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DILATIONS AND SUBNORMAL OPERATORS WITH RICH SPECTRUM

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0. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite-dimensional complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. The paper [4] which proved the existence of invariant subspaces for all subnormal operators initiated the study of dual subalgebras of $\mathcal{L}(\mathcal{H})$. The main ideas of this theory and a detailed bibliography as of 1984 can be found in [1]. More recent results can be found in [11], [9], [6], [7], and [8].

In the theory of dual algebras, systems of equations have played a central role. One consequence has been many dilation theorems for general classes of operators, cf [1, chapter V] for some examples. However, these theorems apply only to contractions. In this paper some of these dilation theorems are generalized to certain subnormal operators, which need not be contractions.

1. PRELIMINARIES

For $T \in \mathcal{L}(\mathcal{H})$ denote by $\sigma(T)$ (resp: $\sigma_{le}(T)$) the spectrum of T (resp: left essential spectrum of T); recall that T is a contraction if $\|T\| \leqslant 1$. A contraction T is absolutely continuous if the unitary part of T is absolutely continuous (or acts on the space (0)). If $\mathcal{K}$ is another Hilbert space then $\mathcal{H} \oplus \mathcal{K} = \{u \oplus v : u \in \mathcal{H}, v \in \mathcal{K}\}$ is a Hilbert space with $\|u \oplus v\|^2 = \|u\|^2 + \|v\|^2$. Moreover if $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{K})$, then $T \oplus S \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ is given by $T \oplus S(u \oplus v) = T(u) \oplus S(v)$. If $\mathcal{M}$ is a subspace of $\mathcal{H}$ then $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. If $\mathcal{M}$ and $\mathcal{N}$ are subspaces of $\mathcal{H}$ then $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap \mathcal{N}^\perp$. If $S \in \mathcal{L}(\mathcal{H})$ and $\mathcal{K}$ is a supspace of $\mathcal{H}$ such that $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces for S and $\mathcal{M} \supseteq \mathcal{N}$ then $\mathcal{K}$ is a semi-invariant subspace for S. Moreover, $S_\mathcal{K} \in \mathcal{L}(\mathcal{K})$ denotes the operator $P_\mathcal{K}S|\mathcal{K}$. It is well-known that if $\mathcal{K}$ is semi-invariant for S then $(S^k)_\mathcal{K} = (S_\mathcal{K})^k$ for any k in $\mathbb{N}$. Also, if T is unitarily equivalent to $S_\mathcal{K}$ for some semi-invariant subspace $\mathcal{K}$ of S then T is called a compression of S, or equivalently, S a dilation of T.

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ denote the unit circle. Let m denote Lebesgue arc-length measure on $\mathbb{T}$. Let $H^\infty(\mathbb{D}) = \{f$ analytic on D: $\sup\{|f(\lambda)| : : |\lambda| < 1\} < \infty\}$. If $f \in H^\infty(\mathbb{D})$ then $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in \mathbb{D}\}$. If $V \subset \mathbb{D}$, then NTL(V) is the set of all $e^{it} \in \mathbb{T}$ such that there exists a sequence $\{\lambda_n\}_{n=1}^\infty \subset V$ with $\lambda_n \to e^{it}$ nontangentially. It is well-known that NTL(V) is a Borel subset of $\mathbb{T}$. A set $V \subset \mathbb{D}$ is called dominating for $\mathbb{T}$ if $m(\mathbb{T}\backslash \text{NTL}(V)) = 0$. It is well-known that V is dominating for $\mathbb{T}$ if and only if $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in V\}$, for all f belonging to $H^\infty(\mathbb{D})$ (cf. [3, Theorem 3]).

Much of this work takes place in the context of dual algebras, and the notation is as in [1]. Some of the notation and definitions are reviewed here for the convenience of the reader. The Banach algebra $\mathcal{L}(\mathcal{H})$ can be regarded as the dual of $\mathcal{C}_1(\mathcal{H})$, the trace class operators on $\mathcal{H}$, via the pairing $\langle T, L\rangle = \text{tr}(TL), T \in \mathcal{L}(\mathcal{H}), L \in \mathcal{C}_1(\mathcal{H})$. The weak* or ultraweak topology on $\mathcal{L}(\mathcal{H})$ is the topology induced by this pairing. A dual algebra $\mathcal{A}$ is a weak* closed, unital subalgebra of $\mathcal{L}(\mathcal{H})$. Let ${}^\perp\mathcal{A}$ denote the preannihilator of the dual algebra $\mathcal{A}$, that is ${}^\perp\mathcal{A} = \{\mathcal{L} \in \mathcal{C}_1(\mathcal{H}) : \langle A, L\rangle = = 0$ for all $A \in \mathcal{A}\}$. Then $\mathcal{A}$ may be identified with the dual of the Banach space $Q_\mathcal{A} = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}$ via the pairing

$$(1.1) \qquad \langle A, [L]_\mathcal{A}\rangle = \text{tr}(AL), A \in \mathcal{A}, L \in \mathcal{C}_1(\mathcal{H})$$

where $[L]_\mathcal{A}$ denotes the coset of L in $Q_\mathcal{A}$. The weak* topology induced by this pairing on $\mathcal{A}$ coincides with the relative weak* topology on $\mathcal{A}$ (cf. [1, Proposition 1.19]). For x and y belonging to $\mathcal{H}$, $x \otimes y$ denotes the rank-one operator in $\mathcal{C}_1(\mathcal{H})$ defined by $(x \otimes y)(u) = (u, y)x$, for $u \in \mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$ then $\text{tr}(A(x \otimes y)) = (Ax, y)$. Note that (1.1) implies that if $A \in \mathcal{A}$ and x and y belong to $\mathcal{H}$, then

$$(1.2) \qquad \langle A, [x \otimes y]_\mathcal{A}\rangle = (Ax, y)_\mathcal{H}.$$

If $T \in \mathcal{L}(\mathcal{H})$, then $\mathcal{A}_T$ denotes the ultraweakly closed subalgebra of $\mathcal{L}(\mathcal{H})$ generated by T and the identity. We write Q_T instead of $Q_{\mathcal{A}_T}$ and the coset of L in Q_T is written $[L]_T$. The following definitions are taken from [1].

DEFINITION 1.3. Let $\mathcal{A}$ be a dual algebra, and let n and m be cardinal numbers such that $1 \leqslant m, n \leqslant \aleph_0$. We say that $\mathcal{A}$ has property $(\mathbb{A}_{m,n})$ if for every array $\{[L_{i,j}] : 0 \leqslant i < m, 0 \leqslant j < n\}$ of elements of $Q_{\mathcal{A}}$ there exist sequences $\{x_i : 0 \leqslant i < m\}$ and $\{y_j : 0 \leqslant j < n\}$ such that $[L_{ij}] = [x_i \otimes y_j]$ for $0 \leqslant i < m, 0 \leqslant j < n$.

Property $(\mathbb{A}_{n,n})$ is usually written as property $(\mathbb{A}_n)$.

