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ANALITICITY IN TRIANGULAR ALGEBRAS

BARUCH SOLEL and BELISARIO A. VENTURA

1. INTRODUCTION

In this paper we consider the relationship between two classes of non self-adjoint subalgebras of an AF algebra: the strongly maximal triangulars and the analytic triangulars. The specific examples that are given concern UHF algebras.

Let $\mathcal{A}$ be an AF (resp. UHF) algebra with Strătilă-Voiculescu masa $\mathcal{D}$. A subalgebra $\mathcal{T}$ of $\mathcal{A}$ is called triangular (TAF) (resp. TUHF) if $\mathcal{T} \cap \mathcal{T}^* = \mathcal{D}$, and is called strongly maximal triangular if it is triangular and $\mathcal{T} + \mathcal{T}^*$ is dense in $\mathcal{A}$ (cf. [7, §3], [2, Theorem 4.2] and [12, Theorem 2.1]). If R is the AF groupoid associated with $\mathcal{A}$ and $\mathcal{D}$, and X is the unit space of R, then corresponding to a strongly maximal triangular subalgebra $\mathcal{T}$ of $\mathcal{A}$, there is an open subset P of R such that $\mathcal{T} = C(P)$, the collection of all elements of $\mathcal{A}$ that are supported on P. P is a partial order on every equivalence class, that is, $P \circ P \subseteq P$, $P \cap P^{-1} = X$ and $P \cup P^{-1} = R$ ([2, Theorem 4.2]).

Let d be a continuous 1-cocycle on R with values in $\mathbb{R}$, the set of real numbers, and let α be the one parameter group of automorphisms of $\mathcal{A}$ induced by d (cf. [9, Proposition II.5.1]). The analytic subalgebra $\mathcal{A}_\alpha$ of $\mathcal{A}$ associated with α is defined by

$$\mathcal{A}_\alpha = \overline{\{A \in C_c(R) : d \text{ is non-negative on the support of } \mathcal{A}\}}$$

where $C_c(R)$ is the collection of elements of $\mathcal{A}$ with compact support in R. Equivalently, $\mathcal{A}_\alpha = \mathcal{A}^\alpha([0,\infty))$, the α-spectral subspace of $\mathcal{A}$ corresponding to $[0,\infty)$ (cf. [5, Chapter 8]).

If $\mathcal{A}_\alpha$ is triangular, then $\mathcal{A}_\alpha$ is strongly maximal triangular (cf. [13, Theorem 5.2]), and in this case the open subset of $\mathbb{R}$ corresponding to $\mathcal{A}_\alpha$ is $d^{-1}([0,\infty))$. We show that the converse is not true, by constructing explicit examples of non-analytic strongly maximal TUHF algebras. Moreover, we show that any UHF algebra that is not finite-dimensional contains a non-analytic TUHF algebra. These results parallel similar ones obtained by P. Muhly, K.-S. Saito and B. Solel in the context of triangular subalgebras of a von Neumann algebra admiting a Cartan subalgebra (cf. [3], [4]).

We also consider a family of examples of strongly maximal TUHF algebras introduced by J. Peters, Y. Poon and B. Wagner ([7, Theorem 4.5] and [6, Theorem 2.24]). We use these examples to construct non-analytic maximal nest UHF algebras. Therefore we prove that among the strongly maximal TUHF algebras, there are nest algebras that are not analytic. Recall that in the context of non self-adjoint subalgebras of von Neumann algebras the nest subalgebras are not only analytic but they also correspond to inner flows (cf. [1, Theorem 4.2.3], [4, Corollary 3.4]).

The next section establishes the notation and the results that are needed for the construction of non-analytic strongly maximal TUHF algebras. These results concern analytic TAF algebras that correspond only to unbounded cocycles. Their properties play a crucial role in the criteria for non-analyticity (Theorem 3.1) that we establish for strongly maximal TUHF algebras in Section 3. In that section we use those criteria to analyze the examples and prove the result described in the two previous paragraphs.

2. ANALYTIC TAF'S AND UNBOUNDED COCYCLES

In this section we consider certain analytic TAF algebras that correspond only to unbounded cocycles. We begin by introducing the notation and terminology.

Let $\mathcal{A}$ be an AF algebra with diagonal $\mathcal{D}$ and let $\mathbb{R}$ be the AF groupoid associated with $\mathcal{A}$. We denote by X the unit space of R. Recall that $X = \hat{\mathcal{D}}$, the Gelfand spectrum of the diagonal. Recall also that if $\mathcal{A}$ is UHF, then R is minimal and that every point $u \in X$ has a dense orbit (cf. [9, p. 128]).

We write $[\omega]$ for the equivalence class of $\omega \in X$, and given an R-set $t \subseteq R$, we denote by $r(t)$ the range of t, that is

$$r(t) = \{\omega \in X : (\omega, u) \in t \text{ for some } u \in X\}.$$

Given $\mathcal{A}$ and $\mathcal{D}$ we can always choose a set of matrix units consistent with the diagonalization of $\mathcal{A}$ by $\mathcal{D}$ (cf. [11, Proposition I.1.8]). Each matrix unit is supported on a compact open subset of R which is the graph of a partial homeomorphism of X. Without any loss of generality we can assume that each matrix unit is the characteristic function of its support. The collection of supports of the matrix units form a base for the topology of R.

All the 1-cocycles on R considered in this paper are continuous and real valued. We recall that if $\mathcal{T}$ is the subalgebra associated with the flow induced by the cocycle d, and if $\mathcal{T}$ is triangular, then $\mathcal{T} = C(P)$, with $P = d^{-1}([0,\infty))$. P satisfies that $P \cup P^{-1} = R$ and $P \cap P^{-1} = X$. In particular, $d^{-1}(\{0\}) = X$.

Note that if R is a UHF groupoid, then a cocycle d is bounded on R if and only if d is a coboundary ([9, Proposition I.4.5 and Theorem I.4.10]).

We define now the class of analytic TAF algebras that we will be dealing with in this section. We have borrowed the term $\mathbb{Z}$-analytic from J. Peters, Y. Poon and B. Wagner [8].

