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STRUCTURE OF BANACH ALGEBRAS WITH TRIVIAL CENTRAL COHOMOLOGY

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INTRODUCTION

This paper considers homological questions of the theory of Banach algebras and their representations associated with central cohomologies. Here the adjective "central" points out the class of admissible morphisms in the relative category of Banach modules over a Banach algebra.

The study of some central homological characteristics is given in M. J. Liddel's paper [12] and the joint papers of J. Phillips and I. Raeburn [13, 14]. In the latter of these papers one of the important problems of the central homology was discussed, that is, the description of the structure of C^*-algebra for which all the central cohomology groups vanish. In [13] it was proved that separable unital C^*-algebras with continuous trace satisfy this property. In [14] it was made an attempt to outline a proof of the converse assertion, which was founded on a deep result of Elliot, Akemann and Pederson on the structure of separable unital C^*-algebras having the trivial one-dimensional cohomology group with coefficients in the algebra itself; but for the full proof of the converse assertion it was missed a closing link. Recently, when this paper have been already written, the present author was informed, that the missing link was indeed found by Phillips and Raeburn. Their argument used D. Voiculescu's double commutant theorem and some new original ideas. However, the way chosen by them required rather strict assumptions on C^*-algebras in consideration: they supposed an algebra in question to be separable and a priori unital.

This paper, which gives, in particular, the solution of the discussed problem for arbitrary C^*-algebras is founded on somewhat other ideas. We prove that all the central cohomology groups of Banach algebras A for $n \geqslant 1$ vanish (i.e. A is central contractible) iff A is a central biprojective algebra with an identity; and, for some class of Banach algebras which contains all C^*-algebras, we study the structure of such central biprojective algebras with an identity.

In [12] M. J. Liddell considered in equivalent terms the structure of central biprojective algebras with an identity under various topological assumptions on the algebra. Some methods of this work will be used in the present paper.

We call a Banach algebra A clear if it has the following property: for each closed two-sided ideals M of the algebra A from the contractibility of the Banach algebra A/M it follows, that A/M is the finite direct sum of full matrix algebras. The class of clear algebras include approximative admissible Banach algebras [5]. In particular, all C^*-algebras are certainly clear.

Before formulating the main theorem we remind [11] that a Banach algebra A is called n-homogeneous if for every primitive ideal I of A a Banach algebra A/I is isomorphic to $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the algebra of all complex matrices of order n. Note that in view of Lemma 4.8 [13] (which is valid also without condition that the algebra is separable), a C^*-algebra A with an identity is a C^*-algebra with continuous trace iff A is the finite direct sum of homogeneous C^*-algebras of finite rank. Now we formulate our main theorem.

THEOREM. *Let A be a Banach algebra, and $Z(A_+)$ be the center of A_+ (A_+ denotes the unitization of A). Then the properties written below are connected among themselves by the following way. For all algebras* (i) *is equivalent to* (ii). *For clear algebras* (iii) *follows from* (ii). *For C^*-algebras all properties are equivalent to the property that A has an identity and continuous trace.*

(i) *All the central (i.e. $Z(A_+)$) cohomology groups of A for $n \geqslant 1$ vanish.*

(ii) *A has an identity and is central biprojective.*

(iii) *A has an identity and is the finite direct sum of homogeneous Banach algebras of finite rank.*

The main concepts of the relative homological theory for Banach B-algebras are introduced in Sections 1 and 2. The equivalence of conditions (i) and (ii) of the main theorem is proved in Section 2. In this section the vanishing $\mathcal{H}^3_B(A, X)$ for any B-biprojective Banach B-algebra A with coefficients in arbitrary Banach (A, B)--bimodule X is also proved. The properties of closed maximal two-sided ideals of central contractible clear algebras are described in Section 3. For such algebras we establish that:

(i) each primitive ideal is a closed maximal two-sided ideal;

(ii) there exists an one to one correspondence between sets of closed maximal two-sided ideals of the algebra and its center;

(iii) for each maximal closed ideal M of the algebra A/M is a full matrix algebra.

In Section 4 we prove, that the central biprojectivity of the enveloping algebra $A^e = A \underset{Z(A_+)}{\hat{\otimes}} A^{op}$ follows from the central contractibility of Banach algebra A, here $\underset{Z(A_+)}{\hat{\otimes}}$ is a sign of the projective tensor product of left and right Banach $Z(A_+)$-modules [16], A^{op} is an opposite algebra of A with a multiplication ab equal to the "previous" ba. In the concluding section in Theorem 5.2 it is shown that a central contractible clear algebra A is a projective Banach $Z(A)$-bimodule. In Theorem 5.4 we prove that such an algebra is the finite direct sum of homogeneous Banach algebras; thereby, returning to the formulation of the main theorem it is shown that for clear algebras the property (ii) implies (iii). The assertion of main theorem that for C^*-algebras the property (iii) implies (i) could be extracted from the proof of Theorem 4.20 [13]. Following on a chosen way we prove in Theorem 5.5 that for C^*-algebras the property (iii) implies (ii).

1. B-ALGEBRAS

In first sections of the present paper we rely mainly on [2, 8, 4, 6] and introduce notions of the relative homological theory of Banach B-algebras, everywhere B is a commutative Banach algebra with an identity e_B. When B is $\mathbf{C}$, we omit the sign B in the corresponding definitions using well-known definitions [6, 10].

We will consider symmetrical Banach bimodules over B, in other words, Banach B-bimodules X, for which $b \cdot x = x \cdot b$ for any $b \in B$ and $x \in X$. Everywhere afterwards "a Banach B-module" will mean an unital symmetrical Banach B-bimodule. The category of unital symmetrical Banach B-bimodules is denoted by B-unsmod. For $X, Y \in B$-unsmod the Banach space of morphisms from X to Y is denoted by ${}_Bh_B(X,Y)$.

For an arbitrary Banach algebra A, not necessarily possessing an identity, we denote by A_+ the Banach algebra obtained by adjoining an identity to A. If there is an identity in A itself, it is denoted by e, and the adjoined identity is denoted by e_+.

We say that a Banach algebra A is a Banach B-algebra if A is a Banach B-module and for any $a_1, a_2 \in A$, $b \in B$ the following condition holds $(a_1a_2) \cdot b = a_1(a_2 \cdot b) = (a_1 \cdot b)a_2$.

For a Banach B-algebra A a left (right) Banach A-module X will be called a left (right) Banach (A, B)-module if, in addition, X is a Banach B-module and for any $a \in A$, $b \in B$ and $x \in X$ the following conditons $(a \cdot x) \cdot b = a \cdot (x \cdot b) = (a \cdot b) \cdot x$ (respectively, $(x \cdot a) \cdot b = x \cdot (a \cdot b) = (x \cdot b) \cdot a$) hold.

For a Banach B-algebra A a Banach A-bimodule X will be called a Banach (A, B)-bimodule if, in addition, X is a Banach B-module and for any $a \in A$, $b \in B$

and $x \in X$ the following conditions: $(a \cdot x) \cdot b = a \cdot (x \cdot b) = (a \cdot b) \cdot x$ and $(x \cdot a) \cdot b = x \cdot (a \cdot b) = (x \cdot b) \cdot a$ hold.

A morphism of left (right, bi-) Banach A-modules will be called a morphism of left (respectively, right, bi-) Banach (A, B)-modules if, in addition, it is a morphism of Banach B-modules.