Let $L^1 = L^1(\mathbb{T})$. It is well-known that $L^\infty = L^\infty(\mathbb{T})$ is the dual space of L^1 under the pairing $\langle f, g\rangle = (2\pi)^{-1}\int_{\mathbb{T}} fg\,\mathrm{d}m, f \in L^\infty, g \in L^1$. Furthermore, $H^\infty = H^\infty(\mathbb{T})$ is a weak*-closed subspace of L^∞, and ${}^\perp(H^\infty)$ is the subspace $H_0^1 = \{f \in L^1 : \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}nt}\mathrm{d}t = 0$ for $n = 0, 1, 2, \ldots\}$. It follows (cf. [1, Proposition 1.19]) that H^∞ is the dual space of L^1/H_0^1, where the duality is given by the pairing: $\langle f, [g]\rangle = (2\pi)^{-1}\int_{\mathbb{T}} fg\,\mathrm{d}m, f \in H^\infty, [g] \in L^1/H_0^1$. If $T \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction, and $f \in H^\infty$, then we can define $f(T)$ using the Sz.-Nagy-Foiaş functional calculus. Let $\Phi_T : H^\infty \to \mathcal{A}_T$ be the map given by $\Phi_T(f) = f(T)$; then there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L^1/H_0^1$ such that $\phi_T^* = \Phi_T$ (cf. [5, Theorem 3.2] or [14, Theorem III.1.2]).

DEFINITION 1.4. $\mathbb{A}(\mathcal{H})$ is the set of all absolutely continuous contractions $T \in \mathcal{L}(\mathcal{H})$ such that Φ_T is an isometry; $\mathbb{A}$ is written instead of $\mathbb{A}(\mathcal{H})$ when no confusion will result.

Note that if $T \in \mathbb{A}$ then both ϕ_T and Φ_T are invertible.

DEFINITION 1.5. If n and m are cardinal numbers such that $1 \leqslant n, m \leqslant \aleph_0$, then $\mathbb{A}_{m,n}(\mathcal{H})$ is the set of all absolutely continuous contractions $T \in \mathcal{L}(\mathcal{H})$ such that $T \in \mathbb{A}(\mathcal{H})$ and $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. $A_{n,n}(\mathcal{H})$ is usually written as $\mathbb{A}_n(\mathcal{H})$. When no confusion will result $\mathbb{A}_{m,n}$ is used instead of $\mathbb{A}_{m,n}(\mathcal{H})$.

If $T \in \mathbb{A}$ and $\lambda \in \mathbb{D}$, then $[C_\lambda]_T = \phi_T^{-1}([P_\lambda])$, where $[P_\lambda] \in L^1/H_0^1$ and $P_\lambda(\mathrm{e}^{\mathrm{i}t})$ is the usual Poisson kernel function, $P_\lambda(\mathrm{e}^{\mathrm{i}t}) = (1 - |\lambda|^2)|1 - \overline{\lambda}\mathrm{e}^{\mathrm{i}t}|^{-2}$. It is well-known that if f belongs to $H^\infty(\mathbb{D})$, then $\langle f(T), [C_\lambda]_T\rangle = f(\lambda)$.

If W is a set of vectors in $\mathcal{H}$, then $\vee W$ denotes the smallest (closed) subspace of $\mathcal{H}$ containing W.

The proof of the following elementary proposition about functions in H^∞ is left to the reader.

PROPOSITION 1.6. *If* $h \in H^\infty(\mathbb{D})$ *and* $\lambda_0 \in \mathbb{D}$, *then there exists a function* $g \in H^\infty(\mathbb{D})$ *such that* $h(\lambda) = h(\lambda_0) + (\lambda - \lambda_0)g(\lambda)$ *and* $\|g\|_\infty \leqslant 2(1 - \lambda_0)^{-1}\|h\|_\infty$.

2. SOME DILATION THEOREMS

The following proposition is the basic tool of this paper.

Proposition 2.1. Let $S \in \mathcal{L}(\mathcal{H})$ be a subnormal operator such that $\sigma_{le}(S) \cap \mathbb{D}$ is dominating for $\mathbb{T}$. Let $A \in \mathcal{L}(\mathcal{H})$ be any absolutely continuous contraction, and L be any countable subset of $\mathcal{H}$; then there exist subspaces $\mathcal{M}$ and $\mathcal{N}$ invariant for S with $\mathcal{M} \supset \mathcal{N}$ and a closed one-to-one linear transformation $X : \mathcal{D}(X) \to \mathcal{M} \ominus \mathcal{N}$ such that:

a) *$\mathcal{D}(X)$ is a dense linear manifold in $\mathcal{H}$ containing L,*

b) *The range of X is dense in $\mathcal{M} \ominus \mathcal{N}$,*

c) *If $x \in \mathcal{D}(X)$, then $Ax \in \mathcal{D}(X)$,*

d) *$S_{\mathcal{M}\ominus\mathcal{N}}Xx = XAx$ for all $x \in \mathcal{D}(X)$.*

Before proving Proposition 2.1 some of its implications will be examined. The conclusions of this theorem are identical to a theorem about operators in $\mathbb{A}_{\aleph_0}$ (cf. [1, Theorem 5.3]). Moreover, Proposition 2.1 is a very weak dilation theorem. However for certain special cases of the operator A stronger results can be obtained. This next theorem and proof are similar to [1, Proposition 5.4].

THEOREM 2.2. *Suppose S in $\mathcal{L}(\mathcal{H})$ is a subnormal operator with $\sigma_{le}(S) \cap \mathbb{D}$ dominating for $\mathbb{T}$ and $\{\lambda_k\}_{k=1}^{\infty}$ is any sequence of (not necessarily disjoint) points from $\mathbb{D}$. Then there exists a semi-invariant subspace $\mathcal{K}$ for S such that $S_{\mathcal{K}}$ is unitarily equivalent to a normal operator N in $\mathcal{L}(\mathcal{H})$ whose matrix relative to some orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ is the diagonal matrix Diag$(\{\lambda_k\}_{k=1}^{\infty})$.*

Proof. Let A be any operator in $\mathcal{L}(\mathcal{H})$ such that each λ_k for $0 < k < \infty$ is an eigenvalue for A of infinite multiplicity. Let L be a set of vectors in $\mathcal{H}$ which contains an infinite number of linearly independent eigenvectors corresponding to each λ_k. Let $\mathcal{M}, \mathcal{N}$ and X be as in Proposition 2.1. Let $T = S_{\mathcal{M}\ominus\mathcal{N}}$. Using conclusion d) of Proposition 2.1 it follows that each λ_k is an eigenvalue of T of infinite multiplicity. Therefore one can construct easily by induction an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{M} \ominus \mathcal{N}$ such that $Tf_k = \lambda_k f_k$ for $0 < k < \infty$. Let $\mathcal{K} =_{\vee} \{f_k : 0 < k < \infty\}$. Then $\mathcal{K}$ is an invariant subspace for T and $T|\mathcal{K}$ is unitarily equivalent to N. Regarding $\mathcal{K}$ as a subspace of $\mathcal{H}$ one easily sees that $\mathcal{K}$ is semi-invariant for S and $S_{\mathcal{K}}$ is unitarily equivalent to N.

The following corollary and its proof are similar to [1, Corollary 5.5]

COROLLARY 2.3. *Let S be as in Theorem 2.2. Then there exists $\mathcal{K}$, a semi-invariant subspace for S, such that* $\dim(\mathcal{K}) = \aleph_0$ *and $S_{\mathcal{K}} = 0$.*

Proof. Let $\lambda_k = 0$ for $0 < k < \infty$ and apply Theorem 2.2.