DEFINITION 2.1. Let $\mathcal{T} = C(P)$ be a strongly maximal triangular subalgebra of $\mathcal{A}$. $\mathcal{T}$ is *$\mathbb{Z}$-analytic* if there is a cocycle d such that

(1) $P = d^{-1}([0,\infty))$,
(2) $d(R) \subseteq \lambda\mathbb{Z}$, for some $\lambda \in \mathbb{R}$.

PROPOSITION 2.2. *Let $\mathcal{T} = C(P)$ be an analytic unital TAF algebra with respect to the cocycle d. Let α be the flow induced by d. Then α is periodic and outer if and only if $d(R) \subseteq \lambda\mathbb{Z}$, for some $\lambda \in \mathbb{R}$, and d is unbounded.*

Proof. If λ is periodic, then clearly $d(R) \in \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}$. To prove the first implication we now show that if d is bounded, then d is a coboundary (and then α is inner by [9, Proposition II.5.3]). So assume d is bounded, so that $d(R)$ is finite. For every $t \in d(R)$ the set $d^{-1}(t)$ is a closed and open R-set in R. For every $x \in X$ we write

$$g(x) = \min\{d(x,y) \colon (x,y) \in R\}.$$

If $d(R) = \{t_1 < t_2 < \ldots < t_k\}$ then

$$g^{-1}(t_j) = r\left(d^{-1}(t_j)\right) \setminus \bigcup_{m=1}^{j-1} r\left(d^{-1}(t_m)\right);$$

and $g^{-1}(t_j)$ is closed and open. Thus g is continuous. Given $(x,y) \in R$, if $g(x) = d(x,z)$ then $g(y) = d(y,z)$ and $g(y) - g(x) = d(y,z) - d(x,z) = d(y,x)$. Hence d is a coboundary.

Conversely, suppose $d(R) \subseteq \lambda\mathbb{Z}$ for some $\lambda \in \mathbb{R}$ and d is unbounded. Since X is compact, every coboundary on R is bounded. It follows that d is not a coboundary and α is outer (cf. [5, 8.1.6]), so $\mathrm{Sp}(\alpha) \subseteq \lambda\mathbb{Z}$ in this case. We claim that if ξ is in the annihilator of $\lambda\mathbb{Z}$, then α_ξ is the identity automorphism. This will show that α is periodic. Let ξ be in the annihilator of $\mathrm{Sp}(\alpha)$. If $A \in \mathcal{A}^\alpha(\{\tau\})$, the α-spectral subspace of $\mathcal{A} = C^*(R)$ corresponding to $\tau \in \mathrm{Sp}(\alpha)$ (cf. [5, 8.1.3]), then

$$\alpha_\xi(A) = \exp(\mathrm{i}\xi\tau)A = A,$$

by [5, Corollary 8.1.8]. Since $\sum\{\mathcal{A}^{\alpha}(\tau): \tau \in \mathrm{Sp}(\alpha)\}$ is dense in $\mathcal{A} = C^*(R)$ (cf. 5, Theorem 8.1.4(viii)]), we obtain that α_ξ is the identity automorphism. ■

LEMMA 2.3. *Let R be an AF groupoid. Let F be a compact open subset of X, the unit space of R, and let N be a positive integer. Suppose that for every $w \in F$, $F \cap [w]$ contains no more than N elements. Then, there is a non-empty compact open subset $F_1 \subseteq F$ such that $(F_1 \times F_1) \cap R$ is compact (in the topology of R).*

Proof. We can assume that there is some $\omega \in F$ such that $F \cap [\omega]$ has precisely N elements. Then, for such ω, there are N disjoint compact open R-sets $\{\tau_j: 1 \leqslant j \leqslant N\}$ contained in $(F \times F) \cap R$ such that

$$F \cap [\omega] = \{\tau_j(\omega): 1 \leqslant j \leqslant N\}.$$

Write $F_1 = \bigcap\{r(\tau_j): 1 \leqslant j \leqslant N\}$. Then F_1 is compact and open and for every $u \in F_1$, $F \cap [u] = \{\tau_j(u): 1 \leqslant j \leqslant N\}$. Hence $(F_1 \times F_1) \cap R \subseteq \bigcup\{\tau_j: 1 \leqslant j \leqslant N\}$ and $(F_1 \times F_1) \cap R$ is compact. ■

We establish now the main result of this section.

THEOREM 2.4. *Let $\mathcal{T} = C(P)$ be a $\mathbb{Z}$-analytic TAF algebra with respect to the cocycle d. Let d_0 be a cocycle with $d_0^{-1}([0,\infty)) = P$ $(= d^{-1}([0,\infty)))$. If $d|(F \times F) \cap R$ is unbounded for every non-empty compact open subset $F \subseteq X$, then so is d_0.*

Proof. Let k the minimal positive value obtained by d. Since d is continuous, $d^{-1}(k)$ is a non-empty open subset of $\mathbb{R}$. Fix a compact open R-set $t \subseteq d^{-1}(k)$, and write $F = r(t)$. It follows from the assumption on d and its continuity that there is no open compact subset $F_1 \subseteq X$ such that $(F_1 \times F_1) \cap R$ is compact. We conclude from Lemma 2.3 that $\sup\{|F \cap [w]|: w \in F\} = \infty$ (where $|A|$ is the cardinality of the set A).

Let $a = \min(d_0|t)$ be the minimum value of the continuous cocycle d_0 on the compact open set t. Notice that $a \geqslant 0$, since $t \subseteq P = d^{-1}([0,\infty)) = d_0^{-1}([0,\infty))$. Moreover $a > 0$ because $t \cap X = \emptyset$.

