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HANKEL OPERATORS ON THE BERGMAN SPACE OF MULTIPLY-CONNECTED DOMAINS

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1. INTRODUCTION

Let Ω be a bounded multiply-connected domain in the complex plane $\mathbb{C}$ whose boundary $\partial\Omega$ consits of finitely many simple closed smooth analytic curves Γ_i, $(i = 1, 2, \ldots, n)$, where Γ_i $(i = 1, 2, \ldots, n)$ are positively oriented with respect to Ω and $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$. We further assume that Γ_1 is the boundary of the unbounded component of $\mathbb{C} \setminus \Omega$. Let D_1 be the bounded component of $\mathbb{C} \setminus \Gamma_1$ and D_i $(i = 1, 2, \ldots, n)$ be the unbounded component of $\mathbb{C} \setminus \Gamma_i$, respectively, so that $\Omega = \bigcap_1^n D_j$.

For $\mathrm{d}v = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$, we consider the usual L^2-space $L^2(\Omega) = L^2(\Omega, \mathrm{d}v)$. The Bergman space $H^2(\Omega)$, consisting of holomorphic L^2-functions, is a closed subspace of $L^2(\Omega)$. The Bergman projection P is the orthogonal projection from $L^2(\Omega)$ onto $H^2(\Omega)$ defined by

$$PF(z) = \int K(z, w) f(w) \mathrm{d}v(w),$$

where $K(z, w)$ is the Bergman reproducing kernel of $H^2(\Omega)$. For $f \in L^2(\Omega)$, the Hankel operator H_f from H^2 into L^2 is defined by $H_f g = (I - P)(fg)$. In general, H_f is only densely defined and may be unbounded.

In [3], D. Békollé, C. A. Berger, L. A. Coburn and K. H. Zhu proved that for the bounded symmetric domains D in $\mathbb{C}^n$, if $f \in L^2(D)$, then the Hankel operators $H_f, H_{\overline{f}}$ are bounded (compact) if and only if f has bounded (vanishing) mean oscillation on D (at the boundary ∂D). In this paper, using their theorems about the Hankel operators on the unit disk, we extend the results in [3] to the multiply-connected domains Ω given at the beginning of this paper. It is proved that for $f \in L^2(\Omega)$, the Hankel operators $H_f, H_{\overline{f}}$ are bounded if and only if f has bounded mean oscillation on Ω (i.e. $f \in \mathrm{BMO}(\Omega)$); the Hankel operators $H_f, H_{\bar{f}}$ are compact if and only if f has vanishing mean oscillation at the boundary $\partial\Omega$ (i.e. $f \in \mathrm{VMO}(\Omega)$). The mean oscillations will be defined in the next section. Since our theory is conformally invariant, by the well-known fact (see [7]) that any bounded multiply-connected domain whose boundary consists of finitely many simple closed smooth analytic curves is conformally equivalent to a *canonical bounded multiply-connected domain* whose boundary consists of finitely many circles, we only need to prove our theorems for the *canonical bounded multiply-connected domains.*

2. DEFINITIONS AND NOTATIONS

Let D be the bounded multiply-connected domain given at the beginning of Section 1, i.e. $D = \bigcap_1^n D_i$ with D_1 the bounded component of $\mathbb{C} \setminus \Gamma_1$ and D_j ($j = 2, 3, \dots, n$) the unbounded component of $\mathbb{C} \setminus \Gamma_j$. We will use Δ to denote the punctured disk $\Delta = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Let Ω be any one of the domains $D, \Delta, D_i (i = 1, \dots, n)$. Let $\partial\Omega$ denote the set of non-isolated boundary points of Ω. Following [3], for $k_a(z) = K(z,a)/K(a,a)^{\frac{1}{2}}$ we define the Berezin transform of $f \in L^2(\Omega)$ by $\tilde{f}(a) = \int_\Omega f(w)|k_a(w)|^2 \mathrm{d}v(w)$. Since $k_a(z)$ is bounded on Ω for each fixed $a \in \Omega$ (see [5, p. 59]) and $k_a(z) \in C^\infty(\Omega)$ as a function of a for each fixed $z \in \Omega$, $\tilde{f}$ is well defined for all $f \in L^1(\Omega)$ and $\tilde{f} \in C^\infty(\Omega)$.

DEFINITION 2.1. For $f \in L^2(\Omega)$, let $\mathrm{MO}(f,z) = (|f|^2)\tilde{}(z) - |\tilde{f}(z)|^2$. We say $f \in \mathrm{BMO}(\Omega)$ if $\mathrm{MO}(f,z)$ is bounded on Ω; we say $f \in \mathrm{VMO}(\Omega)$ if $\lim_{z \to \partial\Omega} \mathrm{MO}(f,z) = 0$, where by $z \to \partial\Omega$ we mean the distance function $d(z,\partial\Omega) = \inf\{|z-w| : w \in \partial\Omega\} \to 0$. For any $E \subset \Omega$, we write

$$\|f\|_{\mathrm{BMO}(E)} = \sup_{z \in E} \mathrm{MO}(f,z)^{\frac{1}{2}} \quad \text{and} \quad \|f\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}(\Omega)}.$$

REMARK. By direct computation, it's easy to see that

$$\mathrm{MO}(f,z) = \int_\Omega |f(z) - \tilde{f}(z)|^2 |k_z(w)|^2 \mathrm{d}v(w) =$$

$$= \frac{1}{2} \iint_\Omega |f(w) - f(u)|^2 |k_z(w)|^2 |k_z(u)|^2 \mathrm{d}v(w) \mathrm{d}v(u).$$

For $z \in \Omega$ and $\xi \in \mathbb{C}$, let $B_\Omega(z,\xi)$ be the infinitesimal form of the Bergman metric on Ω. For $z, w \in \Omega$, let $\beta_\Omega(z,w)$ be the Bergman distance between them. For details of the Bergman metric and the Bergman distance, see [11, p. 45]. We will use $E_\Omega(z,r)$ to denote the Bergman ball centered at $z \in \Omega$ with radius r, i.e. $E_\Omega(z,r) = \{w \in \Omega : \beta_\Omega(z,w) < r\}$.