COROLLARY 2.4. *Let S be as in Theorem 2.2. Then there exists $N \in A_{\aleph_0}$ and $\mathcal{K}$, a semi-invariant subspace for S, such that $S_{\mathcal{K}}$ is unitarily equivalent to N.*

Proof. Let $\{\lambda_k\}_{k=1}^{\infty}$ be any sequence in $\mathbb{D}$ that is dominating for $\mathbb{T}$. Apply Theorem 2.2 to obtain $\mathcal{K}$, a semi-invariant subspace for S, such that $S_{\mathcal{K}}$ is unitarily equivalent to $N = \mathrm{Diag}(\{\lambda_k\}_{k=1}^{\infty})$. We see that N is a completely non-unitary normal contraction and $\sigma(N) \cap \mathbb{D}$ is dominating for $\mathbb{T}$. Therefore $N \in A_{\aleph_0}$, (cf. [11, Theorem 1.13]).

Corollary 2.3 and Corollary 2.4 have many important consequences. To see how Corollary 2.3 can be used look at [1, Proposition 9.1, Theorem 9.2, Corollary 9.3, Corollary 9.4, Corollary 9.6]. Corollary 2.4 allows us to apply the dilation theory of $A_{\aleph_0}$. We give an example.

THEOREM 2.5. *Let S be as in Theorem 2.2, and let $\{A_j\}_{j=1}^{\infty}$ be any sequence of strict contractions acting on Hilbert spaces of dimension less than or equal $\aleph_0$. Then there exists a semi-invariant subspace $\mathcal{K}$ for S such that $S_{\mathcal{K}}$ is unitarily equivalent to $\bigoplus_{j=1}^{\infty} A_j$.*

Proof. Apply Corollary 2.4 to S to obtain a semi-invariant subspace $\mathcal{M}$ for S such that $S_{\mathcal{M}}$ belongs to $A_{\aleph_0}$. Let $T = S_{\mathcal{M}}$. Now apply [1, Theorem 5.11] to obtain a semi-invariant subspace $\mathcal{K}$ for T such that $T_{\mathcal{K}}$ is unitarily equivalent to $\bigoplus_{j=1}^{\infty} A_j$. Regarding $\mathcal{K}$ as a subspace of $\mathcal{H}$ it is easy to see that $\mathcal{K}$ is semi-invariant for S and $S_{\mathcal{K}}$ is unitarily equivalent to $T_{\mathcal{K}}$ and the proof is complete.

The following notation will hold for the remainder of this section. Let $S \in \mathcal{L}(\mathcal{H})$ be a subnormal operator with $\sigma_{le}(S) \cap \mathbb{D}$ dominating for $\mathbb{T}$. Let $M \in \mathcal{L}(\mathcal{K})$ be the minimal normal extension of S. Let $E(\cdot)$ be the projection-valued spectral measure for M and $N = M|_{E(\mathbb{D})\mathcal{K}}$. It is an easy consequence of the spectral theorem that N is normal and a completely non-unitary contraction. It will be shown later that in fact $N \in A_{\aleph_0}(E(\mathbb{D})\mathcal{K})$.

This next proposition is the essential ingredient in the proof of Proposition 2.1.

PROPOSITION 2.6. *If $\{[L_{ij}]_N : 0 \leqslant i,j < \infty\}$ is an arbitrary array of vectors in Q_N, then there exist sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_j\}_{j=0}^{\infty} \subset \mathcal{H}$ such that $[L_{ij}]_N = [E(\mathbb{D})x_i \otimes E(\mathbb{D})y_j]$ for $0 \leqslant i,j < \infty$.*

The proof of this proposition will require several lemmas and is given after the proof of Proposition 2.1

Proof of Proposition 2.1. Let $\{e_i\}_{i=0}^{\infty}$ be a sequence that is dense in $\mathcal{H}$ and contains L. Let $[L_{ij}]_N = \phi_N^{-1}\phi_A([e_i \otimes e_j]_A)$, for $0 \leqslant i, j < \infty$. Since $N \in \mathbf{A}(E(\mathbb{D})\mathcal{K})$, Φ_N^{-1} must exist. Apply Proposition 2.6 to obtain sequences $\{x_i\}_{i=0}^{\infty}, \{y_j\}_{j=0}^{\infty} \subset \mathcal{H}$ such that $[L_{ij}]_N = [E(\mathbb{D})x_i \otimes E(\mathbb{D})y_j]_N$ for $0 \leqslant i, j < \infty$. Suppose $0 \leqslant i, j, k < \infty$, then $(A^k e_i, e_j) = \langle A^k, [e_i \otimes e_j]_A\rangle$ (from 1.2)

$= \langle N^k, \phi_N^{-1}\phi_A([e_i \otimes e_j]_A)\rangle$ since $A^k = \Phi_A \Phi_N^{-1}(N^k), \phi_A^* = \Phi_A$, and $\phi_N^* = \Phi_N$.

$= \langle N^k, [L_{ij}]_N\rangle = \langle N^k, [E(\mathbb{D})x_i \otimes E(\mathbb{D})y_j]_N\rangle$ from the constructions of $[L_{ij}]_N$ and the sequences x_i and y_j

$= (N^k\ E(\mathbb{D})x_i, E(\mathbb{D})y_j)_{E(\mathbb{D})\mathcal{K}}$ from (1.2)

$= (M^k\ E(\mathbb{D})x_i, E(\mathbb{D})y_j)_{\mathcal{K}}$ since $N = M|_{E(\mathbb{D})\mathcal{K}}$

$= (E(\mathbb{D})M^k x_i, E(\mathbb{D})y_j)_{\mathcal{K}}$ since $E(\mathbb{D})$ is a spectral projection of M.

$= (M^k x_i, E(\mathbb{D})y_j)_{\mathcal{K}}$ since $E(\mathbb{D})$ is self-adjoint.

$= (S^k x_i, E(\mathbb{D})y_j)_{\mathcal{K}}$ since $x_i \in \mathcal{H}$ and $S = M|_{\mathcal{H}}$

$= (S^k x_i, P_{\mathcal{H}}E(\mathbb{D})y_j)_{\mathcal{H}}$ since $S^k x_i \in \mathcal{H}$

Let $\tilde{y}_j = P_{\mathcal{H}}E(\mathbb{D})y_j$. Therefore $(A^k e_i, e_j) = (S^k x_i, \tilde{y}_j)$ for $0 \leqslant i, j, k < \infty$. Let $\mathcal{M} =_\vee \{S^k x_i : 0 \leqslant i, k < \infty\}$ and $\mathcal{M}_* =_\vee \{S^{*k}\tilde{y}_j : 0 \leqslant j, k < \infty\}$. Let $\mathcal{N} = \mathcal{M} \cap \mathcal{M}_*^{\perp}$. Note that $\mathcal{M}$ and $\mathcal{N}$ are invariant for S and $\mathcal{M} \supset \mathcal{N}$. Write $x_i = = z_i + w_i$, where $z_i \in \mathcal{M} \ominus \mathcal{N}$ and $w_i \in \mathcal{N}$, then $(S^k x_i, \tilde{y}_j) = (S^k z_i, \tilde{y}_j) + (S^k w_i, \tilde{y}_j)$. However, $S^k w_i \in \mathcal{N} \subset \mathcal{M}_*^{\perp}$ and $\tilde{y}_j \in \mathcal{M}_*$ which implies that:

$$(S^k x_i, \tilde{y}_j) = (S^k z_i, \tilde{y}_j) = (A^k e_i, e_j). \tag{2.7}$$

Let $T = S_{\mathcal{M}\ominus\mathcal{N}}$, so $S^k z_i = T^k z_i + v_{ik}$ where $v_{ik} \in \mathcal{N}$, hence:

$$(S^k z_i, y) = (T^k z_i, y) \quad \text{for all} \quad y \in \mathcal{M}_*. \tag{2.8}$$