Let n be an arbitrary positive integer and let $w \in F$ be such that $|F \cap [\omega]| > n$. We can find elements $\omega_1, \dots, \omega_n$ of $F \cap [\omega]$ such that they are all different. Using the order induced by P on $[\omega]$, we can write

$$\omega_1 < \omega_2 < \dots < \omega_n.$$

For every $1 \leqslant m < n$ we have that $d(\omega_m, t(\omega_m)) = k$ (since $t \subset d^{-1}(k)$) and that $d(\omega_m, \omega_{m+1}) \geqslant k$ (because k is the minimal positive value attained by d). Hence $\omega_{m+1} \geqslant t(\omega_m)$ and, therefore, $d_0(\omega_m, \omega_{m+1}) \geqslant d_0(\omega_m, t(\omega_m)) \geqslant a$. Since this holds for all $1 \leqslant m < n$, $d_0(\omega_1, \omega_n) \geqslant a(n-1)$. It follows that d_0 is unbounded on $(F \times F) \cap R$, and therefore d_0 is unbounded on R.

If B is a non-empty compact open subset of X we can apply the result to $R \cap \cap (B \times B)$ to find that $d_0 | (B \times B) \cap R$ is unbounded. ∎

If R is a UHF groupoid, it is easy to check that for every compact open subset $F \subseteq X$ and every $\omega \in F$, $F \cap [\omega]$ is infinite. Using this instead of Lemma 2.3 in the proof of Theorem 2.4, we get the following.

COROLLARY 2.5. *Suppose $\mathcal{T} = C(P)$ is a $\mathbb{Z}$-analytic TUHF algebra with respect to the cocycle d, and let d_0 be a cocycle with $d_0^{-1}([0, \infty)) = P$. If d is unbounded and $F \subseteq X$ is open, then $d_0 | (F \times F) \cap R$ is unbounded.*

This result was also proved in [10] using different methods.

REMARK 2.6. J. Peters, Y. T. Poon and B. Wagner have proved that the TUHF algebras $\mathcal{T}_{\{x_0\}}$, introduced in [7, Example 1.3] are $\mathbb{Z}$-analytic with respect to an unbounded cocycle.

We shall need the folowing result. First note that a subset $Y \subseteq X$ is said to be increasing if whenever $x \in Y$ and $(x, y) \in P$, then $y \in Y$.

COROLLARY 2.7. *Let $\mathcal{T} = C(P)$ be a TAF $\mathbb{Z}$-analytic algebra with respect to a cocycle d such that $d | (F \times F) \cap R$ is unbounded for every compact open set $F \subseteq X$. (If $\mathcal{T}$ is TUHF it is enough to assume that d is unbounded on R.) Let d_0 be a cocycle with $d_0^{-1}([0, \infty)) = P$ and let $Y \subseteq X$ be an increasing (or decreasing) subset such that $\overline{Y}$ is open. Then $d_0 | (Y \times Y) \cap R$ is unbounded.*

Proof. Since d_0 is continuous and its restriction to $(U \times U) \cap R$ is unbounded for every open non-empty subset $U \subseteq X$, it is left to show that $(Y \times Y) \cap R$ is dense in $(\overline{Y} \times \overline{Y}) \cap R$. For this take $(x, y) \in (\overline{Y} \times \overline{Y}) \cap P$ and let τ be a compact open R-set that contains (x, y) and is contained in P. Take a sequence $\{x_n\} \subseteq Y$ such that $x_n \to x$. Then $\tau(x_n) \to \tau(x) = y$. As Y is decreasing and $\tau \subseteq P$, $\tau(x_n) \in Y$. Hence (x, y) is in the closure of $(Y \times Y) \cap R$. ∎

The following definition, when taken together with non-trivial $\mathbb{Z}$-analyticity will provide a class of analytic triangular UHF algebras that is crucial for the result of the next section.

DEFINITION 2.8. Let $\mathcal{T}$ be a TAF algebra and let X be the Gelfand spectrum of the diagonal $\mathcal{T} \cap \mathcal{T}^*$. *$\mathcal{T}$ well-orders a base for the topology* if there is a base of compact open sets for the topology of X such that for every F in this base there is a unique point $\omega_0 \in F$ that is the largest in $F \cap [\omega_0]$, and a unique point $\omega_1 \in F$ that is the smallest in $F \cap [\omega_1]$.

Strongly maximal TUHF algebras that well order a base for the topology of X are fairly common. They include the TUHF algebras that are the inductive limit of a sequence of upper triangular matrix algebras, as the next proposition shows.

PROPOSITION 2.9. *Let $\mathcal{A}$ be a UHF algebra and let $\{\mathcal{A}_n : n = 1, 2, \ldots\}$ be an increasing sequence of subalgebras of $\mathcal{A}$ such that $\mathcal{A} = \left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right)^{-}$ and $\mathcal{A}_n \simeq M_{N(n)}(\mathbb{C})$. Let $\left\{E_{ij}^{(n)} : 1 \leqslant i, j \leqslant N(n),\ n = 1, 2, \ldots\right\}$ be a system of matrix units for $\mathcal{A}$, consistent with $\{\mathcal{A}_n : n = 1, 2, \ldots\}$. If $\mathcal{T}$ is a triangular subalgebra of $\mathcal{A}$ which is strongly maximal with respect to $\{\mathcal{A}_n\}$, then $\mathcal{T}$ well-orders $\left\{\hat{E}_{ii}^{(n)} : 1 \leqslant i \leqslant N(n),\ n = 1, 2, \ldots\right\}$, where $\hat{E}_{ii}^{(n)}$ denotes the support of $E_{ii}^{(n)}$.*

Proof. Without loss of generality we can assume that

$$\mathcal{T} \cap \mathcal{A}_n = \operatorname{span}\left\{E_{ij}^{(n)} : 1 \leqslant i \leqslant j \leqslant N(n)\right\}.$$

Consider $E_{ii}^{(n)}$, for some positive integers n and i with $1 \leqslant i \leqslant N(n)$. Let k be a positive integer. Using the linear ordering induced by $\mathcal{T} \cap \mathcal{A}_{n+k}$ on $\left\{E_{jj}^{(n+k)} : 1 \leqslant j \leqslant N(n+k)\right\}$, which equals $(\mathcal{A}_{n+k} \cap \mathcal{D})\hat{}$ (the Gelfand spectrum of $\mathcal{A}_{n+k} \cap \mathcal{D}$), define $P_i^{(n+k)}$ (resp. $Q_i^{(n+k)}$) to be the largest (resp. the smallest) element of $E_{ii}^{(n)}$, viewed as a subset of $(\mathcal{A}_n \cap \mathcal{D})\hat{}$.