The category of left Banach (A, B)-modules over a B-algebra A and their morphisms is denoted by (A, B)-mod; mod-(A, B) and (A, B)-mod-(A, B) denote the corresponding categories of right (A, B)-modules and (A, B)-bimodules. In connection with modules as well as algebras the word "Banach" will usually be omitted. For $X, Y \in (A, B)$-mod (mod-(A, B), (A, B)-mod-(A, B)) the Banach space of morphisms from X to Y is denoted by ${}_{(A,B)}h(X, Y)$ ($h_{(A,B)}(X, Y)$, ${}_{(A,B)}h_{(A,B)}(X, Y)$).

Consider the B-algebra $A_B = A \oplus B$ for which a multiplication and a B-module structure are determined by the following formulae

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2 + a_1 \cdot b_2 + b_1 \cdot a_2, b_1 b_2);$$

$$b_1 \cdot (a, b) = (b_1 \cdot a, b_1 b) = (a, b) \cdot b_1; \quad a_1, a_2, a \in A, \ b_1, b_2, b \in B,$$

with some norm which makes A_B into a Banach algebra such that $\|(a, 0)\| = \|a\|$; $a \in A$, and $\|(0, b)\| = \|b\|$; $b \in B$ (for example, $\|(a, b)\| = \max\{\|a\|, \|b\|, (a, b) \in A_B\}$). Note, that an element $(0, e_B)$ is identity in A_B.

Further we shall consider a B-algebra A_B as an (A, B)-bimodule over a B-algebra A with structures of left and right A-modules and B-module defined by

$$a \cdot (a_1, b_1) = (a, 0)(a_1, b_1) = (aa_1 + a \cdot b_1, 0),$$

$$(a_1, b_1) \cdot a = (a_1, b_1)(a, 0) = (a_1 a + b_1 \cdot a, 0); \quad a \in A, \ (a_1, b_1) \in A_B,$$

and, respectively

$$b_1 \cdot (a, b) = (b_1 \cdot a, b_1 b) = (a, b) \cdot b_1; \quad (a, b) \in A_B, \ b_1 \in B.$$

On the other hand, one can easily see that any (A, B)-module can be considered as (A_B, B)-module. Indeed, for example, for any left (A, B)-module X a structure of a left A_B-module is determined by

$$(a, b) \cdot x = a \cdot x + b \cdot x; \quad (a, b) \in A_B, \ x \in X.$$

Then any morphism of (A, B)-modules $\varphi \in {}_{(A,B)}h(X, Y)$ is a morphism of (A_B, B)-modules.

For $X \in B$-unsmod the spaces $A_B \underset{B}{\hat{\otimes}} X$, $X \underset{B}{\hat{\otimes}} A_B$ and $A_B \underset{B}{\hat{\otimes}} X \underset{B}{\hat{\otimes}} A_B$ can be considered in (A, B)-mod, mod-(A, B) and (A, B)-mod-(A, B), respectively, with the operations $a\cdot(x\otimes y)=a\cdot x\otimes y$ or (and) $(x\otimes y)\cdot a=x\otimes y\cdot a$; $b\cdot(x\otimes y)=b\cdot x\otimes y=$ $=x\otimes y\cdot b=(x\otimes y)\cdot b$. These modules are called free left (right, bi) (A,B)-modules over A.

Let $\mathcal{K}$ be any of the categories considered above. A complex $\cdots\leftarrow X_{n-1}\overset{\varphi_{n-1}}{\longleftarrow} X_n\overset{\varphi_n}{\longleftarrow}X_{n+1}\overset{\varphi_{n+1}}{\longleftarrow}\cdots$ of modules (in $\mathcal{K}$) is said to be B-admissible if it is splittable, regarded as a complex of B-modules (i.e., it has a contracting homotopy consisting of morphisms of B-modules).

We call $P\in(A,B)$-mod (A,B)-projective if the complex ${}_{(A,B)}h(P,\mathcal{X})$ is exact for every B-admissible complex $\mathcal{X}$ in (A,B)-mod. All free modules and all retracts of them (i.e., direct module summands) are (A,B)-projective; on the other hand, every B-admissible epimorphism of any module onto an (A,B)-projective module is a retract. As a consequence, any X is (A,B)-projective iff the morphism $\pi_X: A_B\hat{\otimes}_B X\to X: a\otimes x\mapsto a\cdot x$ (which is called the canonical morphism for X and always is a B-admissible epimorphism) is a retraction. The same term will be used for its restriction $\pi_X: A\hat{\otimes}_B X\to X$. For a fixed X we often write π instead of π_X.

For $X\in(A,B)$-mod a complex

$$0\leftarrow X\overset{\varepsilon}{\leftarrow}P_0\overset{\varphi_0}{\leftarrow}P_1\overset{\varphi_1}{\leftarrow}P_2\leftarrow\cdots\qquad(0\leftarrow X\leftarrow\mathcal{P})$$

over X is called a resolution of X if it is B-admisible, and an (A,B)-projective resolution if in addition all the modules in $\mathcal{P}$ are (A,B)-projective.

Every left (A,B)-module X has some standard (A,B)-projective resolutions. One of them is analogous to a non-normalized Bar-resolution [4]

$$0\leftarrow X\overset{\pi_X}{\leftarrow}A_B\hat{\otimes}_B X\overset{\partial_0}{\leftarrow}A_B\hat{\otimes}_B A_B\hat{\otimes}_B X\overset{\partial_1}{\leftarrow}A_B\hat{\otimes}_B A_B\hat{\otimes}_B A_B\hat{\otimes}_B X\overset{\partial_2}{\leftarrow}\cdots,\qquad(0\leftarrow X\leftarrow\mathcal{B})$$

where

$$\partial_{n-1}(a_1\otimes a_2\otimes\cdots\otimes a_{n+1}\otimes x)=$$
$$=\sum_{i=1}^{n}(-1)^{i+1}a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_{n+1}\otimes x+(-1)^{n+2}a_1\otimes\cdots\otimes a_n\otimes a_{n+1}\cdot x;$$

$a_i\in A_B$; $i=1,2,\ldots,n+1$, $x\in X$, and another is analogous to a normalized standard complex [2; IX, Section 6]

$$0\leftarrow X\overset{\pi_X}{\leftarrow}A_B\hat{\otimes}_B X\overset{\partial_0'}{\leftarrow}A_B\hat{\otimes}_B A\hat{\otimes}_B X\overset{\partial_1'}{\leftarrow}A_B\hat{\otimes}_B A\hat{\otimes}_B A\hat{\otimes}_B X\overset{\partial_2'}{\leftarrow}\cdots\qquad(0\leftarrow X\leftarrow\mathcal{N})$$

where

$$\partial_{n-1}'((a,b)\otimes a_1\otimes\cdots\otimes a_n\otimes x)=$$
$$=(a,b)\cdot a_1\otimes\cdots\otimes a_n\otimes x+\sum_{i=1}^{n-1}(-1)^i(a,b)\otimes a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n\otimes x+$$
$$+(-1)^n(a,b)\otimes a_1\otimes a_2\otimes\cdots\otimes a_n\cdot x;$$

$(a,b) \in A_B$, $a_i \in A$; $i = 1, \ldots, n$, $x \in X$.

The cohomology of the complex ${}_{(A,B)}h(\mathcal{P}, Y)$ depends only on modules X and Y and does not depend on a concrete choise of a resolution $0 \leftarrow X \leftarrow \mathcal{P}$. Accordingly to "classical" tradition, we denote the n-th cohomology of the complex ${}_{(A,B)}h(\mathcal{P}, Y)$ by $\mathrm{Ext}^n_{(A,B)}(X,Y)$, actually it is complete seminormed space and it is B-module in algebraic sense. Obviously, $\mathrm{Ext}^0_{(A,B)}(X,Y) = {}_{(A,B)}h(X,Y)$.

