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FULL MULTIPLICITY ERGODIC ACTIONS OF COMPACT GROUPS ON VON NEUMANN ALGEBRAS

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1. INTRODUCTION

This paper considers the full multiplicity ergodic actions of the groups U(2), SU(2) $\times$ B, and SO(3) $\times$ B on von Neumann algebras, where B is a compact abelian group. An action α of a group G on a von Neumann algebra $\mathcal{M}$ is *ergodic* if the fixed point algebra consists only of the scalars. Ergodic actions of compact abelian groups have been completely classified in terms of bicharacters of dual groups by various authors (e.g. [1], [9]).

The first major breakthrough in understanding ergodic actions of non-abelian compact groups on von Neumann algebras was accomplished by the Finiteness Theorem of Høegh-Krohn, Landstad and Størmer [5]. They showed that if $\alpha : G \to \to \mathrm{Aut}(\mathcal{M})$ is an ergodic action, then the unique G-invariant faithful normal state on $\mathcal{M}$ is a trace and the multiplicity of an irreducible representation $\pi \in \hat{G}$ in $\mathcal{M}$ is $\leqslant \dim \pi$. In particular, $\mathcal{M}$ is a finite von Neumann algebra. They also showed that $\mathcal{M}$ is hyperfinite. In fact $\mathcal{M}$ has to be a type I or type II algebra. The ergodic actions on type I von Neumann algebras are the so called classical actions [12, Section 11]. This left open the problem of whether any non-abelian classical compact group can act ergodically on the hyperfinite II_1 factor. Subsequently Antony Wassermann [12] developed some general machinery for studying ergodic actions. Wassermann's Multiplicity Map Theorem [12, Theorem 17] and the associated multiplicity diagrams play a crucial role in the study of ergodic actions. See section 2 for more details on the general theory of ergodic actions of compact groups on von Neumann algebras. In [14] Wassermann classified the ergodic actions of SU(2) (and hence SO(3)). He showed that SU(2) (and hence SO(3)) can only act ergodically on type I von Neumann algebras. In particular, SU(2) (and hence SO(3)) has no ergodic actions on the hyperfinite II_1 factor.

For an ergodic action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$, the crossed product is type I with atomic center (see [12, Corollary 1 to Theorem 4]). An important class of ergodic actions are the *full multiplicity* ergodic actions. These are the ergodic actions for which the crossed product is a type I factor. The expression "full multiplicity" is used because an ergodic action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is a full multiplicity ergodic action if and only if every $\pi \in \hat{G}$ appears with maximum (full) multiplicity $\dim \pi$ in $\mathcal{M}$ [12, Theorem 15]. In [13] and [6] a theory is developed for full multiplicity ergodic actions of non-abelian compact groups which parallels the classification theory of ergodic actions of compact abelian groups. There it is shown that full multiplicity ergodic actions are classified by cocyles and bicharacters of the group dual, with analogues of the non-degeneracy criteria for the action to be on a factor. The quantum Yang-Baxter equations appear as a consequence of the bicharacter relations [13, Lemma 26].

In this paper we show that every full multiplicity ergodic action of U(2) is induced from a full multiplicity ergodic action of a maximal torus (Theorem 2). We show that every full multiplicity ergodic action of $SU(2) \times B$ is induced from a full multiplicity ergodic action $\mathbf{T} \times B$ (Theorem 1(a)). In particular, every full multiplicity ergodic action of $SU(2) \times \mathbf{T}^n$ is induced from a full multiplicity ergodic action of a maximal torus. Consequently U(2) and $SU(2) \times B$ have no full multiplicity ergodic actions on a factor. We also show that $SO(3) \times B$ has no full multiplicity ergodic actions on a factor (Theorem 1(b)). In [14] and [15] Wassermann has obtained results on full multiplicity ergodic actions of the groups $SU(2), SO(3), SU(2) \times SU(2)$ and SU(3).

The $SU(2) \times B$ and $SO(3) \times B$ cases will be dealt with by an application of the commutation relations between the fundamental unitary eigenmatrices. In the U(2) case we obtain an ergodic action of the finite covering group $SU(2) \times \mathbf{T}$ (this is the finite cover approach) and determine the possible multiplicity diagrams which will be very tractable because of the constraints. We then pass to a semidual action and use the method mentioned in the $SU(2) \times B$ case above.

