)\right]^-$.)

3. Suppose $\{T_k\}_{k=1}^{\infty}$ is a bounded sequence of operators ($T_k \in \mathcal{L}(\mathcal{H}_k), k = 1, 2, \ldots$), for each $k \geqslant 1$ there exist dense linear manifolds $\mathcal{X}_k$ (invariant under T_k) and $\mathcal{Y}_k$, and a linear map S_k of $\mathcal{Y}_k$ into itself such that

$$\|T_k^n x_k\| \to 0 \text{ for all } x_k \in \mathcal{X}_k \text{ and } \|S_k^n y_k\| \to 0 \text{ for all } y_k \in \mathcal{Y}_k \ (n \to \infty),$$

and $T_k S_k = 1_{\mathcal{Y}_k}$.

Define $T = \sum \bigoplus_{k=1}^{\infty} T_k \in \mathcal{L}(\mathcal{H}) \left(\mathcal{H} = \sum \bigoplus_{k=1}^{\infty} \mathcal{H}_k\right)$, $\mathcal{X}$ = the algebraic direct sum of the $\mathcal{X}_k$'s, $\mathcal{Y}$ = the algebraic direct sum of the $\mathcal{Y}_k$'s, and $S = \sum \bigoplus_{k=1}^{\infty} S_k$.

It is not difficult to check that

$$T\mathcal{X} \subset \mathcal{X},\ S\mathcal{Y} \subset \mathcal{Y},\ TS = 1_{\mathcal{Y}},$$

and $\|T^n x\| \to 0$ for all $x \in \mathcal{X}$ and $\|S^n y\| \to 0$ for all $y \in \mathcal{Y}$ ($n \to \infty$).

The Kitai-Gethner-Shapiro criterion implies that T is hypercyclic [6], [7], [15]. However, H. N. Salas [17] has recently constructed a bilateral weighted shift T such that both T and T^* are hypercyclic, but $T \oplus T^*$ is not even cyclic. (Thus, either T or T^*, or both, cannot satisfy the Kitai-Gethner-Shapiro criterion!)

PROBLEM 1. Suppose $T \in HC(\mathcal{H})$. Is $T \oplus T$ hypercyclic?

4. An obvious consequence of 1) is that if A and B are quasisimilar (i.e., $XA = BX$ and $AY = YB$, for two quasiaffinities X and Y), then A is hypercyclic (supercyclic) if and only if B is so. A *normal operator* N cannot be supercyclic [13]; therefore, an operator T quasisimilar to an operator N cannot be supercyclic either. (Operators quasisimilar to normal operators were characterized by C. Apostol in [1].)

However, a hypercyclic operator can be decomposable (in the sense of Foiaş). Let B be the bilateral weighted shift (with respect to an orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$) with weights $\beta_0 = \beta_1 = \beta_{-1} = 1$,

$$\beta_k = \frac{\log k}{\log(k+1)}, \quad \beta_{-k} = \frac{\log(k+1)}{\log k} \quad (k \geqslant 2).$$

Then $(\beta_0\beta_1\cdots\beta_{n-1})^{-1} = \beta_{-1}\beta_{-2}\cdots\beta_{-(n-1)} = \log n \to \infty$ $(n \to \infty)$ "rather slowly." B is invertible, $\sigma(B) = \partial\mathbf{D}$, and $\sigma_p(B) = \sigma_p(B^*) = \emptyset$ because $\sum_{n=2}^{\infty}(\log n)^{\pm 2} = \infty$ (see [18]); moreover, B is decomposable [16]. (Here $\sigma_p(\cdot)$ denotes the point spectrum.)

Define $\mathcal{X} = \left\{\sum_{k=-m}^{m} \lambda_k e_k : m \geqslant 1\right\}$; $\mathcal{X}$ is a dense linear manifold invariant under B and under $B^{-1}, BB^{-1} = 1$, and

$$\|B^n x\| \to 0 \quad \text{and} \quad \|B^{-n}x\| \to 0 \ (n \to \infty)$$

for all x in $\mathcal{X}$.

It follows from [7], [15, Theorem 2.2] that B is hypercyclic (and so is B^{-1} because the inverse of an invertible hypercyclic operator is also hypercyclic).