We say that a left (A,B)-module X has a B-homological dimension n if $\mathrm{Ext}^{n+1}_{(A,B)}(X,Y) = 0$ for any $Y \in (A,B)$-mod, but there exists $Z \in (A,B)$-mod such that $\mathrm{Ext}^n_{(A,B)}(X,Z) \neq 0$. The B-homological dimension of left (A,B)-module X will be denoted $\mathrm{dh}_{(A,B)}X$. The number (or ∞) $\mathrm{dg}_B A = \sup\{\mathrm{dh}_{(A,B)}X : X \in$ $\in (A,B) - \mathrm{mod}\}$ is called the global B-homological dimension of A.

For $X \in (A,B)$-mod the following properties are equivalent:

(i) $\mathrm{dh}_{(A,B)}X \leqslant n$;

(ii) there exists an (A,B)-projective resolution $0 \leftarrow X \leftarrow \mathcal{P}$ such that $P_{n+1} = 0$;

(iii) if $0 \leftarrow X \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_{n-1} \leftarrow Q \leftarrow 0$ is resolution of the (A,B)-module X in which $P_0, \ldots, P_{n-1}$ are (A,B)-projective, then so is Q. As a consequence, for $X \in (A,B)$-mod $\mathrm{dh}_{(A,B)}X = 0$ iff X is (A,B)-projective.

Regarding a right module and a bimodule as left modules (see above), we carry over the definitions with the indicated facts to the categories mod-(A,B) and (A,B)-mod-(A,B).

2. B-COHOMOLOGY GROUPS AND BIPROJECTIVITY OF B-ALGEBRAS

Recall [14] the definition of groups B-cohomology of a B-algebra. Let A be a B-algebra and X be an (A,B)-bimodule.

DEFINITION 2.1. Consider the complex

$$0 \to C^0_B(A,X) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{n-1}} C^n_B(A,X) \xrightarrow{\delta^n} C^{n+1}_B(A,X) \to \cdots, \qquad (C_B(A,X))$$

where $C^0_B(A,X)$ is the B-module X, $C^n_B(A,X)$ for $n > 0$ is the B-module of all continuous n-B-linear maps $f : A \times \cdots \times A \to X$ also, δ^n acts by the formulae $(\delta^0 x)(a) = a \cdot x - x \cdot a$ and for $n > 0$

$$(\delta^n f)(a_1, \ldots, a_{n+1}) = a_1 \cdot f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) +$$

$$+ (-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1};$$

$f \in C^n_B(A,X)$, $a_k \in A$; $k = 1, \ldots, n+1$.

It is easily shown that $\delta^{n+1} \circ \delta^n = 0$; $n \geqslant 0$. This complex is called the standard cohomological complex for an (A, B)-bimodule X. The n-th cohomology of $C_B(A, X)$ is called the n-dimensional B-cohomology group of A with coefficients in X and is denoted by $\mathcal{H}^n_B(A, X)$.

The results concerning study of B-cohomology groups are contained, for example, in [13, 14].

One can transfer to objects in discussion the well known construction of the pure homological algebra. Let $X, Y \in (A, B)$-mod. Let us consider ${}_Bh_B(X, Y)$ as an (A, B)-bimodule with actions $[a \cdot u](x) = a \cdot u(x)$, $[u \cdot a](x) = u(a \cdot x)$, $[b \cdot u](x) = b \cdot u(x)$ and $[u \cdot b](x) = u(b \cdot x)$; $a \in A$, $b \in B$, $u \in {}_Bh_B(X, Y)$, $x \in X$. We remark that $(b \cdot u)(x) = b \cdot u(x) = u(b \cdot x) = (u \cdot b)(x)$.

LEMMA 2.1. *Let A be a B-algebra. For any left (A, B)-modules X and Y there exists a topological isomorphism*

$$\mathrm{Ext}^n_{(A,B)}(X, Y) = \mathcal{H}^n_B(A, {}_Bh_B(X, Y)).$$

Proof. If we compute the latter Ext with the help of the resolution $\mathcal{N}(X)$, we come the cohomology of a complex which is isomorphic to $C_B(A, {}_Bh_B(X, Y))$. ■

LEMMA 2.2. *Let A be a B-algebra. Then for any (A, B)-bimodule X there exists a topological isomorphism*

$$\mathrm{Ext}^n_{(A,B)-(A,B)}(A_B, X) = \mathcal{H}^n_B(A, X).$$

Proof. If we compute the above mentioned Ext with the help of the resolution $\mathcal{N}(A_B)$, we come to considering the cohomology of the complex ${}_{(A,B)}h_{(A,B)}(\mathcal{N}(A_B), X)$. The latter is isomorphic to the standard complex $C_B(A, X)$. ■

DEFINITION 2.2. *A B-algebra A is called B-biprojective if it is a projective (A, B)-bimodule.*

LEMMA 2.3. *A B-algebra A with an identity is B-biprojective iff there exists a morphism of (A, B)-bimodules $\rho : A \to A \underset{B}{\hat{\otimes}} A$ such that for the canonical morphism $\pi : A \underset{B}{\hat{\otimes}} A \to A$ we have $\pi \circ \rho = 1_A$.*

Proof. Let a B-algebra A is B-biprojective. Then by the definition for the canonical morphism $\pi : A \underset{B}{\hat{\otimes}} A \to A$ there exists a morphism of (A, B)-bimodules $\rho : A \to A \underset{B}{\hat{\otimes}} A$ such that $\pi \circ \rho = 1_A$.

Conversely, let there exists a morphism of (A, B)-bimodules $\rho : A \to A \underset{B}{\hat{\otimes}} A$ such that $\pi \circ \rho = 1_A$. Then, in view of that $A \underset{B}{\hat{\otimes}} A$ is a projective (A, B)-bimodule, for any B-admissible epimorphism $\sigma \in {}_{(A,B)}h_{(A,B)}(X, Y)$; $X, Y \in (A, B)$-mod-(A, B), and for any morphism $\psi \in {}_{(A,B)}h_{(A,B)}(A, Y)$ there exists a morphism $\gamma \in {}_{(A,B)}h_{(A,B)}(A \underset{B}{\hat{\otimes}} A, X)$ such that $\sigma \circ \gamma = \psi \circ \pi$

$$\begin{array}{ccc} A \underset{B}{\hat{\otimes}} A & \underset{\rho}{\overset{\pi}{\rightleftarrows}} & A \\ \gamma \downarrow & & \downarrow \psi \\ X & \overset{\sigma}{\longrightarrow} & Y \end{array}$$

Consequently, the morphism $\varphi = \gamma \circ \rho \in {}_{(A,B)}h_{(A,B)}(A, X)$ satisfies the identity $\sigma \circ \varphi = (\sigma \circ \gamma) \circ \rho = \psi \circ (\pi \circ \rho) = \psi$. ∎

LEMMA 2.4. *A B-algebra A with an identity is B-biprojective iff the centre of the (A, B)-bimodule $A \underset{B}{\hat{\otimes}} A$ contains an element u (called a B-splitting element for A) such that $\pi(u) = e$.*

Proof. If A is B-biprojective and if $\rho : A \to A \underset{B}{\hat{\otimes}} A$ from Lemma 2.3, then $u = \rho(e)$ is a B-splitting element for A. If there is a B-splitting element, then $\rho : A \to A \underset{B}{\hat{\otimes}} A$, given by $a \mapsto a \cdot u = u \cdot a$, is a morphism of (A, B)-bimodules such that $\pi \circ \rho = 1_A$. ∎

DEFINITION 2.3. The B-bidimension of a B-algebra A is the number (or ∞) $\mathrm{db}_B A = \inf\{n : \mathcal{H}_B^{n+1}(A, X) = 0$ for all $X \in (A, B)$-mod-$(A, B)\}$.