For any fixed $r > 0$, let

$$\hat{f}(z) = \frac{1}{|E_\Omega(z,r)|} \int\limits_{E_\Omega(z,r)} f(w)\mathrm{d}v(w) \text{ and } \mathrm{MO}_r(f,z) = (|f - \hat{f}_r(z)|^2)\hat{}_r(z),$$

where $|E_\Omega(z,r)|$ is the usual Lebesgue measure of $E_\Omega(z,r)$. Then $\hat{f}_r(z)$ and $\mathrm{MO}_r(f,z)$ are continuous functions on Ω. Now, we can define the spaces $\mathrm{BMO}_r(\Omega)$ and $\mathrm{VMO}_r(\Omega)$:

DEFINITION 2.2. For $f \in L^2(\Omega)$, f is said to be in $\mathrm{BMO}_r(\Omega)$ provided that $\mathrm{MO}_r(f,z)$ is bounded on Ω; f is said to be in $\mathrm{VMO}_r(\Omega)$ provided that $\mathrm{MO}_r(f,z) \to$ $\to 0$ as $z \to \partial\Omega$.

For $f \in L^2(D)$, we define $f_i \in L^2(D_i)$ $(i = 1, 2, \ldots, n)$ by letting

$$f_i(z) = \begin{cases} f(z) & \text{if } z \in D \\ 0 & \text{if } z \in D_i \setminus D \end{cases}.$$

The notation H_{f_i} $(i = 1, \ldots, n)$ will be used to denote the Hankel operators from $H^2(D_i)$ into $L^2(D_i)$.

DEFINITION 2.3. Let $f \in H^2(\Omega)$, we say $f \in \mathcal{B}(\Omega)$ provided that $|f'(z) \cdot$ $\cdot d(z, \partial\Omega)| \in L^\infty(D)$; we say $f \in \mathcal{B}_0(\Omega)$ provided that $f'(z)d(z,\partial\Omega) \to 0$ as $z \to \partial\Omega$.

The letters C and M will be used to denote constants, they may change from line to line.

Now we can state our main theorems:

THEOREM A. *For $f \in L^2(D)$, the following statements are equivalent:*

1) *the Hankel operators H_f and $H_{\overline{f}}$ are bounded.*
2) $f \in \mathrm{BMO}(D)$.
3) $f \in \mathrm{BMO}_r(D)$ *for some* $r > 0$.
4) $f \in \mathrm{BMO}_r(D)$ *for all* $r > 0$.

THEOREM B. *For $f \in L^2(D)$, the following statements are equivalent:*

1) *the Hankel operators H_f and $H_{\overline{f}}$ are compact.*
2) $f \in \mathrm{VMO}(D)$.
3) $f \in \mathrm{VMO}_r(D)$ *for some* $r > 0$.
4) $f \in \mathrm{VMO}_r(D)$ *for all* $r > 0$.

THEOREM C. $\mathcal{B}(D) = \mathrm{BMO}(D) \cap H^2(D)$; $\mathcal{B}_0(D) = \mathrm{VMO}(D) \cap H^2(D)$.

In Section 3, we will give some estimates about the Bergman kernel and the Bergman metric on the canonical multiply-connected domain D; in Section 4, we will establish the relationships between the mean oscillations on D and the mean oscillations on D_i ($i = 1, 2, \ldots, n$); Section 5 is devoted to giving the relationships between the Hankel operators H_f and H_{f_i} ($i = 1, 2, \ldots, n$); in Section 6, we prove our main theorems for the canonical bounded multiply-connected domains; in Section 7, we will prove that our theory is conformally invariant and, consequently, get the main theorems for all the bounded multiply-connected domains which are conformally equivalent to one of our canonical multiply-connected domains.

3. THE ESTIMATES ABOUT THE BERGMAN KERNEL AND THE BERGMAN METRIC

From now on, we assume that D is a canonical bounded multiply-connected domain given in section 1 so that $D = \bigcap_1^n D_i$ with $D_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $D_j = \{z \in \mathbb{C} : |z - a_j| > r_j\}$ for $j = 2, 3, \ldots, n$, respectively. Here $a_j \in D_1$ and $0 < r_j < 1$ satisfying $|a_i - a_k| > r_i + r_k$ if $i \neq k$ and $1 - |a_j| > r_j$. We still use Δ to denote the punctured unit disk $D_1 \setminus \{0\}$. We will use $K^i(z, w)$, $K(z, w)$ and $K^0(z, w)$ to denote the Bergman kernels on D_i ($i = 1, 2, \ldots, n$), D and Δ, respectively. We write $k_w^i(z) = K^i(z, w)/K^i(w, w)^{\frac{1}{2}}$ for $i = 0, 1, \ldots, n$.

LEMMA 3.1. *There is an isometric isomorphism Ψ from $L^2(\Delta)$ onto $L^2(D_1)$ such that $\Psi(H^2(\Delta)) = H^2(D_1)$. For any $(z, w) \in \Delta \times \Delta$, $K^0(z, w) = K^1(z, w)$. The Bergman metric of Δ is the restriction to Δ of the Bergman metric of the unit disk D_1.*

Proof. See [10]. ■

Let $E(z, w) = K(z, w) - \sum_1^n K^i(z, w)$ for all $(z, w) \in D \times D$. We state some results about the localization of the Bergman kernel near the boundary of D. For details see [1,5].