We claim that $P_i^{(n,k+1)} \in \hat{P}_i^{(n,k)}$ (resp. $Q_i^{(n,k+1)} \in \hat{Q}_i^{(n,k)}$) viewed as a collection of minimal projections in $\mathcal{A}_{n+k} \cap \mathcal{D}$. Indeed, if $P_i^{(n,k+1)} \notin \hat{P}_i^{(n,k)}$ (resp. $Q_i^{(n,k+1)} \notin \hat{Q}_i^{(n,k)}$), then $P_i^{(n,k+1)}$ (resp. $Q_i^{(n,k+1)}$) belongs to a minimal projection E in $\mathcal{A}_{n+k} \cap \mathcal{D}$, which is a subprojection of $E_{ii}^{(n)}$. By definition of $P_i^{(n,k)}$ (resp. $Q_i^{(n,k)}$), E is smaller (resp. larger) than $P_i^{(n,k)}$ (resp. $Q_i^{(n,k)}$), in the $\mathcal{T} \cap \mathcal{A}_{n+k}$ order for $(\mathcal{A}_{n+k} \cap \mathcal{D})\hat{}$. Let V be the matrix unit of $\mathcal{T} \cap \mathcal{A}_{n+k}$ such that VEV^* (resp. $VQ_i^{(n,k)}V^*$) equals $P_i^{(n,k)}$ (resp. E). Then $VP_i^{(n,k+1)}V^*$ (resp. $V^*Q_i^{(n,k+1)}V$) belongs to $\hat{P}_i^{(n,k)}$ (resp. $\hat{E}$), and therefore to $\hat{E}_{ii}^{(n)}$, viewed as a subsets of $(\mathcal{A}_{n+k+1} \cap \mathcal{D})\hat{}$. Also $VP_i^{(n,k+1)}V^*$ (resp. $V^*Q_i^{(n,k+1)}V$) is larger (resp. smaller) than $P_i^{(n,k+1)}$ (resp. $Q_i^{(n,k+1)}$) in the $\mathcal{T} \cap \mathcal{A}_{n+k+1}$ order for $\mathcal{A}_{n+k+1} \cap \mathcal{D}$, contradicting the definition of $P_i^{(n,k+1)}$ (resp. $Q_i^{(n,k+1)}$).

Let $x_i^{(n)}$ and $y_i^{(n)}$ be the elements of $\hat{E}_{ii}^{(n)}$ (now viewed as a subset of X) such that

$$\left\{x_i^{(n)}\right\} = \bigcap_{k=1}^{\infty} \hat{P}_i^{(n,k)} \quad \text{and} \quad \left\{y_i^{(n)}\right\} = \bigcap_{k=1}^{\infty} \hat{Q}_i^{(n,k)}.$$

We want to verify that $x_i^{(n)}$ (resp. $y_i^{(n)}$) is the largest (resp. the smallest) in $\hat{E}_{ii}^{(n)} \cap \left[x_i^{(n)}\right]$ (resp. $\hat{E}_{ii}^{(n)} \cap \left[y_i^{(n)}\right]$).

If $\omega \in \hat{E}_{ii}^{(n)}$ and $\omega \neq x_i^{(n)}$, then there is some k such that $\omega \notin \hat{P}_i^{(n+k)}$ (resp. $\hat{Q}_i^{(n,k)}$). Thus, if $\omega \in \left[x_i^{(n)}\right]$ (resp. $\left[y_i^{(n)}\right]$), it follows that ω is smaller (resp. larger) than $x_i^{(n)}$ (resp. $y_i^{(n)}$), and that $x_i^{(n)}$ (resp. $y_i^{(n)}$) is the largest (resp. smallest) in $\hat{E}_{ii}^{(n)} \cap \left[x_i^{(n)}\right]$ (resp. $\hat{E}_{ii}^{(n)} \cap \left[y_i^{(n)}\right]$). On the other hand, if $w \notin \left[x_i^{(n)}\right]$ (resp. $\left[y_i^{(n)}\right]$), it follows that w is not the largest (resp. smallest) in $\hat{E}_{ii}^{(n)} \cap [w]$, and that $\hat{E}_{ii}^{(n)} \cap [w]$ has neither a largest nor a smallest element. Notice that $x_1^{(1)}$ is the largest of $\left[x_1^{(1)}\right]$ and $y_{N(n)}^{(1)}$ is the smallest of $\left[y_{N(n)}^{(1)}\right]$. ∎

REMARK 2.10. It seems likely that the converse of Proposition 2.9 is false. However, we have not been able to find an example that will support this assertion.

THEOREM 2.11. *Let $\mathcal{T} = C(P)$ be a TAF algebra such that*

(1) $\mathcal{T}$ is non-trivial $\mathbb{Z}$-analytic with respect to a cocycle d;

(2) $\mathcal{T}$ well-orders a base for the topology of X; and

(3) there is a non-empty compact open subset F in this base such that for every compact open $F_1 \subseteq F$ and every $\omega \in F_1$, $F_1 \cap [\omega]$ is infinite.

Let $Y \subseteq X$ be an increasing (or decreasing) subset such that $F \cap Y$ is dense in F. Then, for every cocycle d_0 with $d_0^{-1}([0,\infty)) = P$, $d_0|(Y \times Y) \cap R$ is unbounded.

Proof. Let $F' \subseteq F$ be a compact open set in the base for the topology of X that is well-ordered by $\mathcal{T}$. We show that we can choose some $\omega \in F' \cap Y$ such that $F' \cap Y \cap [\omega]$ is an infinite set.