From Lemma 2.1 it follows that $\mathrm{dg}_B A \leqslant \mathrm{db}_B A$.

DEFINITION 2.4. A B-algebra A is called B-contractible if $\mathrm{db}_B A = 0$.

THEOREM 2.1. *Let A be a B-algebra. Then the following conditions are equivalent:*

(i) *A is B-contractible.*

(ii) *A_B is a projective (A, B)-bimodule.*

(iii) *A has an identity and is a projective (A, B)-bimodule.*

Proof. Lemma 2.2 implies immediately that properties (i) and (ii) are equivalent. Suppose these properties are satisfied; then, if we consider B as a trivial left (A, B)--module, Lemma 2.1 ensures that as every left (A, B)-module B is (A, B)-projective. This implies that the short exact sequence of left (A, B)-modules

$$0 \leftarrow B \leftarrow A_B \leftarrow A \leftarrow 0$$

splits, and hence that A has a right identity e_1. In exactly the same way, if B is considered as a right (A, B)-module, we get that there is a left identity e_2 in A.

This implies that e_1 and e_2 coincide and must be the "usual" identity in A. In this way all that is left to prove is that for a B-algebra with an identity, A_B is a projective (A,B)-bimodule iff A is a projective (A,B)-bimodule. This is implied by Lemma 2.4: if u_{A_B} is a B-splitting element for A_B, then $e \cdot u_{A_B} = u_{A_B} \cdot e$ is a splitting element for A; conversely if u_A were a B-splitting element for A, then $u = u_A + ((0,e_B) - (e,0)) \otimes ((0,e_B) - (e,0))$ would be a B-splitting element for A_B. ■

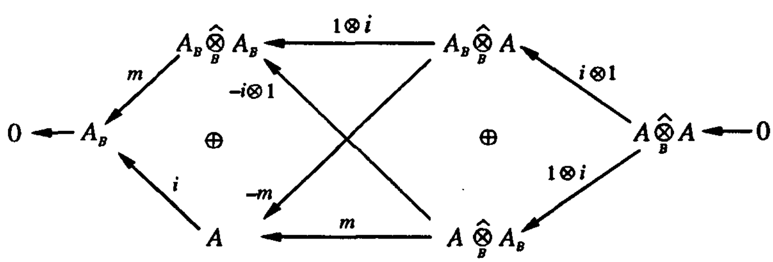
THEOREM 2.2. *Let A be a B-algebra. Then for any (A,B)-bimodule X, up to a topological isomorphism*

$$\mathcal{H}^n_B(A,X) = \mathcal{H}^n_B(A_B,X).$$

Proof. By Lemma 2.2 $\mathcal{H}^n_B(A,X) = \mathrm{Ext}^n_{(A,B)-(A,B)}(A_B,X)$. If we compute the latter Ext with the help of the resolution $\mathcal{B}(A_B)$, we come to considering the cohomology of the complex ${}_{(A,B)}h_{(A,B)}(\mathcal{B}(A_B),X)$. The latter is isomorphic to the standard complex $C_B(A_B,X)$. ■

THEOREM 2.3. *Let A be a B-biprojective B-algebra. Then, for any (A,B)--bimodule X, $\mathcal{H}^3_B(A,X) = 0$.*

Proof. Let us show that for $A_B \in (A,B)$-mod-(A,B) there exists an (A,B)--projective resolution of length two; hence, by force of Lemma 2.2, the theorem will be proved. Such resolution is the analogue of entwining resolution from [6]



where i is a morphism of inclusion and m is a morphim of multiplication. ■

3. SOME PROPERTIES OF CENTRAL CONTRACTIBLE ALGEBRAS AND THEIR IDEALS

DEFINITION 3.1. The enveloping algebra of a B-algebra A with an identity is called the algebra $A^e = A \hat{\otimes}_B A^{op}$, where A^{op} is the opposite algebra of A.

We recall that $Z(A)$ is the center of A.

NOTATION 3.1. $K = \{\xi \in A^e : a \cdot \xi = \xi \cdot a$ for each $a \in A\}$.

NOTATION 3.2. Let A be a B-algebra and let I be a subset of A. Then IA is the linear cover of the set $\{xa; x \in I,\ a \in A\}$ and $\overline{I}$ is the closure of I.

Let us define the map $\eta : A^e \to {}_Bh_B(A, A)$, putting for an elementary tensor $a_1 \otimes a_2 \in A^e$, $\eta(a_1 \otimes a_2)(a) = a_1 a a_2$; $a \in A$.

LEMMA 3.1. [12; 4.1, 4.5]. *Let K and η be defined as above. Then*

(i) *η is a B-algebra homomorphism;*

(ii) *for each $a \in A$, the map $A^e \to A : \xi \to \eta(\xi)(a)$ is an (A, B)-bimodule morphism;*

(iii) *for each $\xi \in K$, $a \in A$, $\eta(\xi)(a) \in Z(A)$.*

Proof. (i) It is clear.

(ii) For any $a_1, a_2 \in A$, $\xi \in A^e$, it is easily seen that $a_1[\eta(\xi)(a_2)] = \eta(a_1 \cdot \xi)(a_2)$ and $[\eta(\xi)(a_2)]a_1 = \eta(\xi \cdot a_1)(a_2)$.

(iii) It follows from the equalities $a_1[\eta(\xi)(a)] = \eta(a_1 \cdot \xi)(a) = \eta(\xi \cdot a_1)(a) = [\eta(\xi)(a)]a_1$ for $\xi \in K$ and $a, a_1 \in A$. ∎

Further we shall use the next form equivalent definitions of B-contractibility of A (see Theorem 2.1): A has an identity and is a projective (A, B)-bimodule. We shall call such algebras for $B = Z(A_+)$ by *central contractible.*

LEMMA 3.2. [12; 4.6]. *Let A be a $Z(A_+)$-contractible algebra. Then there exists a $Z(A)$-module projection $\alpha : A \to Z(A)$.*

Proof. By Lemma 2.3 there exists the morphism of $(A, Z(A_+))$-bimodules $\rho: A \to A^e$ such that $\pi \circ \rho = 1_A$. Then $\rho(e) \in K$ and, consequently by Lemma 3.1, $\eta(\rho(e))$ is a $Z(A)$-module morphism of A onto $Z(A)$. Besides, $\eta(\rho(e))(z) = z[\eta(\rho(e))(e)] = z\pi\rho(e) = z$, $z \in Z(A)$. Thus, $\eta(\rho(e))$ is a $Z(A)$-module projection of A onto $Z(A)$. ∎

LEMMA 3.3. [12; 4.7]. *Let A be a $Z(A_+)$-contractible algebra, let I be an ideal of $Z(A)$. Then $\overline{IA} \cap Z(A) = \overline{I}$.*

Proof. By lemma 3.2. $A = A_+ \oplus Z(A)$, where A_1 and $Z(A)$ are $Z(A)$-modules. Then $\overline{IA} = \overline{IA_1} \oplus \overline{IZ(A)}$ and $\overline{IA_1} \subset A_1$. Thus, $\overline{IA} \cap Z(A) = \overline{IZ(A)} \cap Z(A) = \overline{I}$. ∎