LEMMA 3.2. (1) *$E(z, w) \in L^\infty(D \times D)$.* (2) *There are annuli $G_i \subset D$, $1 \leqslant i \leqslant n$, with $G_i \cap G_j = \varnothing$ if $i \neq j$ such that for any $(z, w) \in G_i \times D \cup D \times G_i$ we have $|K(z, w)| \leqslant C \cdot |K^i(z, w)|$, and $|K^i(z, w)| \leqslant |K(z, w)| + C$.* (3) *For $z \in D$, $|K^i(z, z)| \leqslant$ $\leqslant |K(z, z)|$. Where the annuli are given by $G_i = \{z : R_i > |z - a_i| > r_i\}$ for some $R_i > r_i$, $2 \leqslant i \leqslant n$, and $G_1 = \{z : R_1 < |z| < 1\}$ for some $R_1 < 1$.*

For any set $\Omega \in \mathbb{C}$ and any $\delta > 0$, we write $\Omega(\delta) = \{z \in \Omega : \mathrm{d}(z, \partial\Omega) < \delta\}$.

LEMMA 3.3. *For any $r > 0$, there are constants δ, C and M such that for*

$1 \leqslant i \leqslant n$, *we have*

(1) *For any* $z \in D_i(\delta)$ *and* $\xi \in \mathbb{C}$, $M^{-1} \cdot B_i(z,\xi) \leqslant B_D(z,\xi) \leqslant M \cdot d(z,\partial D)^{-1}|\xi|$;

(2) *If* $z \in D_i(\delta)$, *then* $E_i(z, C^2 \cdot r) \subset G_i$ *and* $E_D(z, r/C) \subset E_i(z,r) \subset E_D(z, C \cdot r)$;

where $B_i(z,\xi)$ *and* $B_D(z,\xi)$ *are the infinitesimal forms of the Bergman metrics on* D_i *and* D *respectively, and* $E_i(z,R)$ *and* $E_D(z,R)$ *are the Bergman metric balls with respect to the Bergman metrics of* D_i *and* D, *respectively;* $D_i(\delta) = \{z \in D_i : d(z, \partial D_i) < \delta\}$.

Proof. The results follow from Lemma 3.2 and the definition of the Bergman metric. ■

REMARK. If we replace r by any positive number $s < r$, Lemma 3.3 still holds with the same constants δ and C.

LEMMA 3.4. *For given* $r, s, \delta > 0$, *there is a constant* M *such that for any* $z \in D$ *we have*

(a) $M^{-1} \leqslant \dfrac{|E_i(z,r)|}{|E_i(z,s)|} \leqslant M \ (i = 1, 2, \ldots, n)$,

(b) *If* $z \in D_i \setminus D_i(\delta)$, *then* $|E_i(z,r)| \geqslant M^{-1} > 0$,

(c) $M^{-1} \leqslant \dfrac{|E_j(z,r)|}{|E_1(r_j/(z-a_j), r)|} \leqslant M \ (j = 2, 3, \ldots, n)$.

Proof. It is easy to check that (a) and (b) hold for $i = 1$ (see [4, Lemma 6]). Note that each D_i is biholomorphically equivalent to Δ and the Bergman metric is invariant under the biholomorphic mappings. By using Lemma 3.1 and the results for $i = 1$, other assertions follow from direct calculations. ■

Note that Theorem A and B hold for the unit disk [3]; and note that each D_i is biholomorphically equivalent to the punctured disk Δ. By using Lemma 3.1, Lemma 3.4 and the transform formula of the Bergman kernels under biholomorphic mappings, one can prove the following theorem by direct computation. For a sketch of the proof, see the proof of Theorem 7.1 except some necessary modifications.

THEOREM 3.5. *Theorem A and Theorem B hold for each domain* D_i, $1 \leqslant i \leqslant n$.

LEMMA 3.6. *For* $f \in H^2(D)$, *we can write it uniquely as* $f(z) = \sum_1^n (P_i f)(z) + (P_0 f)(z)$ *with* $P_i f \in H^2(D_i)$, $P_0 f(z) \in C^\infty(\overline{D}) \cap H^2(D)$, $P_i(P_j f) = 0$ *if* $i \neq j$. *Moreover, there exists a constant* M_1 *such that* $\|P_i f\|_D \leqslant \|P_i f\|_{D_i} \leqslant M_1 \cdot \|f\|_D$ *for* $i = 0, 1, \ldots, n$. *In particular, if* $f \in H^2(D_i)$, *then* $P_i(f) = f$ *and* $\|f\|_{D_i} \leqslant M_1 \cdot \|f\|_D$ *for* $i = 1, 2, \ldots, n$.

Proof. Let f be any function analytic in D. For any $z \in D$, let C_i $(i = 1, 2, \ldots, n)$ be the circles which center at a_i $(a_1 = 0)$ and lie in G_i respectively so that z is exterior to C_i for $1 = 2, \ldots, n$ and interior to C_1. Let

$$F_j(z) = \frac{1}{2\pi\sqrt{-1}} \cdot \int_{C_j} \frac{f(\xi)}{\xi - z} \mathrm{d}\xi, \ 1 \leqslant j \leqslant n.$$

By Cauchy's theorem, the $F_j(z)$ does not depend on the choice of C_j, $1 \leqslant j \leqslant n$. Obviously, each $F_j(z)$ is well defined for all $z \in D_j$ and analytic in D_j.

Define

$$P_1 f = F_1, \ P_j f(z) = F_j(z) - A_j \cdot (z - a_j)^{-1}, \ j = 2, 3, \ldots, n,$$

and

$$P_0 f(z) = \sum_{j=2}^{n} A_j \cdot (z - a_j)^{-1},$$

where $A_j = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{C_j} f(\xi) \mathrm{d}\xi$ for $j = 2, 3, \ldots, n$. The $f(z) = \displaystyle\sum_{i=0}^{n} P_i f(z)$ for all $z \in D$.