2. GENERAL THEORY

In this section we discuss some general theory of ergodic actions of compact groups on von Neumann algebras. Let G be a compact second countable group. All integrals will be with respect to Haar measure. An action of a group G on a von Neumann algebra $\mathcal{M}$ is a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ such that $g \to \alpha_g(x)$ is σ-weakly continuous for $x \in \mathcal{M}$. The fixed point algebra $\{x \in \mathcal{M} : \alpha_g(x) = x,\ \forall g \in G\}$ is denoted by $\mathcal{M}^\alpha$. The action α is *ergodic* if the fixed point algebra consists only of the scalars, i.e. $\mathcal{M}^\alpha = \mathbb{C}1_{\mathcal{M}}$.

Let $\mathcal{M}$ be σ-finite von Neumann algebra. Let $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ be an action. Let $\hat{G}$ be the equivalence class of irreducible unitary representations of G. For $\pi \in \hat{G}$, let V_π be the representation space of π and define

$$E_\pi(x) = \int_G \dim \pi \overline{\mathrm{Tr}\pi(g)} \alpha_g(x) \mathrm{d}g, \quad x \in \mathcal{M}.$$

The *spectral subspace* $\mathcal{M}_\pi$ corresponding to π is defined to be the linear span of all the G-invariant subspaces of $\mathcal{M}$ isomorphic to V_π. E_π is a projection onto $\mathcal{M}_\pi$ which is σ-weakly closed and $\mathcal{M}_\pi \cap \mathcal{M}_\nu = \{0\}$ for $\pi \not\sim \nu$. $\mathcal{M}$ is the σ-weak closure of the direct sum of its spectral subspaces, i.e. $\mathcal{M} = \overline{\bigoplus_{\pi \in \hat{G}} \mathcal{M}_\pi}$. Let φ be a faithful normal state on $\mathcal{M}$ and define $\eta(x) = \int_G \varphi(\alpha_g(x)) \mathrm{d}g$, $x \in \mathcal{M}$. η is a G-invariant faithful normal state on $\mathcal{M}$ and we can define a G-invariant inner product on $\mathcal{M}$ by setting $\langle x, y \rangle = \eta(y^* x)$. $\mathcal{M}_\pi$ is orthogonal to $\mathcal{M}_\nu$ for $\pi \not\sim \nu$. $\mathcal{M}_\pi$ is the σ-weak closure of a direct sum of pairwise orthogonal G-invariant copies of π, and the *multiplicity* of π in $\mathcal{M}$ is defined to be the maximum number of pairwise orthogonal copies of π in $\mathcal{M}_\pi$. An ergodic action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is called a full multiplicity ergodic action if and only if every $\pi \in \hat{G}$ appears with maximum (full) multiplicity $\dim \pi$ in $\mathcal{M}$ (see [12, Theorem 15] for equivalent conditions).

Let H be a closed subgroup of G, and let $\alpha : H \to \mathrm{Aut}(\mathcal{M})$ be an action of H on a von Neumann algebra $\mathcal{M}$. We define the induced algebra as

$$\mathrm{ind}_{H\uparrow G}\mathcal{M} = \{f \in L^\infty(G, \mathcal{M}) : f(gh) = \alpha_h^{-1}(f(g)) \quad h \in H,\ g \in G\},$$

with the induced action $\mathrm{ind}\alpha$ of G given by left translation. See [7], [10], or [12] for more details on induced actions.

If $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is an action, then define the crossed product by $\mathcal{M} \rtimes G = (\mathcal{M} \otimes B(L^2(G)))^{\alpha \otimes \mathrm{Ad}\lambda}$, where λ is left translation on $L^2(G)$. Let $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ be an ergodic action. From [12, Corollary 1 to Theorem 4] we have $\mathcal{M} \rtimes G \cong \bigoplus_i B(H_i)$. Let e_i be a minimal projection in $B(H_i)$ and let $e = \sum_i e_i$. Following [12, Section 8] we set $\mathcal{M}_1 = e(\mathcal{M} \otimes B(L^2(G)))e$ and define the perturbed action α' to be the action of G obtained by restricting $\alpha \otimes \mathrm{Ad}\lambda$ to $\mathcal{M}_1$. We have that $\mathcal{M}_1^{\alpha'} \cong l_n^\infty$ $(1 \leqslant n \leqslant \infty)$. Let $e_1, e_2, e_3, \ldots$ be the minimal projections in $\mathcal{M}_1^{\alpha'}$ and set $c_i = \mathrm{Tr}(e_i)$, $0 < c_i < \infty$. Let $m_{ij}(\pi)$ be the multiplicity of $\pi \in \hat{G}$ in $e_i \mathcal{M}_1 e_j$ and extend

$$M : \hat{G} \to M_n(\mathbb{Z})$$
$$\pi \mapsto (m_{ij}(\pi))$$

additively to a map $M : R(G) \to M_n(\mathbf{Z})$, where $R(G)$ is the representation ring of G. If $\pi \in R(G)$, then the matrix $M(\pi)$ is called a multiplicity matrix. The following *Multiplicity Map* Theorem of Wassermann is fundamental. It provides powerful product formulas for the multiplicities of spectral subspaces. See [12, Theorem 17] for the proof.