LEMMA 3.4. *Let A be a $Z(A_+)$-contractible algebra, let M be a closed two-sided ideal of A, and let $\varphi : A \to A/M$ be the natural epimorphism. Then*

(i) *A/M is $Z(A/M)$-biprojective;*

(ii) *$\varphi(Z(A)) = Z(A/M)$.*

. *Proof.* Define the morphism of $(A/M, Z(A/M))$-bimodules $\tilde{\rho}: A/M \to A/M \underset{Z(A/M)}{\hat{\otimes}} A/M$ putting, for $a+M$ from A/M, $\tilde{\rho}(a+M) = (\varphi \otimes \varphi)\rho(a)$, where $\rho : A \to A \underset{Z(A_+)}{\hat{\otimes}} A$ is the morphism of $(A, Z(A_+))$-bimodules from Lemma 2.3. The morphism is correct defined, since $\tilde{\rho}(u) = \varphi(u) \cdot (\varphi \otimes \varphi)\rho(e) = 0$ for $u \in M$. Besides, $(\pi_{A/M} \circ \tilde{\rho})(a+$ $+M) = (\pi_{A/M} \circ (\varphi \otimes \varphi))\rho(a) = \varphi(\pi_A \rho(a)) = a + M$. The assertion (i) is proved. Let $\bar{z} \in Z(A/M)$, there exists $a \in A$ such that $\varphi(a) = \overline{Z}$. Then, for an element $v = \eta(\rho(e))(a) \in Z(A)$, we have

$$\varphi(v) = \varphi(\eta(\rho(e))(a)) = \eta(\tilde{\rho}(\bar{e}))(\varphi(a)) = \eta(\tilde{\rho}(\bar{e}))(\bar{z}) =$$

$$= \bar{z}[\eta(\tilde{\rho}(\bar{e}))(\bar{e})] = \bar{z}(\pi_{A/M} \circ \tilde{\rho})(\bar{e}) = \bar{z},$$

where $\bar{e} = e + M$. Therefore $\varphi(Z(A)) = Z(A/M)$. ∎

LEMMA 3.5. *Let A be a $Z(A_+)$-contractible clear algebra. Then any primitive ideal I is a maximal closed two-sided ideal, and A/I is a full matrix algebra.*

Proof. A has an identity, hence, by Lemma 2.2.4 [15] $Z(A/I) \cong \mathbf{C}$. Then by Lemma 3.4 A/I is biprojective and, consequently, by the definition of the clear algebra and in view of that A/I is a primitive algebra, A/I is a full matrix algebra. Hence I is a maximal closed two-sided ideal. ∎

Denote by Δ_A the set of closed maximal two-sided ideals of A.

THEOREM 3.1. *Let A be central contractible clear algebra. Then*

(i) *$M \to M \cap Z(A)$ is a bijective correspondence between Δ_A and $\Delta_{Z(A)}$;*

(ii) *for every maximal closed two-sided ideal M, the algebra A/M is a full matrix algebra.*

Proof. Let $m \in \Delta_{Z(A)}$, then by Lemma 3.3 $m = \overline{mA} \cap Z(A)$ and, consequently, $\overline{mA}$ is a proper closed two-sided ideal of A. Hence, by Lemma 3.4 $Z(A/\overline{mA}) =$ $= (Z(A) + \overline{mA})/\overline{mA} \cong Z(A)/(\overline{mA} \cap Z(A)) \cong Z(A)/m \cong \mathbf{C}$, since m is a maximal closed ideal of the commutative Banach algebra $Z(A)$ with an identity. Then by Lemma 3.4 the Banach algebra $A/\overline{mA}$ is biprojective. Hence, by the definition of the clear algebra $A/\overline{mA}$ is a full matrix algebra, since $Z(A/\overline{mA}) \cong \mathbf{C}$. Thus, $\overline{mA}$ is a maximal closed two-sided ideal of A.

For M from Δ_A, $M \cap Z(A)$ is a proper closed ideal of $Z(A)$, since it does not contain the identity. By Lemma 3.4 the simple algebra A/M is a $Z(A/M)$-biprojective, and by Lemma 3.3 A/M has a simple center. Therefore, firstly, by Lemma 3.4 (ii) $Z(A)/(M \cap Z(A)) \cong Z(A/M) \cong \mathbf{C}$, consequently, $M \cap Z(A)$ is a maximal closed ideal of $Z(A)$, and, secondly, by the definition of a clear algebra, A/M is a full matrix algebra. Thus, Theorem 3.1 is proved. ∎

4. SOME PROPERTIES OF THE ENVELOPING ALGEBRA OF A CENTRAL CONTRACTIBLE ALGEBRA

THEOREM 4.1. *Let A be a B-contractible B-algebra. Then*

(i) *A^e is B-biprojective,*

(ii) *if $B = Z(A_+)$, then $Z(A) \to Z(A^e) : b \mapsto b \otimes e$ is an isomorphism of Banach algebras.*

Proof. The map $(a_1, a_2, a_3, a_4) \mapsto a_1 \otimes a_4 \otimes a_2 \otimes a_3$ lifts to a B-module isomorphism $j : (A^e)^e \to (A^e)^e$. Since by Theorem 2.1 the B-algebra A has an identity and is B-biprojective, then by Lemma 2.3 there exists a morphism of (A, B)-bimodules $\rho : A \to A \underset{B}{\hat{\otimes}} A$ such that $\pi \circ \rho = 1_A$. Define a morphism of (A^e, B)-bimodules $\tilde{\rho} : A^e \to A^e \underset{B}{\hat{\otimes}} A^e$, putting $\tilde{\rho} = j \circ (\rho \otimes \rho)$. Then, denoting by $\pi^e : A^e \underset{B}{\hat{\otimes}} A^e \to A^e$ the canonical morphism of the B-algebra A^e, we have $\pi^e \circ \tilde{\rho} = \pi^e \circ j \circ (\rho \otimes \rho) = (\pi \otimes \pi) \circ (\rho \otimes \rho) = 1_{A^e}$. Thus, the assertion (i) is proved.

LEMMA 4.1. [12; 4.5 (c)]. *Let A be a B-contractible B-algebra. Then $\pi(K) = Z(A)$.*

Proof. By Lemma 3.1 (iii), for each $\xi \in K$, we have $\pi(\xi) = \eta(\xi)(e) \in Z(A)$, that is, $\pi(K) \subset Z(A)$. Besides, for any $z \in Z(A)$, $\xi \in K$, $z \cdot \pi(\xi) = \pi(z \cdot \xi)$ and $z \cdot \xi \in K$, therefore, $\pi(K)$ is an ideal of $Z(A)$. Finally, we note that $e = \pi(\rho(e)) \in \pi(K)$, where $\rho : A \to A \underset{B}{\hat{\otimes}} A$ is a morphism of (A, B)-bimodules from Lemma 2.3. ∎

LEMMA 4.2. *Let A be a B-contractible B-algebra. Then $\pi(\rho(e)A^e) = Z(A)$, where $\rho : A \to A \underset{B}{\hat{\otimes}} A$ is the morphism of (A, B)-bimodules from Lemma 2.3.*

Prrof. We note, that $\rho(e)A^e \subset K$, since, for any $a \in A$, $v \in A^e$, we have $a \cdot (\rho(e)v) = (a \cdot \rho(e))v = (\rho(e) \cdot a)v = (\rho(e)v) \cdot a$. Besides, $\pi(\rho(e)\xi) = \pi(\xi)$ for each $\xi \in K$. Therefore by Lemma 4.1 $\pi(\rho(e)A^e) = Z(A)$. ∎