By direct computation, it is easy to check that the operators P_i defined in this way satisfy the conclusions of the lemma. ∎

LEMMA 3.7. *If $\{g_m\}$ is a bounded sequence in $H^2(D)$ and $g_m \to 0$ weakly on $H^2(D)$, then $P_0 g_m \to 0$ uniformly on D.*

Proof. By Lemma 3.6, the operator P_0 is bounded. Then $g_m \to 0$ weakly in $H^2(D)$ implies that $P_0 g_m \to 0$ weakly on $H^2(D)$ and then $P_0 g_m(z) \to 0$ at every point $z \in D$. By the definition of $P_0 g_m$ it is easy to check that the boundedness of $\{\|g_m\|_D\}$ implies that the family of continuous functions $\{P_0 g_m(z)\}$ is uniformly bounded and equicontinuous on $\overline{D}$. By Arzela–Ascoli theorem, it follows that $P_0 g_m \to 0$ uniformly on D. ∎

THEOREM 3.8. [3] *For $f \in \mathrm{BMO}(D)$, there is a constant M_2 such that for any C^1 curve $\gamma : [0, 1] \to D$ we have*

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} (\tilde{f}(\gamma(t))) \right| \leqslant M_2 \cdot \mathrm{MO}(f, \gamma(t))^{1/2} \cdot B_D(\gamma(t), \gamma'(t)).$$

Combining this with Lemma 3.3 we have

COROLLARY 3.9. *If $f \in \mathrm{BMO}(D)$, then for each $\varepsilon > 0$ there is an $M(\varepsilon) > 0$ such that*

$$|\tilde{f}(z)| \leqslant M(\varepsilon) \cdot d(z, \partial D)^{-\varepsilon}.$$

4. BMO(Ω) AND VMO(Ω)

In this section, we establish relationships among $\mathrm{MO}(f,z)$, $\mathrm{MO}(f_i,z)$ and relationships among $\mathrm{MO}_r(f,z)$, $\mathrm{MO}_r(f_i,z)$ for $i=1,2,\dots,n$. Before going on, recall that $G_1=\{z\in D: R_1<|z|<1\}$, $G_i=\{z\in D: r_i<|z-a_i|<R_i\}$ for $i=2,\dots,n$ and $G_i\cap G_j=\varnothing$ if $i\neq j$. For convenience we will write $a_1=0$ and $r_1=1$.

THEOREM 4.1. *Let $f\in L^2(D)$. If $f_i\in L^2(D_i)$ $(i=1,2,\dots,n)$ are the same as that defined in Section 2, then $f\in \mathrm{BMO}(D)$ if and only if $f_i\in\mathrm{BMO}(D_i)$ for all $i=1,2,\dots,n$.*

Proof. By Lemma 3.2, $|k_\lambda(z)|^2\leqslant C\cdot|k_\lambda^1(z)|^2$ for $\lambda\in G_1$. Hence, we have

$$
\begin{aligned}
2\cdot\mathrm{MO}(f,\lambda)&=\iint\limits_{D\times D}|f(z)-f(w)|^2\cdot|k_\lambda(z)|^2\cdot|k_\lambda(w)|^2\mathrm{d}v(z)\mathrm{d}v(w)\leqslant\\
&\leqslant C^2\cdot\iint\limits_{D_1\times D_1}|f_1(z)-f_1(w)|^2\cdot|k_\lambda^1(z)|^2\cdot|k_\lambda^1(w)|^2\mathrm{d}v(z)\mathrm{d}v(w)=\\
&=C^2\cdot 2\,\mathrm{MO}(f_1,\lambda).
\end{aligned}\tag{4.1}
$$

By the same reason, we can prove that for λ in G_i $(i=2,3,\dots,n)$,

$$
\mathrm{MO}(f,\lambda)\leqslant C^2\cdot\mathrm{MO}(f_i,\lambda).\tag{4.2}
$$

Therefore, $\mathrm{MO}(f,\lambda)\leqslant C^2\max\{\mathrm{MO}(f_i,\lambda)\}$ for any $\lambda\in G=\bigcup\limits_1^n G_i$. It is obvious that $\mathrm{MO}(f,\lambda)$ is bounded on $K=D\setminus G$. Thus, $f\in\mathrm{BMO}(D)$ if $f_i\in\mathrm{BMO}(D_i)$ for all $i=1,\dots,n$.

To prove the necessity, we assume $f\in\mathrm{BMO}(D)$. For $\lambda\in D_1$, note that

$$
\begin{aligned}
2\cdot\mathrm{MO}(f_1,\lambda)&=\iint\limits_{D_1\times D_1}|f_1(z)-f_1(w)|^2\cdot|k_\lambda^1(z)|^2\cdot|k_\lambda^1(w)|^2\mathrm{d}v(z)\mathrm{d}v(w)=\\
&=\iint\limits_{D\times D}|f(z)-f(w)|^2\cdot|k_\lambda^1(z)\cdot k_\lambda^1(w)|^2\mathrm{d}v(z)\mathrm{d}v(w)+\\
&\quad+2\cdot\iint\limits_{(D_1\setminus D)\times D}|f(w)\cdot k_\lambda^1(z)\cdot k_\lambda^1(w)|^2\mathrm{d}v(z)\mathrm{d}v(w).
\end{aligned}\tag{4.3}
$$

By Lemma 3.2, if $\lambda\in G_1$, then there is a constant C such that

$$
|k_\lambda^1(z)|\leqslant C\cdot[|k_\lambda(z)|+K^1(\lambda,\lambda)^{-1/2}].\tag{4.4}
$$

It is easy to see that for $z\in D_1\setminus D$, there is a constant M' such that

$$
|k_\lambda^1(z)|\leqslant M'\cdot(1-|\lambda|).\tag{4.5}
$$

Combining (4.3) to (4.5) and noting that $|a+b|^2 \leqslant 2\cdot(|a|^2+|b|^2)$, we have

$$2\cdot \mathrm{MO}(f_1,\lambda) \leqslant C\cdot[\mathrm{MO}(f,\lambda)+(1-|\lambda|)^2\cdot(|f|^2)\tilde{}(\lambda)+(1-|\lambda|)^2]. \tag{4.6}$$

Note that $(|f|^2)\tilde{}(\lambda) = \mathrm{MO}(f,\lambda)+|\tilde{f}(\lambda)|^2$. By Corollary 3.9, it follows that

$$(|f|^2)\tilde{}(\lambda) \leqslant \mathrm{MO}(f,\lambda)+M\cdot(1-|\lambda|)^{-1/2}. \tag{4.7}$$

Applying (4.7) to (4.6) yields that

$$2\cdot \mathrm{MO}(f_1,\lambda) \leqslant C\cdot[\mathrm{MO}(f,\lambda)+(1-|\lambda|)] \tag{4.8}$$

for all $\lambda \in G_1$. It is obvious that $\mathrm{MO}(f_1,z)$ is bounded on $D_1\setminus G_1$. Therefore, we get the boundedness of $\mathrm{MO}(f_1,z)$ on D_1, i.e. $f_1\in \mathrm{BMO}(D_1)$.