A similar example can be constructed as follows: define the sequence of weights by

$$\text{for } k \geqslant 0: \quad 1, 1, \ldots, 1, \frac{1}{2}, 1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, \ldots,$$

$$\text{for } k < 0: \quad 1, 1, \ldots, 1, 2, 1, 1, \ldots, 1, 2, 2, 1, 1, \ldots, 1, 2, 2, 2, 1, 1, \ldots$$

(long string of 1's, then $\frac{1}{2}$; another long string of 1's, then two terms equal to $\frac{1}{2}$, etc., and similarly for negative weights, with 1's and 2's).

Assume that the lengths of the strings of 1's are carefully chosen so that $(\beta_0\beta_1\cdots\beta_{n-1})^{-1}$ and $\beta_{-1}\beta_{-2}\cdots\beta_{-n}$ are both $O(e^{n\phi_n})$ for some sequence of positive reals converging to 0, such that $\sum_{n=1}^{\infty}\frac{\phi_n}{n} < \infty$.

In this case, $\sigma(B) = \left\{\lambda \in \mathbf{C} : \frac{1}{2} \leqslant |\lambda| \leqslant 2\right\}$, B and B^{-1} are hypercyclic, and B is not decomposable but, nevertheless, it has a large family of hyperinvariant subspaces [5]. (Of course, we can also require that $\sigma_p(B) = \emptyset$.)

By combining these arguments with the last paragraph of 1) for each $r, r', 0 \leqslant r \leqslant 1 \leqslant r'$, and each $\varepsilon > 0$, it is possible to construct hypercyclic bilateral weighted shifts B such that $\sigma(B)$ is the closed annulus with radii r and r', $\sigma_p(B) = \emptyset$, B has a large supply of hyperinvariant subspaces, and $B = N + K_\varepsilon$, where N is normal and $K_\varepsilon \in \mathcal{K}(\mathcal{H})$, with $\|K_\varepsilon\| < \varepsilon$.

5. Suppose that B is as in the first example of 4). The same proof shows that $B^{(\infty)} = B \oplus B \oplus \cdots$ is hypercyclic. For each open set Ω intersecting $\partial\mathbf{D}$, the decomposable operator B^* has a maximal spectral invariant subspace $\mathcal{N}$ such that $\sigma(B|\mathcal{N}) = \sigma(B^*|\mathcal{N})^* = \Gamma := (\Omega \cap \partial\mathbf{D})^-$.

Let M be a normal operator such that $\sigma(M) \cup \partial\mathbf{D}$ is connected and $\sigma(M) \cap \partial\mathbf{D} = \Gamma$. By the Similarity Orbit Theorem [2,Theorem 9.2], there exists a sequence $\{B_k\}_{k=1}^{\infty}$ of operators similar to $B|\mathcal{N}$ such that $\|M - B_k\| \to 0$ $(k \to \infty)$.

Define $T = \sum_{k=1}^{\infty} \bigoplus B_k$; then $\sigma(T) = \sigma(M)$, T is quasisimilar to $B|\mathcal{N}^{\infty}$ and therefore hypercyclic, and $\sigma_p(T) = \emptyset$.

For each invariant subspace $\mathcal{R}$ of T^*, $T|\mathcal{R}$ is also hypercyclic, and therefore $\sigma(T|\mathcal{R}) \cup \partial\mathbf{D} = \sigma(T^*|\mathcal{R})^* \cup \partial\mathbf{D}$ is connected. Since $\sigma(M)$ can be arbitrarily chosen, we have obtained the following.

PROPOSITION 2. *Let Δ be a nonempty subset of the plane such that $\Delta \cup \partial\mathbf{D}$ is connected, and every point of Δ is connected (in Δ) with some open arc of $\partial\mathbf{D}$ included in Δ. There exists a hypercyclic operator $T \in \mathcal{L}(\mathcal{H})$ such that $\sigma(T) = \sigma_{lre}(T) = \Delta, \sigma_p(T) = \emptyset$, and for each invariant subspace $\mathcal{R}$ of T^*, $\sigma(T^*|\mathcal{R}) \cup \partial\mathbf{D}$ is connected. (Here $\sigma_{lre}(T)$ denotes the complement of $\rho_{s-F}(T)$ in the complex plane.) (For instance, Δ can be the Mandelbrot set.)*

6. Analogous to cyclic operators, there is a relaxed notion of multicyclicity: T is multicyclic of order m (or, T has multiplicity $m \geqslant 1$) if the closed linear span of any set of $(m-1)$ orbits is not dense in $\mathcal{H}$, but there is a set of m orbits whose closed linear span coincides with $\mathcal{H}$.

Do the analogous definitions make sense for the notions of hypercyclic and supercyclic?