Let ω_0 and ω_1 be the unique elements of F' such that ω_0 is the largest in $F' \cap [\omega_0]$ and ω_1 is the smallest in $F' \cap [\omega_1]$. If $\omega \notin [\omega_0] \cup [\omega_1]$, then $F' \cap [\omega]$ is infinite and has no maximal or minimal elements. Since $\omega \in Y$ and Y is increasing (or decreasing) it follows that $F' \cap [\omega] \cap Y$ is infinite.

If we cannot choose $\omega \in F' \cap Y$ such that $\omega \notin [w_1] \cup [\omega_0]$, then $F' \cap Y \subseteq [\omega_1] \cup [\omega_0]$. Suppose Y is increasing. Then $F' \cap Y \cap [\omega_1]$ is infinite. So it is left to check only the case where $F' \cap Y \subseteq [\omega_0]$ and $F' \cap Y \cap [\omega_0]$ is finite. But in this case $F' \cap Y = F' \cap Y \cap [\omega_0]$ is finite. Since $F' \cap Y$ is dense in F', F' is also finite, a contradiction.

A similar argument can be applied to the case when Y is decreasing.

Now, let k be the minimal positive value attained by d on $(F \times F) \cap R$. Fix a compact R-set $t \subseteq D^{-1}(k) \cap (F \times F)$. Since t can be chosen to be the support of a matrix unit, $F' = r(t)$ can be assumed to be in the base for the topology of X that is well ordered by $\mathcal{T}$. Let $\omega \in F' \cap Y$ be such that $F' \cap Y \cap [\omega]$ is an infinite set. Thus the proof of Theorem 2.4 implies that $d_0|((Y \cap F) \times (Y \cap F)) \cap R$ is unbounded, and the conclusion follows. ∎

REMARK 2.12. It follows from [10, Theorem 6.1] that Theorem 2.11 is valid in the setting of simple AF algebras without hypothesis (2) and (3). It seems likely that hypothesis (2) can be eliminated even in the case of an arbitrary AF algebra. However, we have not been able to find such a proof.

We give an example of an analytic triangular UHF algebra to which the previous results apply.

EXAMPLE 2.13. Let $\mathcal{A}$ be a UHF algebra, $\{\mathcal{A}_n : n = 1, 2, \ldots\}$ an increasing sequence of subalgebras of $\mathcal{A}$ such that $\mathcal{A}_n \simeq M_{N(n)}(\mathbb{C})$ and $\mathcal{A} = \left[\bigcup_{n=1}^{\infty} \mathcal{A}_n\right]^{-}$. Let $\left\{E_{ij}^{(n)} : 1 \leqslant i, j \leqslant N(n),\ n = 1, 2, \ldots\right\}$ be a system of matrix units for $\mathcal{A}$, consistent with $\{\mathcal{A}_n : n = 1, 2, \ldots\}$. $\mathcal{T}_n$ is the subalgebra for $\mathcal{A}_n$ spanned by $\left\{E_{ij}^{(n)} : 1 \leqslant i \leqslant j \leqslant N(n)\right\}$. Consider the standard embedding $\sigma : \mathcal{A}_n \mapsto \mathcal{A}_{n+1}$ given by

$$\sigma_n\left(E_{ij}^{(n)}\right) = \sum_{m=0}^{q_n - 1} E_{i+mN(n),\, j+mN(n)}^{(n+1)}$$

where $q_n = N(n+1)/N(n)$ (cf. [7, Example 1.1]. Let $\mathcal{T} = \varinjlim(\mathcal{T}_n, \sigma_n)$. $\mathcal{T}$ is called the standard embedding TUHF algebra.

Recall that $\mathcal{T}$ is analytic (cf. [13, Example 6.1], and that a cocycle corresponding to $\mathcal{T}$ is obtained as follows. Let X be the unit space of the UHF groupoid R corresponding to $\mathcal{A}$ and $\{\mathcal{A}_n : n = 1, 2, \ldots\}$. In this case $X \simeq \prod_{n=1}^{\infty} \{1, 2, \ldots, N(n)\}$. For (x_n), $(y_n) \in X$, define

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} N(n)\,(y_n - x_n);$$

d is clearly unbounded and $d(R) \subseteq \mathbb{Z}$. Thus, any cocycle corresponding to $\mathcal{T}$ is unbounded on R (by Theorem 2.4). Moreover, it is unbounded on $(Y \times Y) \cap R$, where Y is an open subset of X (Corollary 2.5).

As $\mathcal{T}$ is the inductive limit of a sequence of upper triangular matrix algebras, it follows from Proposition 2.9 and its proof, that $\mathcal{T}$ well-orders $\left\{E_{ii}^{(n)} : 1 \leqslant i \leqslant N(n)\right\}$. However, we would like to give a slightly different description of the elements $x_i^{(n)}, y_i^{(n)}$, for $1 \leqslant i \leqslant N(n)$, $n = 1, 2, \ldots$, that are the unique elements in $E_{ii}^{(n)}$ that are the largest of $E_{ii}^{(n)} \cap \left[x_i^{(n)}\right]$ and smallest of $E_{ii}^{(n)} \cap \left[y_i^{N(n)}\right]$, respectively.

Consider $x, y \in X$ such that

$$\{x\} = \bigcap_{n=1}^{\infty} \hat{E}_{11}^{(n)} \quad \text{and} \quad \{y\} = \bigcap_{n=1}^{\infty} \hat{E}_{N(n),N(n)}^{(n)}.$$

Then, $x = x_1^{(1)}$ and $y = y_{N(1)}^{(1)}$ are the maximum of $\hat{E}_{11}^{(1)} \cap [x]$ and the minimum of $\hat{E}_{N(n),N(n)}^{(1)} \cap [y]$, respectively.

Let $\sigma_{ij}^{(n)}$ be the partial homeomorphism of X induced by $E_{ij}^{(n)}$. We can verify that

$$x_j^{(n)} = \sigma_{j1}^{(n)}(x) \quad 1 \leqslant j \leqslant N(n)$$

and

$$y_j^{(n)} = \sigma_{jN(n)}^{(n)}(y) \quad 1 \leqslant j \leqslant N(n)$$

are the maximal element of $E_{jj}^{(n)} \cap \left[x_j^{(n)}\right]$ and the minimal element of $E_{jj}^{(n)} \cap \left[y_j^{(n)}\right]$, respectively (notation from Proposition 2.9).