Similarly, we can prove that

$$2\cdot \mathrm{MO}(f_i,\lambda) \leqslant C\cdot[\mathrm{MO}(f,\lambda)+(|\lambda-a_i|-r_i)] \tag{4.9}$$

whenever $\lambda\in G_i$ and $\mathrm{MO}(f_i,\lambda)$ is bounded on $D_i\setminus G_i$ for $i=2,3,\ldots,n$. Thus, $f_i\in\mathrm{BMO}(D_i)$ when $f\in\mathrm{BMO}(D)$. ∎

THEOREM 4.2. *For $f\in L^2(D)$, let $f_i\in L^2(D_i)$ $(i=1,\ldots,n)$ as defined in Section 2. Then $f\in\mathrm{VMO}(D)$ if and only if $f_i\in\mathrm{VMO}(D_i)$ for all $i=1,2,\ldots,n$.*

Proof. The "if" part comes from (4.1) and (4.2); the "only if" part comes from (4.8) and (4.9) in the proof of Theorem 4.1. ∎

THEOREM 4.3. *For any fixed $r>0$, $f\in\mathrm{BMO}_r(D)$ if and only if $f_i\in$ $\in\mathrm{BMO}_r(D_i)$ for all $i=1,2,\ldots,n$.*

Proof. For each fixed $r>0$, let δ be the same as that given in Lemma 3.3. To prove the sufficiency, for $\lambda\in D$ and $E_D=E_D(\lambda,r)$ we consider (see [4])

$$2\cdot\mathrm{MO}_r(f,\lambda)=|E_D(\lambda,r)|^{-2}\cdot\int_{E_D\times E_D}|f(z)-f(w)|^2\mathrm{d}v(z)\mathrm{d}v(w).$$

If $\lambda\in D_1(\delta)$, by Lemma 3.3 and Lemma 3.4 one has $E_D\subset E_1(\lambda,C\cdot r)$ and

$$|E_D(\lambda,r)|^{-2}\leqslant M\cdot|E_1(\lambda,C\cdot r)|^{-2}.$$

Let $R=C\cdot r$ and $E_1=E_1(\lambda,R)$, then

$$\begin{aligned}2\cdot\mathrm{MO}_r(f,\lambda)&\leqslant M\cdot|E_1(\lambda,R)|^{-2}\cdot\int_{E_1\times E_1}|f_1(z)-f_1(w)|^2\mathrm{d}v(z)\mathrm{d}v(w)=\\&=M\cdot 2\cdot\mathrm{MO}_R(f_1,\lambda).\end{aligned} \tag{4.10}$$

By the same reason, for $\lambda \in D_i(\delta)$, it follows that

$$\mathrm{MO}_r(f,\lambda) \leqslant M \cdot \mathrm{MO}_R(f_i,\lambda) \quad \text{for } i = 2,\ldots,n. \tag{4.11}$$

It is obvious that $\mathrm{MO}_r(f,\lambda)$ is bounded on $D \backslash D(\delta)$. Therefore, $f_i \in \mathrm{BMO}_R(D_i)$ for all $1 \leqslant i \leqslant n$ implies that $f \in \mathrm{BMO}_r(D)$.

In order to prove the necessity, first we prove $f_1 \in \mathrm{BMO}_r(D_1)$. For $\lambda \in D_1(\delta)$, let $s = \dfrac{r}{C}$ and $R = C \cdot r$. For $E_1 = E_1(\lambda, s)$, we consider

$$2\,\mathrm{MO}_s(f_1,\lambda) = |E_1(\lambda,s)|^{-2} \cdot \int\limits_{E_1\times E_1} |f_1(z) - f_1(w)|^2 \mathrm{d}v(z)\mathrm{d}v(w).$$

By Lemma 3.3 and Lemma 3.4, it follows that $E_1 \subset E_D = E_D(\lambda, r)$, and

$$|E_1(\lambda,s)|^{-2} \leqslant M \cdot |E_D(\lambda,r)|^{-2}.$$

Thus,

$$2\cdot \mathrm{MO}_s(f_1,\lambda) \leqslant M\cdot|E_D(\lambda,r)|^{-2}\cdot \int\limits_{E_D\times E_D} |f(z)-f(w)|^2 \mathrm{d}v(z)\mathrm{d}v(w) = 2\cdot M\cdot \mathrm{MO}_r(f,\lambda),$$

i.e.

$$\mathrm{MO}_s(f_1,\lambda) \leqslant M \cdot \mathrm{MO}_r(f,\lambda). \tag{4.12}$$

Since $D_1 \setminus D_1(\delta)$ is compact, the boundedness of $\mathrm{MO}_s(f_1,\lambda)$ on it comes from the continuity of $\mathrm{MO}_s(f_1,\lambda)$. Thus, $f \in \mathrm{BMO}_r(D)$ implies that $f_1 \in \mathrm{BMO}_s(D_1)$. An application of Theorem A in [3] produces that $f_1 \in \mathrm{BMO}_r(D_1)$.