MULTIPLICITY MAP THEOREM (WASSERMANN). (a) *$M : R(G) \to M_n(\mathbf{Z})$ is a ring $*$-homomorphism, i.e. $M(\pi \otimes \nu) = M(\pi)M(\nu)$ and $M(\overline{\pi}) = M(\pi)^t$ for any $\pi, \nu \in R(G)$, where $\overline{\pi}$ is the conjugate representation to π.*

(b) *$M(\pi)\underline{c} = \dim \pi \underline{c}$ for all $\pi \in R(G)$, where $\underline{c}$ is the vector $(c_1, c_2, \ldots)^t$, i.e. $\sum_k m_{ik}(\pi)c_k = \dim \pi c_i$. This equation is called the multiplicity matrix equation.*

We also have that the action $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is equivariantly isomorphic to the action of G obtained by restricting α' to the algebra $e_i\mathcal{M}_1e_i$ for some e_i.

Let τ be any representation of G. There is an associated multiplicity diagram for τ [12, Section 10].

3. FACTS ABOUT COMPACT GROUPS

In this section we gather some information on the compact groups which we will be looking at later on. We denote the group 2×2 unitary matrices with determinant one by

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Denote the circle group by $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. The only proper closed subgroups of $\mathbf{T}$ are the finite cyclic subgroups. Let π_m be the $m+1$ dimensional irreducible representation of SU(2), and let χ_k, $k \in \mathbf{Z}$, be the irreducible representation of $\mathbf{T}$ such that $\chi_k(z) = z^k$. Define $\gamma_{m,k} = \pi_m \otimes \chi_k$. This is an irreducible representation of $\mathrm{SU}(2) \times \mathbf{T}$ and any irreducible representation of $\mathrm{SU}(2) \times \mathbf{T}$ is of this form. Denote the group of 3×3 orthogonal matrices with determinant one by SO(3). There is a double cover $\Phi : \mathrm{SU}(2) \to SU(3)$, this is a group homomorphism with kernel $\{1, -1\}$, the center of SU(2). There are nine types of closed subgroups of SU(2) (unique up to conjugation) [14, page 316]. The only non-trivial closed normal subgroup of SU(2) is $\{1, -1\}$ [4, page 136]. SO(3) has no non-trivial closed normal subgroups. The $\pi_m \in \mathrm{SU}(2)$ which give irreducible representations of SO(3) are those for which m is an even integer. Note that

$$\pi_r \otimes \pi_m = \bigoplus_{j=0}^{2m} \pi_{r-m+2j} \quad \text{for} \quad r \geqslant m.$$

Denote the group of 2×2 unitary matrices by U(2). See [2], [3], [11], or [14] for more details on the above.

4. FULL MULTIPLICITY ERGODIC ACTIONS OF SU(2) × B AND SO(3) × B

Let B be a compact abelian group. In this section we show that every full multiplicity ergodic action of $SU(2) \times B$ is induced from a full multiplicity ergodic action of $\mathbf{T} \times B$ (Theorem 1(a)). In particular, every full multiplicity ergodic action of $SU(2) \times \mathbf{T}^n$ is induced from a full multiplicity ergodic action of a maximal torus. We also show that $SO(3) \times B$ has no full multiplicity ergodic actions on a factor (Theorem 1(b)). First some notation. If $\alpha : G \times B \to \mathrm{Aut}(\mathcal{M})$ is an action then define

$$\xi : G \to \mathrm{Aut}(\mathcal{M}) \quad g \mapsto \alpha_{(g,1)}$$
$$\beta : B \to \mathrm{Aut}(\mathcal{M}) \quad b \mapsto \alpha_{(1,b)}$$

Note that G acts on $\mathcal{M}^\beta$ via ξ. Let G be SU(2) or SO(3). If $\alpha : G \times B \to \mathrm{Aut}(\mathcal{M})$ is a full multiplicity ergodic action, then for $\delta \in \hat{B}$, $\pi_0 \otimes \delta$ has a unitary eigenvector $u_\delta \in \mathcal{M}$ such that $\alpha_{(g,b)}(u_\delta) = \delta(b)u_\delta$. $Z(\mathcal{M})$ will denote the center of $\mathcal{M}$.