Moreover, x and y are the unique elements of X such that x is the largest of $[x]$ and y is the smallest of $[y]$. $[x]$ (resp. $[y]$) is the canonical choice of a decreasing (resp. increasing) set in X. From Theorem 2.11, it follows that any cocycle corresponding to $\mathcal{T}$ is unbounded on $[x]$ and on $[y]$.

3. NON-ANALYTIC TRIANGULAR UHF ALGEBRAS

In this section we apply the results previously obtained to establish a criterion for non-analyticity that is useful in the construction of non-analytic strongly maximal TUHF algebras. As a consequence we show that any UHF algebra has a non-analytic strongly maximal triangular subalgebra. We also obtain examples of strongly maximal TUHF algebras which are nest algebras but are not analytic. In what follows a $\mathbb{Z}$--analytic UHF algebra will be said to be *non-trivial $\mathbb{Z}$-analytic* if the corresponding cocycle is unbounded.

THEOREM 3.1. *Let $\mathcal{T}$ be a strongly maximal triangular subalgebra of the UHF algebra $\mathcal{A}$. Denote by $\leqslant$, the order on X whose graph supports $\mathcal{T}$. If there is a projection $E \in \mathcal{D}$, $E \neq 0, 1$ such that*

(1) $E\mathcal{T}E$ is non-trivial $\mathbb{Z}$-analytic;

(2) $E\mathcal{T}E$ is the inductive limit of a sequence of upper triangular matrix algebras. (Write x (resp. y) for the unique element of $\hat{E}$ that is the largest (resp. smallest) of $[x] \cap \hat{E}$ (resp. $[y] \cap \hat{E}$));

(3) there is an element $z \in X$ such that either $z \in [x]$ and z is smaller than all $\omega \in [x] \cap \hat{E}$ or $z \in [y]$ and z is larger than all $\omega \in [y] \cap \hat{E}$.

Then $\mathcal{T}$ is not analytic.

Proof. Assume $\mathcal{T}$ is analytic and let d be a cocycle on R corresponding to $\mathcal{T}$, that is $d^{-1}([0, \infty)) = P$, where $\mathcal{T} = C(P)$. Then $E\mathcal{T}E = C\left(P \cap \left(\hat{E} \times \hat{E}\right)\right)$ and $d|\left(\hat{E} \times \hat{E}\right) \cap R$ corresponds to $E\mathcal{T}E$.

If $z \in [x]$, consider $[x] \cap \hat{E}$, which is a dense decreasing subset of $\hat{E}$ for the $E\mathcal{T}E$ order of $\hat{E}$. It follows from Corollary 2.7 that $d|\left([x] \cap \hat{E}\right) \times \left([x] \cap \hat{E}\right)$ is unbounded. Now, if $\omega \in [x] \cap \hat{E}$, then z is smaller than ω, by hypothesis (3). Since ω is smaller than x we have that $d(z, x) > d(\omega, x)$. Therefore, $d(z, x)$ is an upper bound for $d|\left([x]\cap\hat{E}\right)\times\left([x]\cap\hat{E}\right)$, a contradiction, since d is unbounded on $\left([x]\cap\hat{E}\right)\times\left([x]\cap\hat{E}\right)$.

A similar argument yields also a contradiction when $z\in[y]$ and z is larger than any $\omega\in[y]\cap\hat{E}$. ∎

We are now in a position to construct a non-analytic strongly maximal TUHF.

EXAMPLE 3.2. Let $\mathcal{A}$ be UHF(2^∞). We express $\mathcal{A}$ as:

$$\varinjlim\left(M_2(\mathbb{C})\otimes M_{2^n}(\mathbb{C}),\mathrm{id}\otimes\sigma_n\right),$$

where σ_n is the standard embedding and id is the identity embedding, and where $I_2\otimes A=\begin{pmatrix}A&0\\0&A\end{pmatrix}$ for $A\in M_{2^n}(\mathbb{C})$. Let us denote $M_2(\mathbb{C})\otimes M_{2^n}(\mathbb{C})$ by $\mathcal{A}_n$.

Let $\mathcal{T}_n$ be the upper triangular matrices in $\mathcal{A}$, and let $\mathcal{T}=\varinjlim(\mathcal{T}_n,\mathrm{id}\otimes\sigma_n)$. We prove that $\mathcal{T}$ is not analytic.

If $E=\begin{pmatrix}1&0\\0&0\end{pmatrix}\otimes I_2\in\mathcal{A}_1$, then identifying $\mathcal{A}_1$ with a subalgebra of $\mathcal{A}$, we have that $E\mathcal{T}E$ is isometrically isomorphic to the standard embedding TUHF subalgebra of the UHF(2^∞) C^*-algebra. From Example 2.13 we know that $E\mathcal{T}E$ is non-trivial $\mathbb{Z}$-analytic and it is the inductive limit of a sequence of upper triangular matrix algebras.

Identify $\mathcal{A}_n$, for $n=1,2,\dots$ with a subalgebra of $\mathcal{A}$. If $\left\{E_{ij}^{(n)}:1\leqslant i,j\leqslant 2^n,\ n=\right.$ $=1,2,\dots\}$ is consistent with $\mathcal{A}$ and $\{\mathcal{A}_n: n=1,2,\dots\}$, then $x\in\hat{E}$ such that $\{x\}=$ $=\bigcap_{n=1}^{\infty}\hat{E}_{11}^{(n)}$ is the unique element of $\hat{E}$ that is the largest of $[x]\cap\hat{E}$.