By the same argument, it follows that if $\lambda \in D_i(\delta)$ for some $i = 2,\ldots,n$, then

$$\mathrm{MO}_s(f_i,\lambda) \leqslant M \cdot \mathrm{MO}_r(f,\lambda). \tag{4.13}$$

However, for $\lambda \in D_i \setminus D_i(\delta)$, by Lemma 3.4 we have $|E_i(\lambda,r)| \geqslant C(\delta) > 0$ for some constant $C(\delta)$ depending on δ. From the definition of $\mathrm{MO}_s(f_i,\lambda)$ we get that $\mathrm{MO}_s(f_i,\lambda)$ is bounded on $D_i \setminus D_i(\delta)$. Hence, $f \in \mathrm{BMO}_r(D)$ implies that $f_i \in \mathrm{BMO}_s(D_i)$ for all $i = 1,2,\ldots,n$. By Theorem 3.5, we have $f_i \in \mathrm{BMO}_s(D_i)$ if and only if $f_i \in \mathrm{BMO}_r(D_i)$ for any $r, s > 0$. This completes the proof. ∎

THEOREM 4.4. *For any fixed $r > 0$, $f \in \mathrm{VMO}_r(D)$ if and only if $f_i \in$ $\in \mathrm{VMO}_r(D_i)$ for all $i = 1,2,\ldots,n$.*

Proof. By Theorem 3.5, the "if" part comes from (4.10) and (4.11), the "only if" part comes from (4.12) and (4.13). ∎

5. THE BOUNDEDNESS AND COMPACTNESS OF HANKEL OPERATORS ON $H^2(\Omega)$

In this section, we establish relationships among the Hankel operators H_f, H_{f_i} $(i = 1, 2, \dots, n)$. We will use the notations and definitions given before except that we will write $K^0(z, w) \equiv E(z, w)$.

DEFINITION 5.1. For $f \in L^2(D)$, we define the operators T_{mk} from $H^2(D)$ into $L^2(D)$ by

$$T_{mk}g(z) = \int_D (f(z) - f(w)) \cdot (P_m g)(w) \cdot K^k(z, w) \mathrm{d}v(w)$$

for all $g \in H^2(D)$ $(m, k = 0, 1, 2, \dots, n)$.

DEFINITION 5.2. We define the operators Q_i $(i = 0, 1, \dots, n)$ from $L^2(D)$ to $L^2(D)$ by

$$Q_i f(z) = \int_D f(w) \cdot |K^i(z, w)| \mathrm{d}v(w).$$

LEMMA 5.3. *Q_i $(i = 1, 2, \dots, n)$ are bounded operators on $L^2(D)$.*

Proof. We claim that there is a constant M such that for each $1 \leqslant i \leqslant n$,

$$\int_D |K^i(z, w)| \cdot K^i(w, w)^{1/4} \mathrm{d}v(w) \leqslant M \cdot K^i(z, z)^{1/4} \text{ for all } z \in D$$

and

$$\int_D |K^i(z, w)| \cdot K^i(z, z)^{1/4} \mathrm{d}v(z) \leqslant M \cdot K^i(w, w)^{1/4} \text{ for all } w \in D.$$

It suffices to prove the first inequality for $i = 1$: Let $\lambda = \dfrac{w - z}{1 - \bar{z} \cdot w}$, then

$$\int_D |K^1(z, w)| \cdot K^1(w, w)^{1/4} \mathrm{d}v(w) \leqslant \int_\Delta \frac{1}{|1 - \bar{z} \cdot w|^2 \cdot (1 - |w|^2)^{1/2}} \mathrm{d}v(w) =$$

$$= (1 - |z|^2)^{-1/2} \cdot \int_\Delta \frac{1}{|1 - \bar{z} \cdot \lambda| \cdot (1 - |\lambda|^2)^{1/2}} \mathrm{d}v(\lambda).$$

By [2, Lemma 4], the integral in the last equation is bounded by a constant M. Hence, we get

$$\int_D |K^1(z, w)| \cdot K^1(w, w)^{1/4} \mathrm{d}v(w) \leqslant M \cdot K^1(z, z)^{1/4}.$$

By the symmetry of z and w, this completes the proof of our claim. An application of Schur's theorem [8] gives the boundedness of the operators Q_i. ■

THEOREM 5.4. *For $f \in L^2(D)$, the Hankel operator H_f is bounded if and only if all T_{mm} are bounded; H_f is compact if and only if all T_{mm} are compact, $1 \leqslant m \leqslant n$.*

Proof. By definition,

$$H_f g(z) = \int_D (f(z) - f(w)) \cdot g(w) \cdot K(z,w) dv(w) =$$

(by Lemma 3.2 and Lemma 3.6)

$$= \int_D (f(z) - f(w)) \cdot \sum_0^n (P_m g)(w) \sum_0^n K^i(z,w) dv(w) = \sum_0^n \sum_0^n T_{mi} g(z). \tag{5.1}$$

Recall that, in this section, we use $K^0(z,w)$ to denote $E(z,w)$.

Now, we claim that $T_0(g) = \left(\sum_0^n T_{0i}\right)(g) = \sum_0^n T_{0i}(g)$ is compact from $H^2(D)$ into $L^2(D)$ and, for $m \neq i$, $m \neq 0$, the operators T_{mi} are compact from $H^2(D)$ to $L^2(D)$.

For the case $i = 0$, since $K^0(z,w) = E(z,w) \in L^\infty(D \times D)$, for any $0 \leqslant m \leqslant n$ it is clear that the operator T_{m0} is compact.

Now we prove that T_0 is compact. By the definition of T_0 we have

$$(*) \quad \begin{aligned} T_0(g)(z) &= \int_D (f(z) - f(w)) \cdot P_0 g(w) \cdot K(z,w) dv(w) = f(z) \cdot P_0(g)(z) - \\ &- \sum_1^n \int_D f(w) \cdot P_0 g(w) \cdot K^i(z,w) dv(w) - \int_D f(w) \cdot P_0 g(w) K^0(z,w) dv(w). \end{aligned}$$

Let $\{g_j\}$ be any bounded sequence in $H^2(D)$ such that $g_j \to 0$ weakly. By Lemma 3.2, Lemma 3.7 and Lemma 5.3 it follows that, after we replace g by g_j in $(*)$, the L^2-norm of each term on the right side of $(*)$ approaches to zero as $j \to \infty$. Consequently, $\|T_0(g_j)\| \to 0$. Thus, $T_0 = \sum_0^n T_{0i}$ is a compact operator.