Define a map X_0 as follows: $X_0\left(\sum_{i=0}^{m} p_i(A)e_i\right) = \sum_{i=0}^{m} p_i(T)z_i$, where $\{p_i\}_{i=0}^{\infty}$ is a set of polynomials with complex coefficients. Since $\{e_i\}_{i=0}^{\infty}$ is dense in $\mathcal{H}$, X_0 is defined on a dense linear manifold of $\mathcal{H}$ which contains L. Then X_0 is a well-defined, closable linear transformation whose closure is one-to-one. The linearity of X_0 is obvious and to prove the rest of the assertions it suffices to show that if $\{\{p_i^{(n)}\}_{i=0}^{m_n}\}_{n=0}^{\infty}$ are sequences of polynomials such that $\lim_n \sum_{i=0}^{m_n} p_i^{(n)}(A)e_i = e$ and $\lim_n \sum_{i=0}^{m_n} p_i^{(n)}(T)z_i = z$, then $e = 0$ iff $z = 0$.

However $e = 0$ iff for every positive integer s and every sequence of polynomials $\{q_j\}_{j=0}^{s}$ it follows that $\left(e, \sum_{j=0}^{s} q_j(A^*)e_j\right) = 0$.

However $\left(e, \sum_{j=0}^{s} q_j(A^*)e_j\right) = 0$ iff

$$\lim_n \left(\sum_{i=0}^{m_n} p_i^{(n)}(A)e_i, \sum_{j=0}^{s} q_j(A^*)e_j \right) = 0 \text{ iff}$$

$\lim_n \sum_{i=0}^{m_n} \sum_{j=0}^{s} (\overline{q}_j(A)p_i^{(n)}(A)e_i, e_j) = 0$ ($\overline{q}_j$ is the polynomial whose coefficients are the complex conjugates of the coefficients of q_j) iff

$$\lim_n \sum_{i=0}^{m_n} \sum_{j=0}^{s} (\overline{q}_j(S)p_i^{(n)}(S)z_i, \tilde{y}_j) = 0 \text{ (using (2.7)) iff}$$

$$\lim_n \left(\sum_{i=0}^{m_n} p_i^{(n)}(S)z_i, \sum_{j=0}^{s} q_j(S^*)\tilde{y}_j \right) = 0 \text{ iff}$$

$$\lim_n \left(\sum_{i=0}^{m_n} p_i^{(n)}(T)z_i, \sum_{j=0}^{s} q_j(S^*)\tilde{y}_j \right) = 0 \text{ (using (2.8)) iff}$$

$$(z, \sum_{j=0}^{s} q_j(S^*)\tilde{y}_j) = 0.$$

Hence $e = 0$ iff for every positive integer s and every sequence of polynomials $\{q_j\}_{j=0}^{s}$, $\left(z, \sum_{j=0}^{s} q_j(S^*)\tilde{y}_j \right) = 0$. However this last condition is true iff $z \in \mathcal{M}_*^{\perp}$. However $z \in \mathcal{M} \ominus \mathcal{N}$, so $z \in \mathcal{M}_*^{\perp}$ iff $z = 0$. One can now conclude that X_0 is well-defined, closable and the closure of X_0 is one-to-one. Let X denote the closure of X_0. It is easy to see that L is contained in $\mathcal{D}(X)$ and that $\mathcal{D}(X)$ is dense in $\mathcal{H}$. We now show that the range of X_0 is dense in $\mathcal{M} \ominus \mathcal{N}$ which will imply that the range of X is dense in $\mathcal{M} \ominus \mathcal{N}$. The range of X_0 is $\left\{ \sum_{i=0}^{s} p_i(T)z_i \right\}$ where s ranges over all positive integers and $\{p_i\}_{i=0}^{s}$ ranges over all sets of polynomials. However $p_i(T)z_i = P_{\mathcal{M}\ominus\mathcal{N}}(p_i(S)x_i)$ and $\left\{ \sum_{i=0}^{s} p_i(S)x_i \right\}$ is dense in $\mathcal{M}$. Therefore the range of X_0 is dense in $\mathcal{M} \ominus \mathcal{N}$. Now suppose $w \in \mathcal{D}(X)$. Then there exist sequences of polynomials $\{\{p_i^{(n)}\}_{i=0}^{m_n}\}_{n=1}^{\infty}$ such that $\lim_n \sum_{i=0}^{m_n} p_i^{(n)}(A)e_i = w$ and $\lim_n \sum_{i=0}^{m_n} p_i^{(n)}(T)z_i = Xw$. Note that $Aw = \lim_n \sum_{i=0}^{m_n} Ap_i^{(n)}(A)e_i \in \mathcal{D}(X)$. So $TXw = \lim_n \sum_{i=0}^{m_n} Tp_i^{(n)}(T)z_i = XAw$. This completes the proof of Theorem 2.1.

In order to prove Proposition 2.6., several lemmas are needed. The techniques used are similar to those used to show that a contraction whose essential spectrum is dominating for $\mathbb{D}$ belongs to $\mathbb{A}_{\aleph_0}$ (cf. [12, pp. 7–28] and [5, pp. 130–134]). The following lemma is due to C. Apostol.

LEMMA 2.9. *Suppose $\lambda \in \sigma_{le}(S) \cap \mathbb{D}$ and $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is an orthonormal se-*

quence such that $\|(S-\lambda)x_n\| \to 0$, *then* $\|x_n - E(\mathbb{D})x_n\| \to 0$ *and* $\|(N-\lambda)E(\mathbb{D})x_n\| \to 0$.

Proof. Decompose $\mathcal{K} = E(\mathbb{D})\mathcal{K} \oplus E(\mathbb{C}\setminus\mathbb{D})\mathcal{K}$, and $M = N \oplus N'$ and $x_n = E(\mathbb{D})x_n \oplus z_n$ relative to this decomposition. Note that $\|(S-\lambda)x_n\|^2 = \|(M-\lambda)x_n\|^2 = \|(N-\lambda)E(\mathbb{D})x_n\|^2 + \|(N'-\lambda)z_n\|^2$. This show that $\|(S-\lambda)x_n\| \to 0$ implies that $\|(N-\lambda)E(\mathbb{D})x_n\| \to 0$ (which proves the second assertion in the lemma) and $\|(N'-\lambda)z_n\| \to 0$. Since $\sigma(N') \subset \mathbb{C}\setminus\mathbb{D}$, $N'-\lambda$ must be bounded below. So there exists $c > 0$ such that $\|(N'-\lambda)z_n\|^2 \geqslant c\|z_n\|^2$. Clearly, $\|z_n\|^2 \to 0$. However $\|z_n\|^2 = \|x_n - E(\mathbb{D})x_n\|^2$ which proves the first assertion of the lemma.

PROPOSITION 2.10. *The set* $\sigma_{le}(N) \cap \mathbb{D}$ *is dominating for* $\mathbb{T}$, *hence* $N \in \mathbb{A}(E(\mathbb{D})\mathcal{K})$.