LEMMA 1. (a) *If* $\alpha : G \times B \to \mathrm{Aut}(\mathcal{M})$ *is a full multiplicity ergodic action, and* u_δ *is as above, then* $\mathcal{M}^\beta$ *and* $\{u_\delta : \delta \in \hat{B}\}$ *generate* $\mathcal{M}$ *as a von Neumann algebra.*

(b) *The restrictions of the automorphisms* $\mathrm{Ad}u_\delta$ *to* $Z(\mathcal{M}^\beta)$ *define an action of* $\hat{B}$ *on* $Z(\mathcal{M}^\beta)$ *commuting with the action of* G.

Proof. (a) By [12, Corollary to Theorem 14] we have that $\mathcal{M}_{\pi_m \otimes \delta_0}$ (where δ_0 is the trivial representation of B) and $\mathcal{M}_{\pi_0 \otimes \delta}$ generate $\mathcal{M}$. However $\mathcal{M}_{\pi_m \otimes \delta_0} \subset \mathcal{M}^\beta$ and $\mathcal{M}_{\pi_0 \otimes \delta} = \mathcal{M}^\alpha u_\delta$ since $\alpha_{(g,b)}(x) = \delta(b)x$ implies that $xu_\delta^* \in \mathcal{M}^\alpha$. Furthermore $\mathcal{M}^\alpha \subset \mathcal{M}^\beta$ and this completes the proof of (a).

(b) Notice that $\mathrm{Ad}u_\delta$ (restricted to $Z(\mathcal{M}^\beta)$) commutes with $\mathrm{Ad}u_\tau$, $\tau \in \hat{B}$, since the commutator $u_\delta^* u_\tau^* u_\delta u_\tau \in \mathcal{M}^\beta$ commutes with $Z(\mathcal{M}^\beta)$. ■

In the following Lemma we consider G-equivariant automorphisms of $L^\infty(G/H)$, where H is a closed subgroup of G, G acts by left translation on $L^\infty(G/H)$, $\langle r \rangle$ denotes the subgroup generated by r, $N_G(H)$ denotes the normalizer of H in G, and $C(G/H)$ is the C^*-algebra generated by the spectral subspacs of $L^\infty(G/H)$. We show that any G-equivariant automorphism θ of $L^\infty(G/H)$ is actually right translation by some element in $N_G(H)$.

LEMMA 2. *Every* G*-equivariant automorphism* θ *of* $L^\infty(G/H)$ *has the form* $\theta(f)(xH) = f(xrH)$, $\forall f \in C(G/H)$, *for some* $r \in N_G(H)$. *Furthermore, if* K *is the* *subgroup* $\langle r\rangle H$, *then* $\overline{K} \subset N_G(H)$ *and* $\theta(f) = f$, $\forall f \in C(G/\overline{K})$. *Notice that* r *is uniquely determined up to its class in the Weyl group* $W = N_G(H)/H$.

Proof. $C(G/H)$ is the C^*-algebra generated by the spectral subspaces of $L^\infty(G/H)$ with left translation. Thus $\theta \in \mathrm{Aut}(C(G/H))$ and we have a homeomorphism T of G/H such that

$$\theta(f)(xH) = f(T^{-1}(xH)), \ \ \forall f \in C(G/H), \ \ x \in G.$$

Since θ commutes with left translation, we have that $\theta(f)(xH) = f(xrH)$, for $rH = T^{-1}(H)$, and $T^{-1}(xH) = xrH$ for all $x \in G$. Notice that for $h \in H$, $rH = T^{-1}(H) = T^{-1}(hH) = hrH$, so that $r^{-1}Hr \subset H$. Also $r^{-1}H = T(H) = T(hH) = hr^{-1}H$, so that $rHr^{-1} \subset H$. Thus $rHr^{-1} = H$ and so $r \in N_G(H)$. ■

If we have two actions $\alpha_1 : G \to \mathrm{Aut}(\mathcal{M})$ and $\alpha_2 : G \to \mathrm{Aut}(\mathcal{N})$, then we shall write $(\alpha_1, G, \mathcal{M}) \sim (\alpha_2, G, \mathcal{N})$ whenever $\mathcal{M}$ and $\mathcal{N}$ are equivariantly isomorphic.