Notice that $\hat{E}^{(n+1)}_{2^{n+1}+1,\,2^{n+1}+1}\subseteq\hat{E}^{(n)}_{2^n+1,\,2^n+1}$, so z given by

$$\{z\}=\bigcap_{n=1}^{\infty}\hat{E}^{(n)}_{2^n+1,\,2^n+1}$$

is well defined. Moreover, from the embedding $\mathrm{id}\otimes\sigma_n:\mathcal{A}_n\to\mathcal{A}_{n+1}$ it follows that

$$E^{(n)}_{1,\,2^n+1}=E^{(n+1)}_{1,\,2^{n+1}+1}+E^{(n+1)}_{2^n+1,\,2^{n+1}+2^n+1}.$$

Thus $z=\sigma^{(n)}_{2^n+1,\,1}(x)$ for all n and therefore z belongs to $[x]$.

We claim that z is smaller than any element of $[x]\cap\hat{E}$. Indeed, if $\sigma_{ij}^{(n)}$ is the partial homeomorphism corresponding to $E_{ij}^{(n)}$, then

$$[x]\cap\hat{E}=\bigcup_{n=1}^{\infty}\left\{\sigma_{ij}^{(n)}(x):1\leqslant j\leqslant 2^n\right\}.$$

Given n, $z=\sigma^{(n)}_{2^n+1,\,1}(x)$, so z is clearly smaller than $\sigma^{(n)}_{ji}(x)$ for any j such that $1\leqslant j\leqslant 2^n$.

Therefore z satisfies (3) of Theorem 3.1. It follows from that theorem that $\mathcal{T}$ is not analytic.

More explicitly, if $\mathcal{T}$ were analytic and if d were a cocycle corresponding to $\mathcal{T}$, then from

$$z < \omega < x, \text{ for all } \omega \in [x] \cap \hat{E},$$

(where $<$ denotes the $\mathcal{T}$ order on the unit space of the UHF groupoid corresponding to $\mathcal{A}$ and $\{\mathcal{A}_n : n = 1, 2, \ldots\}$) it would follow that $d(\omega, x) < d(z, x)$ for all $\omega \in [x] \cap \hat{E}$. But this is impossible because $d|\left(\hat{E} \times \hat{E}\right) \cap R$ corresponds to the standard embedding TUHF algebra, and therefore $\{d(\omega, x) : \omega \in [x]\}$ is unbounded.

Adapting the above example to a UHF algebra of arbitrary supernatural number we obtain the following:

THEOREM 3.3. *Every finite dimensional UHF algebra has a non-analytic strongly maximal triangular subalgebra.*

Proof. Let $\mathcal{A}$ be a UHF algebra. Then $\mathcal{A} = \varinjlim \left(M_{N(n)}(\mathbb{C})\right)$, for some sequence $\{N(n) : n = 1, 2, \ldots\}$ of positive integers. Let $K(n)$ be such that $N(n) = N(1)K(n)$. Then $\mathcal{A} = \varinjlim \left(M_{N(1)}(\mathbb{C}) \otimes M_{K(n)}(\mathbb{C}), \mathrm{id} \otimes \sigma_n\right)$ where id and σ_n are as in Example 3.2.

Let $\mathcal{T}_n$ be the upper triangular matrices in $M_{N(1)}(\mathbb{C}) \otimes M_{K(n)}(\mathbb{C})$. We show that $\mathcal{T} = \varinjlim(\mathcal{T}_n, \mathrm{id} \otimes \sigma_n)$ is a non-analytic, strongly maximal subalgebra of $\mathcal{A}$.

As usual, let $\left\{E_{ij}^{(n)} : 1 \leqslant i, j \leqslant N(n),\ n = 1, 2, \ldots\right\}$ be a system of matrix units for $\mathcal{A}$, consistent with $\{\mathcal{A}_n : n = 1, 2, \ldots\}$, where $\mathcal{A}_n = M_{N(1)}(\mathbb{C}) \otimes M_{K(n)}(\mathbb{C})$. Identify $\mathcal{A}_n$ with a subalgebra of $\mathcal{A}$.

Then, if $E = \sum_{i=1}^{K(2)} E_{ii}^{(2)}$, $E\mathcal{T}E$ is isometrically isomorphic to the standard embedding TUHF algebra (with the appropriate supernatural numbers), so $E\mathcal{T}E$ satisfies the hypothesis (1) and (2) of Theorem 3.1. It also satisfies hypothesis (3). The argument is very similar to the one given in Example 3.2 and is omitted. ∎

We now turn to produce examples of non-analytic nest algebras. We first need the following lemma.

LEMMA 3.4. *Let $\mathcal{A}$ be the UHF(2^∞) C^*-algebra; i.e., $\mathcal{A} = \varinjlim M_{2^n}(\mathbb{C})$ and let $\mathcal{D} = \varinjlim D_{2^n}(\mathbb{C})$ be the canonical masa with $\hat{\mathcal{D}} = X$. Let μ be the measure on X induced by the unique normalized trace on $\mathcal{A}$; i.e., for a matrix unit E in $D_{2^n}(\mathbb{C})$, $\mathrm{tr}(E) = \mu(\hat{E})$. Let F be a closed subset of X with the property that for every $x \in X$, $[x] \cap F$ contains at most one point. Then $\mu(F) = 0$.*

Proof. The space X can be identified with $\prod_{i=1}^{\infty}\{0,1\}$, and the equivalence relation introduced by $\mathcal{A}$ on X is given by $x=(x_i)\sim y=(y_i)$ if for some N_0 we have $x_i=y_i$ for all $i\geqslant N_0$. The measure μ is just the product $\prod_{i=1}^{\infty}p$ where $p(\{0\})=p(\{1\})=1/2$. Suppose F has the property that $F\cap[x]$ contains at most one point, for every $x\in X$. For every $j\geqslant 1$ we can define a map τ_j by $\tau_j((x_i))=(y_i)$ where $y_i=x_i$ if $i\neq j$ and $y_j=1-x_j$. Then $x\sim\tau_j(x)$ and τ_j is measure preserving. Hence $\{\tau_j(F): j\geqslant 1\}$ is a family of pairwise disjoint subsets of X with $\mu(\tau_j(F))=\mu(F)$. Since $\mu(X)<$ $<\infty$, $\mu(F)=0$. ■