Proof. Choose $\lambda \in \sigma_{le}(S) \cap \mathbb{D}$ and an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ such that $\|(S-\lambda)x_n\| \to 0$. (Such sequences always exist, cf. [10, Theorem I.8.8].) Consider the sequence $\{E(\mathbb{D})x_n\}_{n=1}^{\infty} \subset E(\mathbb{D})\mathcal{K}$. Since $x_n \to 0$ weakly so does $E(\mathbb{D})x_n$. Also Lemma 2.9 implies that $\|E(\mathbb{D})x_n\| \to 1$, so without loss of generality assume that $E(\mathbb{D})x_n \neq 0$ for all n. Lemma 2.9 also implies that $\|(N-\lambda)E(\mathbb{D})x_n\| \to 0$. Therefore $\lambda \in \sigma_{le}(N)$, (cf. [10, Theorem I.8.8]). This shows that $(\sigma_{le}(S) \cap \mathbb{D}) \subset (\sigma_{le}(N) \cap \mathbb{D})$ and the first statement of the lemma follows easily. The fact that $N \in \mathbb{A}(E(\mathbb{D})\mathcal{K})$ now follows from [13, Theorem 3.1].

Notice that the assumption that $\sigma_{le}(S) \cap \mathbb{D}$ is dominating for $\mathbb{T}$ is important in Proposition 2.10. If S were the unilateral shift of infinite multiplicity then the essential spectrum of S is $\mathbb{D}^-$. Therefore the essential spectrum of S intersected with $\mathbb{D}$ is dominating for $\mathbb{T}$. However M, the minimal normal extension of S, is the bilateral shift of infinite multiplicity. Therefore $M|E(\mathbb{D})\mathcal{K}$ is the zero operator.

The following lemma is similar to [5, Lemma 4.7].

LEMMA 2.11. *The closed absolutely convex hull of* $\{[C_\lambda]_N : \lambda \in \sigma_{le}(S) \cap \mathbb{D}\}$ *is equal to* $\{[L]_N : \|[L]_N\| \leqslant 1\}$.

Proof. Suppose $f \in H^\infty(\mathbb{D})$, then $\|f(N)\| = \|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in \sigma_{le}(S) \cap \mathbb{D}\} = \sup\{|\langle f(N), [C_\lambda]_N\rangle| : \lambda \in \sigma_{le}(S) \cap \mathbb{D}\}$. These equalities are true because $N \in \mathbb{A}$ and $\sigma_{le}(S) \cap \mathbb{D}$ is dominating for $\mathbb{T}$. The result now follows using [5, Proposition 2.8].

LEMMA 2.12. *Let* $\mu \in \sigma_{le}(S) \cap \mathbb{D}$ *and* $\{x_n\}_{n=1}^{\infty} \subset \mathcal{H}$ *be an orthonormal sequence such that* $\|(S-\mu)x_n\| \to 0$. *Then for all* $w \in E(\mathbb{D})\mathcal{K}$ *we have* $\|[E(\mathbb{D})x_n \otimes w]_N\| \to 0$ *and* $\|[w \otimes E(\mathbb{D})]_N\| \to 0$.

Proof. By the Hahn-Banach Theorem there exists a sequence $\{h_n\}_{n=1}^{\infty} \subset H^\infty(\mathbb{D})$

such that $\|h_n\|_\infty = 1$ and $\|[E(\mathbb{D})x_n \otimes w]_N\| = \langle h_n(N), [E(\mathbb{D})x_n \otimes w]_N\rangle =$ $= (h_n(N)E(\mathbb{D})x_n, w)$. Using Proposition 1.6 one finds a sequence $\{g_n\}_{n=1}^\infty \subset H^\infty(\mathbb{D})$ such that $h_n(\lambda) = h_n(\mu) + (\lambda - \mu)g_n(\lambda)$ and $\|g_n\|_\infty \leqslant 2(1-\mu)^{-1}$. Therefore,

$$(h_n(N)E(\mathbb{D})x_n, w) =$$
$$= (h_n(\mu)E(\mathbb{D})x_n, w) + (g_n(N)(N-\mu)E(\mathbb{D})x_n, w)$$
$$\leqslant |h_n(\mu)||(E(\mathbb{D})x_n, w)| + \|g_n(N)\|\,\|w\|\,\|(N-\mu)E(\mathbb{D})x_n\|.$$

However, $|h_n(\mu)| \leqslant 1$ for each n, $E(\mathbb{D})x_n \to 0$ weakly, $\|g_n(N)\| \leqslant 2(1-\mu)^{-1}$ for each n and Lemma 2.9 implies that $\|(N-\mu)E(\mathbb{D})x_n\| \to 0$. This clearly implies the first assertion of the lemma.

Now choose another sequence $\{h_n\}_{n=1}^\infty \in H^\infty(\mathbb{D})$ such that $\|h_n\|_\infty = 1$ and $\|[w \otimes E(\mathbb{D})x_n]_N\| = \langle h_n(N), [w \otimes E(\mathbb{D})x_n]_N\rangle = (h_n(N)w, E(\mathbb{D})x_n)$. Apply Proposition 1.6 again to obtain $\{g_n\}_{n=1}^\infty \in H^\infty(\mathbb{D})$ with the same properties as above. This implies that $(h_n(N)w, E(\mathbb{D})x_n) =$

$$= (h_n(\mu)w, E(\mathbb{D})x_n) + ((N-\mu)g_n(N)w, E(\mathbb{D})x_n)$$
$$= (h_n(\mu)w, E(\mathbb{D})x_n) + (g_n(N)w, (N-\mu)^* E(\mathbb{D})x_n)$$
$$\leqslant |h_n(\mu)||(w, E(\mathbb{D})x_n)| + \|g_n(N)\|\,\|w\|\,\|(N-\mu)^* E(\mathbb{D})x_n\|.$$

However, $|h_n(\mu)| \leqslant 1$ for each n, $E(\mathbb{D})x_n \to 0$ weakly, $\|g_n(N)\| \leqslant 2(1-\mu)^{-1}$ for each n and Lemma 2.9 implies that $\|(N-\mu)^* E(\mathbb{D})x_n\| = \|(N-\mu)E(\mathbb{D})x_n\| \to 0$. This proves the second assertion of the lemma.