CONJECTURE 1. *If there exist $m \geqslant 2$ vectors $y_1, y_2, \ldots, y_m$ such that the union of the multiples of the sets* $\operatorname{Orb}(T; y_1)$, $\operatorname{Orb}(T; y_2), \ldots, \operatorname{Orb}(T; y_m)$ *is dense in $\mathcal{H}$, then the multiples of* $\operatorname{Orb}(T; y_j)$ *is dense in $\mathcal{H}$ for some $j, 1 \leqslant j \leqslant m$. If* $\bigcup_{k=1}^{m} \operatorname{Orb}(T; y_k)$ *is dense in $\mathcal{H}$, then* $\operatorname{Orb}(T; y_j)^- = \mathcal{H}$ *for some $j, 1 \leqslant j \leqslant m$.*

7. We close with two results connected with the linear manifold $\operatorname{Per}(T)$.

PROPOSITION 3. *Let $T \in \mathcal{L}(\mathcal{H})$ (not necessarily cyclic).*

(i) $\operatorname{Per}(T)$ *is the linear span of*

$$\{\ker(T - \alpha) : \alpha \in \sigma_p(T) \cap \exp(2\pi i \mathbf{Q})\}$$

($\mathbf{Q}$ is the field of rational numbers). (In particular, $\operatorname{Per}(T) = \{0\}$ *whenever $\sigma_p(T) = \emptyset$.)*

(ii) $\mathrm{Per}(T)^-$ *is a hyperinvariant subspace of* T.

(iii) $\mathrm{Per}(T)^- = \mathcal{H}$ *if and only if* T *admits an upper triangular matrix representation (with respect to a suitable orthonormal basis of* $\mathcal{H}$*) such that all the diagonal entries of* T *belong to* $\sigma_p(T) \cap \exp(2\pi i\mathbf{Q})$ *and* $\ker(\lambda - T)^2 = \ker(\lambda - T)$ *for each* $\lambda \in \sigma_p(T) \cap \exp(2\pi i\mathbf{Q})$; *moreover, in this case every component of* $\sigma(T)$ *meets* $\partial\mathbf{D}$.

(iv) *If* $T - \alpha$ *is a semi-Fredholm operator of positive index for some* $\alpha \in \partial\mathbf{D}$, $\ker(T - \alpha)^* = \{0\}$, *and* $\mathcal{H} = \vee\{\ker(T - \alpha)^k\}_{k=1}^{\infty}$, *then* $\mathrm{Per}(T)^- = \mathcal{H}$, *and* $\sigma(T)$ *is connected.*

(v) *If*

$$T = \begin{pmatrix} T_1 & T_{12} & T_{13} & \cdots \\ & T_2 & T_{23} & \cdots \\ & & T_3 & \cdots \\ & 0 & & \ddots \end{pmatrix}$$

(with respect to a suitable orthogonal direct sum decomposition $\mathcal{H} = \sum_{k=1}^{\infty} \bigoplus \mathcal{H}_k$*), then* $\mathrm{Per}(T_k)^- = \mathcal{H}_k$ *for all* $k \geqslant 1$, *and* $\sigma(T_k) \cap \sigma(T_j) = \emptyset$ $(k \neq j)$, *then* $\mathrm{Per}(T)^- = \mathcal{H}$.

Proof. (i) As indicated in the introduction, this follows by a simple computation.

(ii) Since $\ker(T - \alpha)$ is a hyperinvariant subspace for all $\alpha \in \sigma_p(T)$, it readily follows from (i) that $\mathrm{Per}(T)^-$ is also a hyperinvariant subspace of T.

(iii) $\mathrm{Per}(T)^- = \mathcal{H}$ implies that $\mathcal{H} = \vee\{\ker(T - \alpha)^k : \alpha \in \sigma_p(T), k \geqslant 1\}$, and therefore T is a triangular operator. Now both implications follow at once from (i) and [3, Section 2] (or [10, Theorem 3.40], [12]).

(iv) If T satisfies the three hypotheses, then there is a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset \subset \exp(2\pi i\mathbf{Q})$ converging to α, such that $T - \alpha_k$ is semi-Fredholm (with $\mathrm{ind}(T - \alpha_k) = \mathrm{ind}(T - \alpha)$) for all $k \geqslant 1$, and

$$\mathcal{H} = \bigvee\{\ker(T - \alpha_k)\}_{k=1}^{\infty}$$

(see [10, Chapter 3]).