THEOREM 3.5. *Given α such that $\frac{2}{3}<\alpha<1$, there exists a strongly maximal triangular nest subalgebra of a UHF algebra, which is not analytic and has α as the supremum of the values of the trace on the non-trivial invariant projections.*

Proof. Write $\beta=3\alpha-2$. Let $\mathcal{A}, X$ and μ be as in Lemma 3.4 and $\mathcal{T}=\mathcal{T}_{(\beta)}$ be the TUHF algebra constructed in [7, Theorem 4.5] and [6, Theorem 2.24] for $\beta=\sum_{n=1}^{\infty}\frac{k_n}{2^n}$ where $\{k_n\}$ is non-terminating. It was shown in [6] that $\mathcal{T}$ is a nest subalgebra, i.e., $\mathcal{T}=\mathcal{A}\cap\operatorname{alg}\mathcal{N}$ for a nest $\mathcal{N}\subseteq\mathcal{D}$. Also it was shown that $\sup\{\operatorname{tr}(P): P\in\mathcal{N}\}=\beta$. We shall first show that if $\mathcal{T}$ is analytic and d is an asociated cocycle, then d is not bounded. If it is bounded, then d is a coboundary; i.e. there is some $g: X\to\mathbb{R}$ continuous such that for $x\sim y$ and $x\neq y$ then $g(x)=g(y)$. For every $P\in\mathcal{N}$ write $g_P=\sup\left\{g(x): x\in\hat{P}\right\}$ and write

$$\tilde{g}=\sup\{g_P: P\in\mathcal{N}\}=\sup\left\{g(x): x\in\bigcup\left\{\hat{P}: P\in\mathcal{N}\right\}\right\}.$$

Let $Y=X\backslash\bigcup\left\{\hat{P}: P\in\mathcal{N}\right\}$. Then Y is closed (as each $\hat{P}$ is open and closed). If there is some $x\in X$ with $g(x)>\tilde{g}$ then there is a closed and open subset $Z\subseteq X$ such that $g>\tilde{g}$ on Z. We can therefore find a non-zero projection $Q\in\mathcal{D}$ such that on $\hat{Q}$, $g>\tilde{g}$ and, consequently, $QP=0$ for every $P\in\mathcal{N}$. Thus $Q\mathcal{A}Q\subseteq\mathcal{T}$; hence $Q\mathcal{A}Q\subseteq\mathcal{D}$ which is impossible. Hence $\tilde{g}=\sup\{g(x): x\in X\}$. Now, if $g(x)<\tilde{g}$ for some $x\in X$, then $g(x)<g_P$ for some $P\in\mathcal{N}$. But $\hat{P}$ is a decreasing set and thus $x\in\hat{P}$. We conclude that on $Y=X\backslash\bigcap\left\{\hat{P}: P\in\mathcal{N}\right\}$, $g=\tilde{g}$. Hence, if $x\sim y$ and $x,y\in Y$, then $g(x)=\tilde{g}=g(y)$ and thus $x=y$. Using Lemma 3.4 we conclude that $\mu(Y)=0$. But $\mu\left(\hat{P}\right)=\operatorname{tr}(P)$ and $\mu\left(\bigcup\left\{\hat{P}: P\in\mathcal{N}\right\}\right)=\sup\{\operatorname{tr}(P): P\in\mathcal{N}\}=\beta<1$. This is a contradiction. Therefore, every cocycle d with the property that $\mathcal{T}_{(\alpha)}$ is supported on $d^{-1}([0,\infty))$ is unbounded.

Now, to produce the example, let $\tilde{\mathcal{A}} = M_3(\mathcal{A})$ and

$$\tilde{\mathcal{N}} = \left\{ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P \end{pmatrix}, \begin{pmatrix} I & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : P \in \mathcal{N} \right\}.$$

Then $\tilde{\mathcal{T}} = \tilde{\mathcal{A}} \cap \operatorname{alg}\tilde{\mathcal{N}}$ can be written matricially as

$$\begin{pmatrix} \mathcal{T} & \mathcal{A} & \mathcal{A} \\ 0 & \mathcal{T} & \mathcal{A} \\ 0 & 0 & \mathcal{T} \end{pmatrix}.$$

$\tilde{\mathcal{T}}$ is a strongly maximal triangular nest subalgebra of a UHF algebra. We also have that the supremum of the values of the normalized trace of the non-trivial invariant projections is α. It is left to show that $\tilde{\mathcal{T}}$ is non-analytic.

Let

$$E_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and define E_2, E_3 similarly. Then $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$ are subsets of $\tilde{X} = \hat{\tilde{\mathcal{D}}} = \tilde{\mathcal{T}} \cap \tilde{\mathcal{T}}^*$. Suppose that there is a cocycle $\tilde{d}$ such that $\tilde{d}^{-1}([0,\infty))$ is the support of $\tilde{\mathcal{T}}$. Then write d for the restriction of $\tilde{d}$ to $\hat{E}_2 \times \hat{E}_2$ and view it as a cocycle associated with $\mathcal{T}$. We know that d is unbounded. In fact, for every $x \in \hat{E}_2$, d is unbounded on $\left([x] \cap \hat{E}_2\right) \times \left([x] \cap \hat{E}_2\right)$. (This is because this set is dense in the equivalence relation on $\hat{E}_2$). On the other hand, for every $x \in \hat{E}_2$, there are $y \in \hat{E}_1 \cap [x]$ and $z \in \hat{E}_3 \cap [x]$. Hence $y \leqslant x \leqslant z$ and if $x_1, x_2 \in \hat{E}_2 \cap [x]$ then $d(x_1, x_2) \leqslant \tilde{d}(x_1, z) \leqslant \tilde{d}(y, z)$. This shows that d is bounded on $\left([x] \cap \tilde{E}_2\right) \times \left([x] \cap \hat{E}_2\right)$ and the contradiction completes the proof. ∎

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BARUCH SOLEL
Department of Mathematics,
Technion-Israel Institute of Technology,
3200 Haifa,
Israel.

BELISARIO A. VENTURA
Department of Mathematics,
California State University,
San Bernardino, CA 92407,
U.S.A.

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