LEMMA 2.13. *Let $\mu \in \mathbb{D}$ and $\{x_n\}_{n=1}^\infty \subset \mathcal{H}$ be an orthonormal sequence such that $\|(S-\mu)x_n\| \to 0$. Then $\|[E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N - [C_\mu]_N\| \to 0$.*

Proof. Use the Hahn-Banach theorem to find a sequence $\{h_n\}_{n=1}^\infty \subset H^\infty(\mathbb{D})$ such that $\|h_n\|_\infty = 1$ and $\|[E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N - [C_\mu]_N\| = \langle h_n(N), [E(\mathbb{D})x_n \otimes$ $\otimes E(\mathbb{D})x_n]_N - [C_\mu]_N\rangle$. Use Proposition 1.6 to obtain $\{g_n\}_{n=1}^\infty \subset H^\infty(\mathbb{D})$ with the indicated properties. Compute:

$$\|[E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N - [C_\mu]_N\| =$$
$$= \langle h_n(\mu)I + (N-\mu)g_n(N), [E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N - [C_\mu]_N\rangle$$
$$= h_n(\mu)\{\langle I, [E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N\rangle - \langle I, [C_\mu]_N\rangle\} +$$
$$+ \langle (N-\mu)g_n(N), [E(\mathbb{D})x_n \otimes E(\mathbb{D})x_n]_N\rangle + \langle (N-\mu)g_n(N), [C_\mu]_N\rangle$$
$$= h_n(\mu)(\|E(\mathbb{D})x_n\|^2 - 1) + (g_n(N)(N-\mu)E(\mathbb{D})x_n, E(\mathbb{D})x_n) + 0$$
$$\leqslant |h_n(\mu)|\,|\|E(\mathbb{D})x_n\|^2 - 1| + \|g_n(N)\|\,\|(N-\mu)E(\mathbb{D})x_n\|\,\|E(\mathbb{D})x_n\|.$$

However, $\|E(\mathbb{D})x_n\| \to 1$, $|h_n(\mu)| \leqslant 1$ for all n, $\|g_n(N)\| \leqslant 2(1-|\mu|)^{-1}$ for all n, and Lemma 2.9 implies that $\|(N-\mu)E(\mathbb{D})x_n\| \to 0$, and the proof is complete.

For a proof of the following lemma consult [12, Lemma 3.8].

LEMMA 2.14. *Let $\{\lambda_k\}_{k=0}^{m-1} \subset \sigma_{le}(S) \cap \mathbb{D}$. Then there exists an orthonormal family $\{e_n^k : 0 \leqslant k < m, 0 \leqslant n < \infty\} \subset \mathcal{H}$ such that $\lim_n \|(S-\lambda_k)e_n^k\| = 0$ for*

$0 \leqslant k < m$.

PROPOSITION 2.15. *Suppose* $[L]_N \in Q_N$ *with* $\|[L]_N\| \leqslant 1, \varepsilon > 0$, *and* $\{\xi_r\}_{r=1}^p$, $\{\eta_r\}_{r=1}^p \subset \mathcal{H}$. *Then there exists* $x, y \in \mathcal{H}$ *such that:*

$$\|x\|, \|y\| \leqslant 1 \tag{2.16}$$

$$\|[L]_N - [E(\mathbf{D})x \otimes E(\mathbf{D})y]_N\| < \varepsilon \tag{2.17}$$

$$\|[E(\mathbf{D})x \otimes E(\mathbf{D})\eta_r]_N\| < \varepsilon, \|[E(\mathbf{D})\xi_r \otimes E(\mathbf{D})y]_N\| < \varepsilon \text{ for } 1 \leqslant r \leqslant p. \tag{2.18}$$

Proof. Apply Lemma 2.11 to obtain $\{\lambda_j\}_{j=1}^m \subset \sigma_{le}(S) \cap \mathbf{D}$ such that $\|[L]_N - \sum_{j=1}^m \delta_j^2 [C_{\lambda_j}]_N\| < \frac{\varepsilon}{3}$ where $\{\delta_j\}_{j=1}^m \subset \mathbf{C}$ and $\sum_{j=1}^m |\delta_j|^2 \leqslant 1$. Since $\{\lambda_j\}_{j=1}^m \subset \sigma_{le}(S) \cap \mathbf{D}$, Lemma 2.14 implies the existence of an orthonormal family $\{e_n^j : 1 \leqslant n < \infty, 1 \leqslant j \leqslant m\} \subset \mathcal{H}$ such that $\lim_n \|(S - \lambda_j)e_n^j\| = 0$. Define $x = \sum_{j=1}^m \delta_j e_{n_j}^j$ and $y = \sum_{j=1}^m \overline{\delta}_j e_{n_j}^j$, where the choice of n_j will be specified. The sequence $\{n_j\}_{j=1}^m$ will be defined inductively. First n_1 is chosen sufficiently large that:

$$\|[C_{\lambda_1}]_N - [E(\mathbf{D})e_{n_1}^1 \otimes E(\mathbf{D})e_{n_1}^1]_N\| < \frac{\varepsilon}{3} \tag{2.19}$$

$$\|[E(\mathbf{D})e_{n_1}^1 \otimes E(\mathbf{D})\eta_r]_N\| < \frac{\varepsilon}{|\delta_1| m} \text{ for } 1 \leqslant r \leqslant p. \tag{2.20}$$

$$\|[E(\mathbf{D})\xi_r \otimes E(\mathbf{D})e_{n_1}^1]_N\| < \frac{\varepsilon}{|\delta_1| m} \text{ for } 1 \leqslant r \leqslant p. \tag{2.21}$$

Condition (2.19) is possible because of Lemma 2.13. Conditions (2.20) and (2.21) are possible because of Lemma 2.12. Suppose $j > 1$ and n_k has been chosen for all $1 \leqslant k < j$, n_j is then chosen sufficiently large that:

$$\|[C_{\lambda_j}]_N - [E(\mathbf{D})e_{n_j}^j \otimes E(\mathbf{D})e_{n_j}^j]_N\| < \frac{\varepsilon}{3} \tag{2.22}$$

$$\|[E(\mathbf{D})e_{n_j}^j \otimes E(\mathbf{D})\eta_r]_N\| < \frac{\varepsilon}{m|\delta_j|} \text{ for } 1 \leqslant r \leqslant p \tag{2.23}$$

$$\|[E(\mathbf{D})\xi_r \otimes E(\mathbf{D})e_{n_j}^j]_N\| < \frac{\varepsilon}{m|\delta_j|} \text{ for } 1 \leqslant r \leqslant p \tag{2.24}$$

$$\|[E(\mathbb{D})e^k_{n_k} \otimes E(\mathbb{D})e^j_{n_j}]_N\| < (3(m^2-m)|\delta_j\delta_k|)^{-1}\varepsilon \text{ for } 1 \leqslant k < j \tag{2.25}$$

$$\|[E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^k_{n_k}]_N\| < (3(m^2-m)|\delta_k\delta_j|)^{-1}\varepsilon \text{ for } 1 \leqslant k < j. \tag{2.26}$$

Condition (2.22) is possible using Lemma 2.13. Conditions (2.23)–(2.26) are possible using Lemma 2.12. To finish the proof it must be shown that x and y as defined above satisfy (2.16)–(2.18).

First note that $\|x\|^2 = \|y\|^2 = \sum_{j=1}^m |\delta_j|^2 \leqslant 1$, since $\{e^j_n : 1 \leqslant j \leqslant m, 1 \leqslant n < \infty\}$ is an orthonormal family. Therefore (2.16) is satisfied.

Now compute $\|[L]_N - [E(\mathbb{D})x \otimes E(\mathbb{D})y]_N\| \leqslant \left\|[L]_N - \sum_{j=1}^m \delta_j^2[C_{\lambda_j}]_N\right\| +$

$$+\left\|\sum_{j=1}^m \delta_j^2[C_{\lambda_j}]_N - \sum_{j=1}^m \delta_j^2[E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N\right\| +$$

$$+\left\|\sum_{j=1}^m \delta_j^2[E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N - [E(\mathbb{D})x \otimes E(\mathbb{D})y]_N\right\|$$

The term $\left\|[L]_N - \sum_{j=1}^m \delta_j^2[C_{\lambda_j}]_N\right\|$ is less than $\frac{\varepsilon}{3}$ because of the choice of the sequence $\{\lambda_j\}$.