Let us continue a proof of Theorem 4.1. By Lemma 4.2, for the algebra A^e, which is $Z(A_+)$-biprojective, as is shown above, we have $\pi^e(\tilde{\rho}(e \otimes e)(A^e)^e) = Z(A^e)$. Hence

$$Z(A^e) = (\pi^e \circ j)(j^{-1}[\tilde{\rho}(e \otimes e)(A^e)^e]) = (\pi^e \circ j)(j^{-1}[j(\rho(e) \otimes \rho(e))(A^e)^e]).$$

Using the equality

$$j^{-1}[(j(a_1 \otimes a_2 \otimes a_3 \otimes a_4))(a'_1 \otimes a'_2 \otimes a'_3 \otimes a'_4)] = j^{-1}[a_1 a'_1 \otimes a'_2 a_4 \otimes a'_3 a_2 \otimes a_3 a'_4] =$$
$$= a_1 a'_1 \otimes a'_3 a_2 \otimes a_3 a'_4 \otimes a'_2 a_4 = (a_1 \otimes a_2)(a'_1 \otimes a'_3) \otimes (a_3 \otimes a_4)(a'_4 \otimes a'_2)$$

$a_i, a'_i \in A$; $i = 1, 2, 3, 4$, it is easily seen that

$$j^{-1}[j(\rho(e) \otimes \rho(e))(A^e)^e] = \rho(e)A^e \underset{Z(A_+)}{\hat{\otimes}} \rho(e)A^e.$$

Therefore,

$$Z(A^e) = (\pi \otimes \pi)(\rho(e)A^e \underset{Z(A_+)}{\hat{\otimes}} \rho(e)A^e) = \pi(\rho(e)A^e) \underset{Z(A_+)}{\hat{\otimes}} \pi(\rho(e)A^e) =$$
$$= Z(A) \underset{Z(A_+)}{\hat{\otimes}} Z(A) \cong Z(A).$$
■

5. CENTRAL CONTRACTIBLE CLEAR ALGEBRAS

LEMMA 5.1 [12; 4.5 (a)]. *Let A be a B-algebra with an identity. Then K is the right annihilator of the ideal* $\ker(\pi : A^e \to A)$ *in the algebra* A^e.

Proof. The equation $a \cdot \xi = \xi \cdot a$, for $a \in A$, $\xi \in A^e$, is equivalent to $(a \otimes e - e \otimes a)\xi = 0$. Therefore K is the right annihilator of the left ideal I in A^e generated by elements $\{a \otimes e - e \otimes a;\ a \in A\}$.

Obviously, $I \subset \ker(\pi : A^e \to A)$. On the other hand, I contains every element of the form

$$a \otimes b - \pi(a \otimes b) \otimes e = a \otimes b - ab \otimes e = (a \otimes e)(e \otimes b - b \otimes e)$$

for any $a, b \in A$. Consequently, I contains every element of the form $\xi - \pi(\xi) \otimes e$ for $\xi \in A^e$. Thus $\pi(\xi) = 0$ implies that $\xi \in I$. We conclude that $I = \ker \pi$. ■

THEOREM 5.1. *Let A be a central contractible clear algebra. Then $A^e K = A^e$.*

Proof. By Lemma 5.1 $(A^e K)$ is a two-sided ideal in A^e. If $(A^e K)$ is proper, then there exists a closed maximal two-sided ideal J in A^e such that $(A^e K) \subset J \subset A^e$ since A has an identity, hence, A^e with an identity.

By Theorem 4.1 A^e is central biprojective. For J from Δ_{A^e}, $m = J \cap Z(A^e)$ is a proper closed ideal in $Z(A^e) = Z(A)$ (Theorem 4.1 (ii)), since m does not contain an identity. By lemma 3.4 a simple algebra A^e/J is $Z(A^e/J)$-biprojective, besides, its center is simple by Lemma 3.3. Therefore, in view of Lemma 3.4 (ii)

$$Z(A)/m \cong Z(A^e)/(J \cap Z(A^e)) \cong Z(A^e/J) \cong \mathbf{C},$$

that is, m is a closed maximal ideal in $Z(A)$. Then by Theorem 3.1 $A/\overline{mA} \cong M_n(\mathbf{C})$ for some $n \in \mathbf{N}$.

Now, if we prove that

$$A^e/\overline{m \cdot A^e} \cong (A/\overline{mA})^e,$$

then we shall obtain that $\overline{m \cdot A^e}$ is a closed maximal two-sided ideal in A^e, and, consequently, $J = \overline{m \cdot A^e}$. Hence, by Lemma 4.1

$$A = AZ(A) = A\pi(K) \subset \pi(A^e K) \subset \pi(\overline{m \cdot A^e}) \subset \overline{\pi(m \cdot A^e)} = \overline{mA}.$$

Thus we have the contradiction.

Finally, we shall prove that

$$A^e/\overline{m \cdot A^e} \cong A/\overline{mA} \,\hat{\otimes}\, (A/\overline{mA})^{op}.$$

We note that, for the bilinear operator

$$\psi : A \times A^{op} \to A/\overline{mA} \,\hat{\otimes}\, (A/\overline{mA})^{op}$$

defined by $\psi(a,b) = (a + \overline{mA}) \otimes (b + \overline{mA})$; $a, b \in A$, the equality $\psi(az, b) = \psi(a, zb)$ holds for any $z \in Z(A_+)$, $a, b \in A$, since m is a closed maximal ideal in $Z(A)$. Besides, it is easily seen that $\|\psi\| \leqslant 1$. Then by the universal property of the projective tensor product [16] there exists a linear continuous operator

$$\tilde{\psi} : A \underset{Z(A_+)}{\hat{\otimes}} A^{op} \to A/\overline{mA} \,\hat{\otimes}\, (A/\overline{mA})^{op}$$

such that $\|\tilde{\psi}\| \leqslant 1$.

It is easily seen that $\tilde{\psi}(\overline{m \cdot A^e}) = 0$, therefore the linear operator

$$\approx{\psi} : A^e/\overline{m \cdot A^e} \to A/\overline{mA} \,\hat{\otimes}\, (A/\overline{mA})^{op}$$

such that

$$\approx{\psi}(w + \overline{m \cdot A^e}) = \tilde{\psi}(w); \quad w \in A^e,$$

is correctly defined.

We define an inverse map of $\approx{\psi}$ by the bilinear continuous operator

$$\varphi : A/\overline{mA} \times (A/\overline{mA})^{op} \to A^e/\overline{m \cdot A^e};$$

$$(a + \overline{mA}, b + \overline{mA}) \mapsto a \otimes b + \overline{m \cdot A^e}.$$

This operator is correctly defined since, for any $u \in \overline{mA}$ and $a, b \in A$, $\varphi(u, b) = 0$ and $\varphi(a, u) = 0$, and is continuous as a bilinear operator defined on a finite-dimensional space. ∎

THEOREM 5.2. *Let A be a central contractible clear algebra. Then there exists $n \in \mathbb{N}$ and morphisms of $Z(A)$-modules $\gamma : A \to Z(A)^n$ and $\beta : Z(A)^n \to A$ such that $\beta \circ \gamma = 1_A$, where $Z(A)^n = \{\{z_i\}_{i=1}^n;\ z_i \in Z(A)\}$.*

Proof. By the previous theorem $A^e K = A^e$, therefore there exist elements $u_i \in A^e$ and $k_i \in K$; $i = 1, 2, \ldots, n$, such that $e \otimes e = \sum_{i=1}^{n} u_i k_i$.