The result follows from (iii).

(v) Assume $\mathrm{Per}(T_k)$ is dense in $\mathcal{H}_k$. Since $\sigma(T_k) \cap \sigma(T_j) = \emptyset$ $(k \neq j)$, an easy inductive argument shows that

$$\mathrm{Per}(T)_n := \mathrm{Per}\begin{pmatrix} T_1 & T_{12} & \cdots & T_{1n} \\ & T_2 & \cdots & T_{2n} \\ & & \ddots & \vdots \\ & 0 & & T_n \end{pmatrix}$$

is dense in $\sum_{k=1}^{n} \bigoplus \mathcal{H}_k$ for all $n = 1, 2, \ldots$ (see the above references).

Since the linear span of the $\mathcal{H}_k$'s is dense in $\mathcal{H}$, and $\mathrm{Per}(T)_n \subset \mathrm{Per}(T)$ $(n \geqslant 1)$, we conclude that $\mathrm{Per}(T)$ is dense in $\mathcal{H}$. ■

As an application to the structure of $HC(\mathcal{H})$, we have the following results. We recall here that the main result of [11] says that $HC(\mathcal{H})^-$ is the class of all operators A in $\mathcal{L}(\mathcal{H})$ satisfying: (1) $\sigma_w(A) \cup \partial\mathbf{D}$ is a connected set. (Here $\sigma_w(A)$ denotes the Weyl spectrum of A.) (2) Every isolated point of $\sigma(A)$ belongs to $\sigma_w(A)$, and (3) $\mathrm{ind}(\lambda - A) \geqslant 0$ for all $\lambda \in \rho_{s-F}(A)$.

PROPOSITION 4. (i) *If $T \in HC(\mathcal{H})$, then either* $\mathrm{Per}(T)^- = \mathcal{H}$, *or* $\mathrm{Per}(T)^-$ *has infinite codimension in $\mathcal{H}$.*

(ii) $\mathrm{Per}(T)^- = \mathcal{H}$ *for all T in a dense subset of $HC(\mathcal{H})$.*

(iii) *If $T \in HC(\mathcal{H})$ and $T - \lambda$ is semi-Fredholm (necessarily of positive index) for all $\lambda \in \partial\mathbf{D}$, then* $\mathrm{Per}(T')^- = \mathcal{H}$ *for all $T' \in HC(\mathcal{H})$ in some neighborhood of T.*

(iv) *If $T \in HC(\mathcal{H})$ and $T - \alpha$ is semi-Fredholm of positive index for some $\alpha \in \partial\mathbf{D}$, but $T - \beta$ is not semi-Fredholm for some other $\beta \in \partial\mathbf{D}$, then* $\mathrm{Per}(T')^-$ *is infinite dimensional for all $T' \in HC(\mathcal{H})$ close enough to T, but T can be approximated by a sequence $\{T_k\}_{k=1}^{\infty}$ in $HC(\mathcal{H})$ such that* $\mathrm{Per}(T-k)^-$ *has infinite codimension in $\mathcal{H}$ for all $k \geqslant 1$.*

(v) *If $T \in HC(\mathcal{H})$ and $\sigma(T) = \sigma_{lre}(T)$, then T can be approximated by a sequence $\{T_k\}_{k=1}^{\infty}$ in $HC(\mathcal{H})$ such that* $\mathrm{Per}(T_k) = \{0\}$ *for all $k \geqslant 1$.*

Proof. (i) By Proposition 3 (ii), $\mathrm{Per}(T)^-$ is hyperinvariant for T, and therefore its orthogonal complement is invariant under T^*. By [11, Proposition 2.2], $H \ominus \mathrm{Per}(T)^-$ cannot be a non-trivial finite dimensional subspace (for, otherwise, $\sigma_p(T^*) \neq \emptyset$).

(ii) This follows immediately from Theorem 1 and Proposition 3 (iv) and (v). (By Theorem 2.1 of [11] and its proof: the operators described in Theorem 1 form a dense subset of $HC(\mathcal{H})$.)

(iii) If $T \in HC(\mathcal{H})$ and $T - \lambda$ is semi-Fredholm for all $\lambda \in \partial\mathbf{D}$, then (by [11, Proposition 2.2]) $\sigma_p(T^*) = \emptyset$, $\mathrm{ind}(\lambda - T) > 0$ and $(T - \lambda)^*$ is bounded below by some constant $\delta > 0$ (for all $\lambda \in \partial\mathbf{D}$).