For the cases of $0 \neq m \neq i \neq 0$, let $E_1 = G_i$ and $E_2 = D \setminus G_i$. Note that

$$\begin{aligned} T_{mi}(g)(z) &= \int_D (f(z) - f(w)) \cdot K^i(z,w) \cdot (P_m g)(w) dv(w) = \\ &= \sum_{j=1}^2 \left[\int_D \chi_{E_j}(w) \cdot f(z) \cdot K^i(z,w) \cdot (P_m g)(w) dv(w) - \right. \\ &\left. - \int_D \chi_{E_j}(w) \cdot f(w) \cdot K^i(z,w) \cdot (P_m g)(w) dv(w) \right], \end{aligned}$$

where χ_{E_j} $(j = 1, 2)$ is the characteristic function of E_j.

By expanding $P_m g(w)$ and $K^i(z,w)$ as series on $\overline{E_1} \cup (\mathbb{C} \backslash D_i)$ and E_i, respectively, one has

$$\int_D \chi_{E_1}(w) \cdot f(z) \cdot K^i(z,w) \cdot (P_m g)(w) \mathrm{d}v(w) = 0.$$

Therefore,

$$T_{mi} g(z) = \int_D \chi_{E_2}(w) \cdot f(z) \cdot K^i(z,w) \cdot (P_m g)(w) \mathrm{d}v(w) +$$

$$+ \sum_{j=1}^{2} \int_D -\chi_{E_j}(w) \cdot f(w) \cdot K^i(z,w) \cdot (P_m g)(w) \mathrm{d}v(w).$$

Let's denote the terms in the last equation by $S(g)(z)$, $T_1(g)(z)$ and $T_2(g)(z)$.

Since $-\chi_{E_2}(w) \cdot f(w) \cdot K^i(z,w) \in L^2(D \times D)$ and $\chi_{E_2}(w) \cdot f(z) \cdot K^i(z,w) \in L^2(D \times \times D)$, it follows that S and T_2 are compact operators from $H^2(D)$ to $L^2(D)$. Considering the operator T_1, let $\{g_k\}$ be any bounded sequence in $H^2(D)$ such that $g_k \to 0$ weakly on $H^2(D)$. Then $P_m g_k \to 0$ weakly on $H^2(D_m)$ and $\{||P_m g_k||_{D_m}\}$ are bounded by Lemma 3.6. It is well-known [5,11] that the boundedness of $\{||P_m g_k||\}$ implies that $\{P_m g_k(z)\}$ is uniformly bounded on any compact subset of D_m. Therefore, $\{P_m g_k\}$ is a normal sequence. Since $P_m g_k \to 0$ weakly implies that $P_m g_k(z) \to 0$ at every point $z \in D_m$, $P_m g_k \to 0$ uniformly on any compact subset of D_m and consequently on $\overline{E_1} = \overline{G_i}$. Note

$$|T_1 g_k(z)| \leqslant \operatorname{Sup}\{|P_m g_k(w)| : w \in E_1\} \cdot |Q_i(|\chi_{E_1} \cdot f|)(z)|. \tag{5.2}$$

An application of Lemma 5.3 to (5.2) yields that

$$||T_1 g_k||_D \leqslant \operatorname{Sup}\{|P_m g_k(w)| : w \in E_1\} \cdot M \cdot ||f||_D \to 0 \text{ as } k \to \infty.$$

Therefore, T_1 is compact from $H^2(D)$ to $L^2(D)$.

By now we proved that T_{mi} are compact operators from $H^2(D)$ to $L^2(D)$ if $m \neq i$, $m \neq 0$ and $i \neq 0$. This completes the proof of our claim.

Now we can complete the proof of our theorem. If T_{mm} are bounded (or compact), by the results proved above, from (5.1) it follows that H_f is bounded (or compact). If H_f is bounded (or compact), applying Lemma 3.6 to (5.1) we get

$$H_f(P_i(g))(z) = \sum_{j=0}^{n} (T_{ij} g)(z) \text{ for } i = 1, 2, \ldots, n. \tag{5.3}$$

Since the boundedness (compactness) of H_f implies the boundedness (compactness) of $H_f P_i$ and since $T_{i,k}$ are compact if $i \neq k$, from (5.3) we see that T_{ii} are bounded (or compact) for $i = 1, 2, \ldots, n$. ■

THEOREM 5.5. *For $f \in L^2(D)$ and each $1 \leqslant m \leqslant n$, T_{mm} is bounded (compact) if and only if H_{f_m} is bounded (compact).*

Proof. It suffices to prove the theorem for $m = 1$. For convenience, we will use the same notation g to denote $g|_D$ for $g \in H^2(D_1)$. Let χ_Ω be the characteristic function of Ω. Note that

$$H_{f_1} g(z) = \int_{D_1} (f_1(z) - f_1(w)) \cdot K^1(z,w) \cdot g(w) \mathrm{d}v(w) =$$

$$= \chi_D(z) \cdot T'_{1,1} g(z) + \chi_D(z) \cdot \int_{D_1} f(z) \cdot \chi_{D_1 \setminus D}(w) \cdot K^1(z,w) g(w) \mathrm{d}v(w) - \tag{5.4}$$

$$-\chi_{D_1 \setminus D}(z) \cdot \int_{D_1} \chi_D(w) \cdot f(w) \cdot K^1(z,w) g(w) \mathrm{d}v(w),$$