Then, for each element $a \in A$

$$a = \pi(e \otimes a) = \pi((e \otimes e)(e \otimes a)) = \sum_{i=1}^{n} \pi[(u_i k_i)(e \otimes a)] =$$
$$= \sum_{i=1}^{n} [\eta(u_i k_i)](\pi(e \otimes a)) = \sum_{i=1}^{n} [\eta(u_i k_i)](a) = \sum_{i=1}^{n} \eta(u_i)[\eta(k_i)(a)] =$$
$$= \sum_{i=1}^{n} \eta(k_i)(a)\eta(u_i)(e) = \sum_{i=1}^{n} \eta(k_i)(a)\pi(u_i).$$

We denote by $a_i = \pi(u_i)$ from A and $f_i = \eta(k_i)$ the morphisms of $Z(A)$-modules from A into $Z(A)$. Then, for each element a from A

$$a = \sum_{i=1}^{n} f_i(a)a_i.$$

Thus, the map $\gamma : A \to Z(A)^n : a \mapsto \{f_i(a)\}_{i=1}^n$ is a $Z(A)$-module embedding of A into the free $Z(A)$-module $Z(A)$ with the left inverse $\beta : Z(A)^n \to A : \{b_i\}_{i=1}^n \mapsto \sum_{i=1}^{n} b_i a_i$.

∎

LEMMA 5.2. *Let A be a central contractible clear algebra, let I be a maximal closed two-sided ideal in A. Then $f_i(I) \subset I \cap Z(A)$; $i = 1, 2, \ldots, n$.*

Proof. By Theorem 3.1 for I there exists a closed ideal m in $Z(A)$ such that $I = \overline{mA}$. Then, since f_i is a morphism of $Z(A)$-modules from A into $Z(A)$, it is easily seen that, for any element a from $I = \overline{mA}$, we have $f_i(a)$ is contained in $m = I \cap Z(A)$.

∎

Theorem 5.3. Let A be a central contractible clear algebra. Then, for each maximal closed two-sided ideal $M_0 \subset \Delta_A$, there exists a neighborhood $U \subset \Delta_A \cong \cong \Delta_{Z(A)}$ such that, for any $M \in U$, $\dim A/M = \text{const}$.

Proof. For each $M \in \Delta_A$, we define the action of $Z(A)$ on $\mathbb{C}^n$ by the formula

$$b_1 \cdot (\lambda_1, \ldots, \lambda_n) \cdot b_2 = (g_{M\cap Z(A)}(b_1)g_{M\cap Z(A)}(b_2)\lambda_1, \ldots, g_{M\cap Z(A)}(b_1)g_{M\cap Z(A)}(b_2)\lambda_n),$$

where $b_1, b_2 \in Z(A)$, $\lambda_i \in \mathbb{C}$; $i = 1, 2, \ldots, n$, $g_{M\cap Z(A)}$ is a multiplicative linear functional corresponding to the maximal closed ideal $M \cap Z(A)$ of the commutative Banach algebra $Z(A)$.

Let $\{\tilde{e}_i\}_{i=1}^n$ be the canonical basis for $Z(A)^n$. Then, the formula $\eta_M\left(\sum_{i=1}^{n} b_i \tilde{e}_i\right) = = (g_{M\cap Z(A)}(b_1), \ldots, g_{M\cap Z(A)}(b_n))$; $b_i \in Z(A)$, $i = 1, 2, \ldots, n$, defines a $Z(A)$-module morphism $\eta_M : Z(A)^n \to \mathbb{C}^n$.

Note that, for any $a \in M$, by Lemma 5.2 $f_i(a) \in M \cap Z(A)$, therefore, $\eta_M(\gamma(a)) =$ $= 0$; besides, $\beta \circ \gamma = 1_A$; where f_i, γ, β are defined in Theorem 5.2. Consequently, there exists a injective linear operator

$$\varphi_M : A/M \to \mathbf{C}^n$$

such the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\gamma\quad} & Z(A)^n \\ \downarrow & & \downarrow \eta_M \\ A/M & \xrightarrow{\quad\varphi_M\quad} & \mathbf{C}^n \end{array}$$

is commutative. From the diagram we see that

$$n = \dim \eta_M(Z(A)^n) = \dim \eta_M(\gamma(A)) + \dim \eta_M(\ker \beta) =$$

$$= \dim \varphi_M(A/M) + \dim \eta_M(\ker \beta) = \dim A/M + \dim \eta_M(\ker \beta).$$

Let us call k elements $u_1, \ldots, u_k$; $k \leqslant n$, of $Z(A)^n$ linearly independent at $m \in$ $\in \Delta_{Z(A)}$ if the elements $\eta_m(u_1), \ldots, \eta_m(u_k)$ form a linearly independent set in $\mathbf{C}^n$.

Then for the proof of theorem it will suffice to show, if arbitrary k elements $u_1, \ldots, u_k$; $k \leqslant n$, of $Z(A)^n$ are linearly independent at the point $m \in \Delta_{Z(A)}$, then there exists a neighborhood $U_m \subset \Delta_{Z(A)}$ of this point such that the elements $u_1, \ldots, u_k$ are linearly independent at each point of this neighborhood.

Suppose, that $u_i = \sum_{j=1}^{n} b_{ij} \tilde{e}_j$; $i = 1, \ldots, k$, $b_{ij} \in Z(A)$, are linearly independent at the point $m \in \Delta_{Z(A)}$. This means that there exists some nonzero $k \times k$ subdeterminant of $(g_m(b_{ij}))$. By continuity of the elements b_{ij}; $i = 1, \ldots, k$, $j = 1, \ldots, n$, there exists a neighborhood U of the point m in the space $\Delta_{Z(A)}$ in which this subdeterminant is nonzero. Hence the elements $\{u_i\}_{i=1}^k$ are linearly independent at each point of U. ■

THEOREM 5.4. *Let A be a central contractible clear algebra. Then the map $t \mapsto \dim A/I_t$ is locally constant on Δ_A, and, consequently, A is the finite direct sum of homogeneous algebras of finite rank.*

Proof. Since Δ_A is compact and $\Delta_A^n = \{t \in \Delta_A : \dim A/I_t = n\}$ is open by Theorem 5.3, there exists a finite covering of Δ_A with open closed sets $\Delta_A^{n_i}$; $i =$ $= 1, \ldots, n$. Then by the Shilov idempotent theorem there exist idempotents e_{n_i} of the algebra $Z(A)$ such that $\operatorname{supp} \hat{e}_{n_i} = \Delta_A^{n_i}$; $i = 1, \ldots, n$, where $\hat{e}_{n_i}$ is the Gelfand transform of the element e_{n_i}, and $e = \sum_{i=1}^{n} e_{n_i}$.

Consequently, $A = A\sum_{i=1}^{n} e_{n_i} = \bigoplus_{i=1}^{n} Ae_{n_i}$, where it is easily seen Ae_{n_i} is n_i--homogeneous algebra (see Lemma 3.5). ■

We recall that, for the C^*-algebra isomorphic to the finite direct sum of n_k--homogeneous C^*-algebras A_{n_k}; $k = 1, \ldots, m$, with an identity, it was shown in the proof of Theorem 4.20 [13] that $\mathrm{db}_{Z(A)}A = 0$, that is, it was proved that the condition (i) follows from the condition (iii) returning to the main theorem formulation. In Theorem 5.5 we give a proof of the fact of the main theorem such that for C^*-algebras the condition (ii) follows from (iii).