Thus, if $\|T - T'\| < \varepsilon$, then $T' - \lambda$ is a semi-Fredholm operator with $\mathrm{ind}(\lambda - -T') = \mathrm{ind}(\lambda - T)$, and $(T' - \lambda)^*$ is bounded below for all $\lambda \in \partial\mathbf{D}$. If, in addition, $T' \in HC(\mathcal{H})$, then $\mathrm{Per}(T') = \mathcal{H}$ by Proposition 3 (iv) (see [10, Chapter 3]). In particular, $\mathrm{Per}(T)^- = \mathcal{H}$.

(iv) Approximate T as in the proof of Theorem 2.1 of [11] by operators $A_2 \in \in \mathcal{L}(\mathcal{H})$, $A_3 \in HC(\mathcal{H})$, such that $A_3 - \beta$ is semi-Fredholm of index 1. Let Γ be a closed tiny arc of the unit circle centered at β such that $A_3 - \lambda$ is Fredholm of index 1 for all $\lambda \in \Gamma$, and let $B|\mathcal{N}$ be constructed as in 5), with $\sigma(B|\mathcal{N}) = \Gamma$.

The Similarity Orbit Theorem [2, Theorem 9.2] indicates that A_2 can be uniformly approximated by operators similar to $A_3 \oplus B|\mathcal{N}$ (by 3), $A_3 \oplus B|\mathcal{N}$ is hypercyclic). Therefore, T can also be approximated by operators of this type.

But $\sigma_p(B|\mathcal{N}) = \emptyset$; therefore, for each W invertible,

$$\mathrm{Per}(W(A_3 \oplus B|\mathcal{N})W^{-1})^- = W\mathcal{H}_3,$$

where $\mathcal{H}_3$ is the space of A_3. Hence, $\mathrm{Per}(W(A_3 \oplus B)W^{-1})^-$ has infinite dimension and infinite codimension in $\mathcal{H}$.

On the other hand, by proceeding as in (iii), it is easy to check that $\vee\{\ker(T' - -\lambda)^k\}_{k=1}^{\infty}$ is an infinite dimensional subspace of $\mathrm{Per}(T')^-$ for each $T' \in HC(\mathcal{H})$ close enough to T.

(v) If $T \in HC(\mathcal{H})$ and $\sigma(T) = \sigma_{lre}(T)$, then given $\varepsilon > 0$ we can approximate T by $T_1 \in HC(\mathcal{H})^-$ so that $\|T - T_1\| < \varepsilon$ and $\sigma(T_1) = \sigma_{lre}(T_1)$ is the closure of an analytic Cauchy domain Ω including $\sigma(T)$ [10], [11].

Let $\Gamma = \partial\mathbb{D} \cap \Omega$, and let $B|\mathcal{N}$ be constructed as in 5) so that $\sigma(B|\mathcal{N}) = \Gamma$. By the Similarity Orbit Theorem [2, Theorem 9.2], T_1 can be approximated by operators similar to $B|\mathcal{N}$. Therefore, there exists W invertible such that $\|T - WB|\mathcal{N}W^{-1}\| < < 2\varepsilon$.

Since $B|\mathcal{N} \in HC(\mathcal{H})$ and $\sigma_p(B|\mathcal{N}) = \emptyset$, $WB|\mathcal{N}W^{-1} \in HC(\mathcal{H})$ and $\mathrm{Per}(WB|\mathcal{N} \cdot \cdot W^{-1}) = \{0\}$. Since ε can be chosen arbitrarily small, we are done. ∎

Of course, if T has the form of Proposition 4 (v), then the approximating sequence $\{T_k\}_{k=1}^{\infty} \in HC(\mathcal{H})$ can also be chosen so that $\mathrm{Per}(T_k)^- = \mathcal{H}$, or $\mathrm{Per}(T_k)^-$ is an infinite dimensional subspace of infinite codimension in $\mathcal{H}$, for each $k \geqslant 1$.

CONJECTURE 2. *If $T \in HC(\mathcal{H})$ and $T - \lambda$ is not a semi-Fredholm operator of positive index for any $\lambda \in \partial\mathbb{D}$, then T can be approximated by a sequence $\{T_k\}_{k=1}^{\infty}$ in $HC(\mathcal{H})$ such that $\mathrm{Per}(T_k) = \{0\}$ for all $k \geqslant 1$.*

This research has been partially supported by a Grant from the National Science Foundation.

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Received September 12, 1990; revised September 25, 1991.