The term $\left\|\sum_{j=1}^m \delta_j^2[C_{\lambda_j}]_N - \sum_{j=1}^m \delta_j^2[E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N\right\| =$

$$= \left\|\sum_{j=1}^m \delta_j^2([C_{\lambda_j}]_N - [E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N)\right\| \leqslant$$

$$\leqslant \sum_{j=1}^m |\delta_j|^2 \|[C_{\lambda_j}]_N - [E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N\| < \frac{\varepsilon}{3} \text{ since}$$

$\sum_{j=1}^m |\delta_j|^2 \leqslant 1$ and conditions (2.19) and (2.22) hold.

The term $\left\|\sum_{j=1}^m \delta_j^2[E(\mathbb{D})e^j_{n_j} \otimes E(\mathbb{D})e^j_{n_j}]_N -\right.$

$$\left. - \left[\left(\sum_{i=1}^m E(\mathbb{D})\delta_i e^i_{n_i}\right) \otimes \left(\sum_{j=1}^m E(\mathbb{D})\delta_j e^j_{n_j}\right)\right]_N\right\| \leqslant$$

$$\leqslant \left\|\sum_{i\neq j}\sum \delta_i\delta_j[E(\mathbb{D})e^i_{n_i} \otimes E(\mathbb{D})e^j_{n_j}]_N)\right\| \leqslant$$

$$\leqslant \sum_{i\neq j}\sum |\delta_j\delta_i| \|[E(\mathbb{D})e^i_{n_i} \otimes E(\mathbb{D})e^j_{n_j}]_N)\| <$$

$$< \sum\sum_{i\neq j} |\delta_i\delta_j|(3(m^2-m)|\delta_j\delta_i|)^{-1}\varepsilon \text{ using conditions (2.25) and (2.26).}$$

$$= \frac{\varepsilon}{3}$$

Now compute $\|[E(\mathbf{D})x\otimes E(\mathbf{D})\eta_r]_N\| = \left\| \left[\left\{ E(\mathbf{D}) \left(\sum_{j=1}^{m} \delta_j e^j_{n_j} \right) \right\} \otimes E(\mathbf{D})\eta_r \right]_N \right\| \leqslant$

$\leqslant \sum_{j=1}^{m} |\delta_j| \|[E(\mathbf{D})e^j_{n_j} \otimes E(\mathbf{D})\eta_r]_N\| < \varepsilon$, using conditions (2.20) and (2.23) and the fact that $\sum_{j=1}^{m} |\delta_j| \leqslant 1$.

A similar computation using conditions (2.21) and (2.24) shows that $\|[E(\mathbf{D})\xi_r\otimes \otimes E(\mathbf{D})y]_N\| < \varepsilon$. This completes the proof of the proposition.

Proof of Proposition 2.6. Note that Proposition 2.15 says that Proposition 2.6 is (roughly speaking) approximately true. To show that Proposition 2.6 is really true requires some results from [2]. Define B: $\mathcal{H}\times\mathcal{H} \to Q_N$ by $B(x,y) = [E(\mathbf{D})x\otimes \otimes E(\mathbf{D})y]_N$. It is easy to see that B is a continuous bilinear map. Now Proposition 2.15 says that B satisfies the property $\Delta_{0,1}$ relative to $\mathcal{H}$ and $\mathcal{H}$, (cf. [2, p. 320, Definition 5.4]). Then [2, p. 332, Theorem 7.2] implies that Proposition 2.6 is true.

The above construction shows that there is a close relationship between N and S. Proposition 2.2 says that N belongs to $\mathbf{A}_{\aleph_0}$. Since S may not be a contraction one cannot expect S to belong to $\mathbf{A}_{\aleph_0}$, but does $\mathcal{A}_S$ have property $(\mathbf{A}_{\aleph_0})$? The following proposition and example show that this need not occur.

PROPOSITION 2.27. *Suppose $1\leqslant n\leqslant \aleph_0, T\in\mathcal{L}(\mathcal{H})$ and $\mathcal{A}_T$ has property $(\mathbf{A}_n)$. Suppose there exist $\lambda\in\mathbf{C}$ and $[L_\lambda]_T\in Q_T$ with the property that $\langle p(T),[L_\lambda]_T\rangle = = p(\lambda)$ for all polynomials p. Then there exist $\mathcal{M}$ and $\mathcal{N}$, invariant subspaces for T, such that $\mathcal{M}\supset\mathcal{N}$, $\dim(\mathcal{M}\ominus\mathcal{N})\geqslant n$, and $T_{\mathcal{M}\ominus\mathcal{N}} = \lambda I_{\mathcal{M}\ominus\mathcal{N}}$.*

Proof. For $0\leqslant i,j<n$, let $[L_{ij}]_T = 0$ if $i\neq j$ and $[L_{ij}]_T = [L_\lambda]_T$ if $i=j$. Since $\mathcal{A}_T$ has property $(\mathbf{A}_n)$ there exist sequences $\{x_i : 0\leqslant i<n\}$ and $\{y_j : 0\leqslant j<n\}$ such that $[L_{ij}]_T = [x_i\otimes y_j]_T$. Let $\mathcal{M} =_\vee \{T^k x_i : k\geqslant 0, 0\leqslant i<n\}$ and $\mathcal{M}_* =_\vee \{T^{*k} y_j : k\geqslant 0, 0\leqslant j<n\}$. Let $\mathcal{N} = \mathcal{M}\cap\mathcal{M}_*^{\perp}$.

Choose $z\in\mathcal{M}\ominus\mathcal{N}$. Since $z\in\mathcal{M}$, there exist sequences of polynomials $\{\{p_i^{(k)}\}_{i=0}^{m_k}\}_{k=1}^{\infty}$ where $m_k<n$ and $z = \lim_k \sum_{i=0}^{m_k} p_i^{(k)}(T)x_i$. Suppose $s<n$ and $\{q_j\}_{j=0}^{s}$ are polynomials, then $\left((T-\lambda)z, \sum_{j=0}^{s} q_j(T^*)y_j\right) =$

$$= \lim_k \left((T-\lambda)\left(\sum_{i=0}^{m_k} p_i^{(k)}(T)x_i\right), \sum_{j=0}^{s} q_j(T^*)y_j \right)$$

$$= \lim_k \sum_{i=0}^{m_k}\sum_{j=0}^{s}((T-\lambda)p_i^{(k)}(T)x_i, q_j(T^*)y_j)$$

$$= \lim_k \sum_{i=0}^{m_k}\sum_{j=0}^{s}\{(Tp_i^{(k)}(T)\overline{q}_j(T)x_i, y_j) - \lambda(p_i^{(k)}(T)x_i, q_j(T)y_j)\}$$

(Define polynomials $a_{ij}^{(k)}(\mu) = \mu p_i^{(k)}(\mu)\overline{q}_j(\mu)$ and $b_{ij}^{(k)}(\mu) = p_i^{(k)}(\mu)\overline{q}_j(\mu)$)

$$= \lim_k \sum_{i=0}^{m_k}\sum_{j=0}^{s}(a_{ij}^{(k)}(T)x_i, y_j) - (b_{ij}^{(k)}(T)x_i, y_j)$$

(If $i \neq j$, then $(a_{ij}^{(k)}(T)x_i, y_j) = (b_{ij}^{(k)}(T)x_i, y_j) = 0$ because $[x_i \otimes y_j]_T = [0]_T$.)