where $T'_{1,1} g(z) \equiv \int_D (f(z) - f(w)) K^1(z,w) g(w) \mathrm{d}v(w)$ is an operator from $H^2(D_1)$ into $L^2(D_1)$. If we denote the last two terms in (5.4) by $T_4 g(z)$ and $T_5 g(z)$, since both $\chi_D(z) \cdot \chi_{D_1 \setminus D}(w) \cdot f(z) \cdot K^1(z,w)$ and $\chi_{D_1 \setminus D}(z) \cdot \chi_D(w) \cdot f(w) \cdot K^1(z,w)$ are in $L^2(D_1 \times D_1)$, it follows that T_4 and T_5 are compact operators from $H^2(D_1)$ to $L^2(D_1)$. Hence, H_{f_1} is bounded (compact) if and only if $\chi_D(z) \cdot T'_{1,1} g(z)$ is bounded (compact) from $H^2(D_1)$ to $L^2(D_1)$. For any $g \in H^2(D_1)$, we have (see Lemma 3.6) $P_1 g = g$ and

$$\|\chi_D \cdot T'_{1,1} g\|_{D_1} = \|T'_{1,1} g\|_D, \quad \|g\|_D \leqslant \|g\|_{D_1} \leqslant M \cdot \|g\|_D.$$

Therefore, $\chi_D \cdot T'_{1,1}$ is bounded (compact) from $H^2(D_1)$ to $L^2(D_1)$ if and only if $T_{1,1} = \chi_D \cdot T'_{1,1} \cdot P_1$ is bounded (compact) from $H^2(D_1)$ to $L^2(D_1)$. The proofs for $m = 2, 3, \ldots, n$ are the same as for $m = 1$. ■

6. THE PROOFS OF THE MAIN THEOREMS ON THE CANONICAL DOMAINS

THEOREM A. *If $f \in L^2(D)$, then the following statements are equivalent:*

1) *The Hankel operators $H_f, H_{\overline{f}}$ are bounded,*
2) $f \in \mathrm{BMO}(D)$,
3) $f \in \mathrm{BMO}_r(D)$ *for some $r > 0$,*
4) $f \in \mathrm{BMO}_r(D)$ *for all $r > 0$,*

5) *The Hankel operators* $H_{f_i}, H_{\overline{f_i}}$ *are bounded from* $H^2(D_i)$ *into* $L^2(D_i)$ *for all* $i = 1, 2, \ldots, n$, *respectively.*

Proof. The equivalence of 1) and 5) comes from Theorem 5.4 and Theorem 5.5; using Theorem 4.1 and Theorem 3.5 we get the equivalence of 2) and 5); the equivalence of 3) and 5) comes from Theorem 4.3 and Theorem 3.5; the equivalence of 3) and 4) follows from Theorem 4.3 and Theorem 3.5. ∎

THEOREM B. *If* $f \in L^2(D)$, *then the following are equivalent:*

1) $H_f, H_{\overline{f}}$ *are compact,*

2) $f \in \mathrm{VMO}(D)$,

3) $f \in \mathrm{VMO}_r(D)$ *for some* $r > 0$,

4) $f \in \mathrm{VMO}_r(D)$ *for all* $r > 0$,

5) $H_{f_i}, H_{\overline{f_i}}$ *are compact from* $H^2(D_i)$ *to* $L^2(D_i)$ *for all* $i = 1, 2, \ldots, n$, *respectively.*

Proof. Combining Theorem 5.4 and Theorem 5.5, we get the equivalence of 1) and 5); the equivalence of 2) and 5) comes from Theorem 4.2 and Theorem 3.5; the equivalence of 3) and 5) comes from Theorem 4.4 and Theorem 3.5; the equivalence of 3) and 4) follows from Theorem 4.4 and Theorem 3.5. ∎

THEOREM C. $\mathcal{B}(D) = \mathrm{BMO}(D) \cap H^2(D) = \mathrm{BMO}_r(D) \cap H^2(D)$;

$$\mathcal{B}_0(D) = \mathrm{VMO}(D) \cap H^2(D) = \mathrm{VMO}_r(D) \cap H^2(D).$$

Proof. The results follow from Theorem A, B and [1, Theorem 4.2]. ∎

7. INVARIANCE OF THE HANKEL OPERATORS AND THE MEAN OSCILLATION

Throughout this section, we will use Ω and D to denote two conformally equivalent bounded domains with smooth boundaries in the complex plane $\mathbb{C}$; Ψ will be a one-to-one conformal mapping from Ω onto D. For any $f \in L^2(D)$, we define a function F on Ω by $F(z) = f(\Psi(z))$. It is obvious that $F \in L^2(\Omega)$ and $\Gamma(f) = F \cdot \Psi'$ is an isometric isomorphism from $L^2(D)$ onto $L^2(D)$ with $\Gamma(H^2(D)) = H^2(\Omega)$. The Bergman kernels on D and Ω will be denoted by $K^D(\cdot,\cdot)$ and $K^\Omega(\cdot,\cdot)$, respectively.

THEOREM 7.1. *Theorem* A *and* B *hold on* D *if and only if Theorem* A *and* B *hold on* Ω.

Proof. By using the transform formula of the Bergman kernels under the biholomorphic mappings it is easy to check that for any $f \in L^2(D)$ and $z \in \Omega$, $\mathrm{MO}(F, z) = \mathrm{MO}(f, \Psi(z))$, and that if we use H_f and H_F to denote the Hankel operators on $H^2(D)$ and $H^2(\Omega)$, respectively, then $H_f = \Gamma^{-1} \circ H_F \circ \Gamma$. Note that the Bergman metrics are invariant under the biholomorphic mappings. Since Ψ and Ψ^{-1} have smooth extensions to the boundaries of Ω and D respectively, it follows that $C^{-1}\,\mathrm{MO}_r(f,\Psi(z)) \leqslant \mathrm{MO}_r(F,z) \leqslant C\cdot \mathrm{MO}_r(f,\Psi(z))$ for any $r>0$ and $z\in\Omega$. Therefore, our results hold. ■

Let Ω be any bounded multiply-connected domain whose boundary consists of finitely many simple closed smooth analytic curves. It is well-known [7, p. 237–238] that such a domain Ω is conformally equivalent to one of our canonical domains. Therefore the main theorems about the Hankel operators hold for Ω.

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