We recall [1, 3.1.5 and 3.3.3] that the spectrum $\hat{A}$ of a C^*-algebra A is a set of equivalence classes of nonzero irreducible representations provided with a topology with a base consisting of sets $V_x = \{S \in \hat{A} : \|T_S(x)\| > 1\}$; $x \in A$, where $T_S : A \to \to \mathcal{B}(H_S)$ is an irreducible representation from S. For each $S \in \hat{A}$ we shall denote by $a(S)$ the element $T_S(a)$ from $\mathcal{B}(H_S)$.

THEOREM 5.5. *Let A be a C^*-algebra isomorphic to the finite direct sum of n_k-homogeneous C^*-algebras A_{n_k}; $k = 1, \ldots, m$, with an identity. Then there exists a morphism of $(A, Z(A))$-bimodules $\rho : A \to A \underset{Z(A)}{\hat{\otimes}} A$ being as in Lemma 2.3.*

Proof. In view of [9] and [17] a n-homogeneous C^*-algebra has the property such that, for each point t_0 of the spectrum $\hat{A}_n$, there exists a neighborhood U_{t_0} of t_0 such that the C^*-algebra $A_n/U_{t_0} = \{a \in A_n : a(t) = 0 \text{ for all } t \in (\hat{A}_n - U_{t_0})\}$ is isomorphic to the C^*-tensor product of the C^*-algebras $C_0(U_{t_0})$ and $M_n(\mathbb{C})$ $C_0(U_{t_0})\tilde{\otimes}M_n(\mathbb{C})$ (see [1; 2.12.15]), where $C_0(U_{t_0})$ is the C^*-algebra of all continuous functions on U_{t_0} vanishing at infinity.

For $n_k \neq 1$ by the conditions of the theorem the spectrum $\hat{A}_{n_k}$ is compact, therefore there exists a finite open covering $\{U_i\}_{i=1}^{m_k}$ of the space $\hat{A}_{n_k}$ such that $A_{n_k}/U_i \cong C_0(U_i)\tilde{\otimes}M_{n_k}(\mathbb{C})$. Denote by h_i^k; $i = 1, \ldots, m_k$, $0 \leqslant h_i^k \leqslant 1$, a decomposition of the identity, subject to $\{U_i\}_{i=1}^{m_k}$ [3]. Put $g_i^k = (h_i^k)^{\frac{1}{4}}$.

Besides, note that for each element $b \in M_{n_k}(\mathbb{C})$ there exists an uniquely representation in the form $b = \sum_{i,j=1}^{n_k} \alpha_{ij}^{(k)} e_{ij}^{(k)}$, where $\alpha_{ij}^{(k)} \in \mathbb{C}$ and $e_{ij}^{(k)}$ is the matrix in $M_{n_k}(\mathbb{C})$ with 1 at the ij-th place and 0 elsewhere.

Let us define for each element $a_k \in A_{n_k}$; $k = 1, \ldots, m$, where $n_k \neq 1$, the element

$$\rho(a_k) = \sum_{i=1}^{m_k} a_k g_i^k \frac{1}{n_k} \left(\sum_{t,j=1}^{n_k} (g_i^k \otimes e_{tj}^{(k)}) \otimes ((g_i^k)^2 \otimes e_{jt}^{(k)}) \right)$$

in $A \underset{Z(A)}{\hat{\otimes}} A$. Using the way of the estimation of the norm of the element $\frac{1}{n_k} \sum_{i,j=1}^{n_k} e_{ij}^{(k)} \otimes$

$\otimes e_{ij}^{(k)}$ in $M_{n_k}(\mathbf{C}) \hat{\otimes} M_{n_k}(\mathbf{C})$ given in [7] we obtain $\|\rho(a_k)\| \leqslant m_k\|a\|$.

For $n_k = 1$ we have $A_1 = C(\hat{A}_1)$ [1; 1.4.1], and, consequently, $A \subset Z(A)$. We put for any element $a_1 \in A_1$

$$\rho(a_1) = \lim_{\mu} a_1 \otimes u_\mu \in A \underset{Z(A)}{\hat{\otimes}} A,$$

where u_μ; $\mu \in \Lambda$, is a bounded approximate identity of the C^*-algebra A_1 [1; 1.7.2].

Now, for each element $a = \sum_{k=1}^{m} a_k$ from $A = \bigoplus_{k=1}^{m} A_{n_k}$, we define the map $\rho : A \to$ $\to A \underset{Z(A)}{\hat{\otimes}} A$ by the formula

$$\rho(a) = \sum_{k=1}^{m} \rho(a_k).$$

For any $a, b \in A$, we can show that $b \cdot \rho(a) = \rho(ba) = \rho(b) \cdot a$ and since, in addition, $\|\rho(a)\| \leqslant \sum_{k=1}^{m} m_k\|a\| \rho$ is a morphism of $(A, Z(A))$-bimodules.

Furthermore, $(\pi \circ \rho)(a) = \sum_{k=1}^{m} \pi\rho(a_k) = \sum_{k=1}^{m} a_k = a$. Thus the theorem is proved. ■

REFERENCES

1. DIXMIER, J., *Les C^*-algebras et leurs representations*, 2nd ed. Paris, Gauthier – Villars, 1969.
2. CARTAN, H.; EILENBERG, S., *Homological algebra*, Princeton, New Jersey, Princeton University Press, 1956.
3. KELLEY, J. L., *General topology*, Van Nostrand, Princeton, New Jersey, 1957.
4. MAC LANE, S., *Homology*, Berlin, Springer–Verlag and New York, Academic Press, 1963.
5. SELIVANOV, Yu. V., *Some questions of homological classification of Banach algebras*, "Candidat" thesis, MSU publishers, Moscow, 1978.
6. HELEMSKII, A. Ya., *Homology in Banach and topological algebras*, MSU publishers, Moskow, 1986 (Russian); English, Dordrecht: Cluwer, 1989.
7. HELEMSKII, A. Ya., A certain method of computing and estimating the global homological dimension of Banach algebras, *Mat. Sb.*, **87(129)**(1972), 122–135 (Russian); *Math. USSR Sb.*, **16**(1972), 125–138.
8. EILENBERG, S., MORE, J. C., Foundation of the relative homological algebra, *Mem. Amer. Math. Soc.*, **55**(1965), 1–39.
9. FELL, J. M. G., The structure of algebras of operator fields, *Acta Math.*, **106**(1961), 233-280.
10. JOHNSON, B. E., Cohomology in Banach algebras, *Mem. Amer. Math. Soc.*, **127**(1972).
11. KAPLANSKY, I., The structure of certain operator algebras, *Trans. Amer. Math. Soc.*, **70**(1951), 219–255.
12. LIDDELL, M. J., Separable topological algebras I, *Trans. Amer. Math. Soc.*, **195**(1974), 31–59.

13. PHILLIPS, J.; RAEBURN, I., Perturbations of C^*-algebras II, *Proc. London Math. Soc.* (3), **43**(1981), 46–72.
14. PHILLIPS, J.; RAEBURN, I., Central cohomology of C^*-algebras, *J. London Math. Soc.* (2), **28**(1983), 363–372.
15. RICKART, C. E., *The general theory of Banach algebras*, van Nostrand, Princeton, New York, 1960.
16. RIEFFEL, M. A., Induced Banach representations of Banach algebras and locally compact groups, *J. Functional Analysis*, **1**(1967), 443–491.
17. TAKESAKI, M.; TOMIYAMA, J., Applications of fibre bundles to a certain class of C^*-algebras, *Tôhoku Math. J.*, **13**(1961), 498–522.

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