$$= \lim_k \sum_{i=0}^{r_k}(a_{ii}^{(k)}(T)x_i, y_i) - \lambda(b_{ii}^{(k)}(T)x_i, y_i)$$

(Note that $r_k = \min(m_k, s)$)

$$= \lim_k \sum_{i=0}^{r_k}(\langle a_{ii}(T), [L_\lambda]\rangle - \lambda\langle b_{ii}(T), [L_\lambda]\rangle)$$

(because $[x_i \otimes y_i]_T = [L_\lambda]_T$)

$$= \lim_k \sum_{i=0}^{r_k}(a_{ii}^{(k)}(\lambda) - \lambda b_{ii}^{(k)}(\lambda)) = 0.$$

Hence $(T-\lambda)z$ is perpendicular to $\mathcal{M}_*$, which means that $(T-\lambda)z \in \mathcal{N}$. Therefore, $P_{\mathcal{M}\ominus\mathcal{N}}(T-\lambda)z = 0$ for all $z \in \mathcal{M}\ominus\mathcal{N}$. This means that $P_{\mathcal{M}\ominus\mathcal{N}}(T-\lambda I_{\mathcal{H}})|_{\mathcal{M}\ominus\mathcal{N}} = 0$, or $T_{\mathcal{M}\ominus\mathcal{N}} = \lambda I_{\mathcal{M}\ominus\mathcal{N}}$.

To complete the proof it must be shown that $\dim(\mathcal{M}\ominus\mathcal{N}) \geqslant n$. Let $w_i = P_{\mathcal{M}\ominus\mathcal{N}}x_i$. Then $\{w_i : 0 \leqslant i < n\}$ is a linearly independent family of vectors in $\mathcal{M}\ominus\mathcal{N}$. To see this suppose $m < n$, and $\sum_{i=0}^{m}\alpha_i w_i = 0$, where $\alpha_i \in \mathbb{C}$. Obviously, $\sum_{i=0}^{m}\alpha_i x_i \in \mathcal{N} \subset \mathcal{M}_*^{\perp}$. In particular one sees that $\left(\sum_{i=0}^{m}\alpha_i x_i, y_j\right) = 0$ for $0 \leqslant j < n$. However, $\left(\sum_{i=0}^{m}\alpha_i x_i, y_j\right) = \sum_{i=0}^{m}\alpha_i(x_i, y_j) = \sum_{i=0}^{m}\alpha_i\langle I, [x_i \otimes y_j]_T\rangle = \sum_{i=0}^{m}\alpha_i\langle I, [L_{ij}]_T\rangle = \alpha_j\langle I, [L_\lambda]_T\rangle = \alpha_j$ for $0 \leqslant j < m$. Therefore, $\alpha_j = 0$ for $0 \leqslant j < m$. This shows that $\{w_i\}$ is linearly independent.

Example 2.28. Let A denote Lebesgue area measure on $\mathbb{C}$. Let $L^2(A, \mathbb{D})$ denote the Hilbert space of all measurable functions on $\mathbb{D}$ which are square-integrable and M_z the operator on $L^2(A, \mathbb{D})$ defined by $(M_z f)(z) = zf(z)$. Let $N = M_z \oplus 2$ acting on $L^2(A, \mathbb{D}) \oplus \mathbb{C}$. It is easy to see that N is subnormal (in fact N is normal) and $\sigma_{le}(N) = \mathbb{D}^-$. We now show that $\mathcal{A}_N$ does not have property $(\mathbb{A}_2)$. Suppose the contrary is true in order to obtain a contradiction. The element $[L]_N = [(0\oplus 1)\otimes(0\oplus 1)]_N$ has the property that $\langle p(N), [L]_N\rangle = p(2)$ for all polynomials p. Therefore Proposition 2.27

implies that there exists $\mathcal{K}$ such that: $\mathcal{K} = \mathcal{M} \ominus \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are invariant for N, $\mathcal{M} \supset \mathcal{N}, N_{\mathcal{K}} = 2I_{\mathcal{K}}$, and $\dim(\mathcal{K}) = n \geqslant 2$. Let $\{f_i \oplus \alpha_i\}_{0 \leqslant i < n}$ be an orthonormal basis for $\mathcal{K}$, with f_i not the zero function for $i \geqslant 1$. (Note that at most one of the f_i can be the zero function.) Compute: $N(f_i \oplus \alpha_i) = zf_i \oplus 2\alpha_i = (2f_i \oplus 2\alpha_i) + (g_i \oplus \beta_i)$, where $g_i \oplus \beta_i \in \mathcal{N}$. This implies that $\beta_i = 0$ and $g_i(z) = (z-2)f_i(z)$ almost everywhere on $\mathbf{D}$. Since $g_i \oplus 0 \in \mathcal{N}$ and $f_i \oplus \alpha_i \in \mathcal{M} \ominus \mathcal{N}$ it follows that g_i is orthogonal to f_i, i.e. $(M_z - 2)f_i$ is orthogonal to f_i. Let $h_i = \|f_i\|^{-1} f_i$ for $i \geqslant 1$. So $((M_z - 2)h_i, h_i) = 0$, or $(M_z h_i, h_i) - 2 = 0$, which implies that $(M_z h_i, h_i) = 2$. However, $\|M_z\| = \|h_i\| = 1$. This is a contradiction.

REFERENCES

1. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., *Dual Algebras with applications to invariant subspaces and dilation theory*, CBMS Regional Conf. Ser. in Math., no. 56, Amer. Math. Soc., Providence, R.I., 1985.
2. BERCOVICI, H.; FOIAŞ, C.; PEARCY, C., Banach space methods and dual operator algebras, *J. Funct. Anal.*, **78**(1988) no. 2, 306–345.
3. BROWN, L.; SHIELDS, A.; ZELLER, K., On absolutely convergent sums, *Trans. Amer. Math. Soc.*, **96**(1960), 162–183.
4. BROWN, S., Some invariant subspaces for subnormal operators, *Integral Equations Operator Theory*, **1**(1978), 310–333.
5. BROWN, S.; CHEVREAU, B.; PEARCY, C., Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1**(1979), 123–136.
6. BROWN, S.; CHEVREAU, B.; PEARCY, C., On the structure of contraction operators, II, *J. Funct. Anal.*, **76**(1988), 30–55.
7. CHEVREAU, B.; EXNER, G.; PEARCY, C., Structure theory and reflexivity of contraction operators, *Bull. Amer. Math. Soc.*, **19**(1988), 299–301.
8. CHEVREAU, B.; EXNER, G.; PEARCY, C., On the structure of contraction operators, III, *Michigan Math. J.*, **36**(1989) 29–62.
9. CHEVREAU, B.; PEARCY, C. On the structure of contraction operators, I, *J. Funct. Anal.*, **76**(1988), 1–29.
10. CONWAY, J. B., *Subnormal Operators*, Pitman, Boston, Mass., 1981.
11. EXNER G.; SULLIVAN P., Normal operators and the classes $\mathbb{A}_n$, *J. Operator Theory*, **19**(1988) 81–94.
12. ROBEL G., *Ph. D. Thesis*, University of Michigan, 1982.
13. SULLIVAN, P., Subnormal operators and the class $\mathbb{A}$, *J. Operator Theory*, **18**(1987), 237–248.
14. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.

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