THEOREM 1. (a) *Every full multiplicity ergodic action of* $\mathrm{SU}(2)\times B$ *is induced from a full multiplicity ergodic action of* $\mathrm{T}\times B$. *In particular, every full multiplicity ergodic action of* $\mathrm{SU}(2)\times \mathrm{T}^n$ *is induced from a full multiplicity ergodic action of a maximal torus.*

(b) $\mathrm{SO}(3)\times B$ *has no full multiplicity ergodic actions on a factor.*

Proof. (a) Let $\alpha : \mathrm{SU}(2)\times B \to \mathrm{Aut}(\mathcal{M})$ be a full multiplicity ergodic action. Thus $\xi : \mathrm{SU}(2) \to \mathrm{Aut}(\mathcal{M}^\beta)$ is a full multiplicity ergodic action and so by [14, Theorem 2],

$$(\xi, \mathrm{SU}(2), \mathcal{M}^\beta) \overset{\Psi}{\sim} (\lambda, \mathrm{SU}(2), L^\infty(\mathrm{SU}(2))),$$

where λ is left translation. Let u_δ be as in Lemma 1. From Lemma 1 we see that $\mathrm{Ad}u_\delta \in \mathrm{Aut}(\mathcal{M}^\beta)$ commutes with ξ_g, and also that $\mathrm{Ad}u_\delta$ commutes with $\mathrm{Ad}u_\tau$, $\tau \in \hat{B}$. Set $\theta_\delta = \Psi \circ \mathrm{Ad}u_\delta \circ \Psi^{-1}$. Thus θ_δ commutes with left translation on $L^\infty(\mathrm{SU}(2))$, so by Lemma 2, with $H = \{1\}$, we get that $\theta_\delta(f)(x) = f(xr_\delta)$, $\forall f \in C(\mathrm{SU}(2))$, for some $r_\delta \in \mathrm{SU}(2)$. Also $r_\delta r_\tau = r_\tau r_\delta$ since θ_δ commutes with θ_τ. Let K be the abelian subgroup generated by $\{r_\delta : \delta \in \hat{B}\}$. Hence $\theta_\delta(f) = f$ for $f \in L^\infty(\mathrm{SU}(2)/\overline{K})$. Thus $\Psi^{-1}(L^\infty(\mathrm{SU}(2)/\overline{K}))$ commutes with u_δ and $\mathcal{M}^\beta$ and so by Lemma 1

$$\Psi^{-1}(L^\infty(\mathrm{SU}(2)/\overline{K})) \subset Z(\mathcal{M}).$$

Identifying $L^\infty(\mathrm{SU}(2)\times B/\overline{K}\times B)$ with $L^\infty(\mathrm{SU}(2)/\overline{K})$ and using [10, Theorem 10.5] and the properties of induced actions [12, Theorem 5], it follows that α is induced from an ergodic action of $\overline{K}\times B$. However $\overline{K}\times B$ is conjugate to a closed subgroup of $\mathrm{T}\times B$ since $\overline{K}$ is abelian. Thus after inducing in stages [12, Theorem 5], α will be induced from an ergodic action of $\mathbb{T}\times B$. Using [12, Theorem 5(b) and Theorem 15], it follows that α is induced from a full multiplicity ergodic action of $\mathbb{T}\times B$.

(b) Let $\alpha : \mathrm{SO}(3)\times B \to \mathrm{Aut}(\mathcal{M})$ be a full multiplicity ergodic action. Let u_δ be as in Lemma 1. Then by Lemma 1, u_δ and $\mathcal{M}^\beta$ generate $\mathcal{M}$. SO(3) acts ergodically with full multiplicity on $\mathcal{M}^\beta$ and so by [14, Theorem 6],

$$(\xi, \mathrm{SO}(3), \mathcal{M}^\beta) \sim \mathrm{ind}_{H\uparrow \mathrm{SO}(3)}\mathrm{End}(V),$$

where $H = \{1\}$ or D_2. Therefore

$$(\xi, \mathrm{SO}(3), Z(\mathcal{M}^\beta)) \overset{\Psi}{\sim} (\lambda, \mathrm{SO}(3), L^\infty(\mathrm{SO}(3)/H)).$$

Consider $\mathrm{Ad}u_\delta \in \mathrm{Aut}(Z(\mathcal{M}^\beta))$. Using θ_δ as in (a) above, it follows from Lemma 2 that

$$\theta_\delta(f)(xH) = f(xr_\delta H) \quad \forall f \in C(\mathrm{SO}(3)/H),$$

for some $r_\delta \in N_{\mathrm{SO}(3)}(H)$. We get $r_\delta r_\tau H = r_\tau r_\delta H$ because θ_δ commutes with θ_τ. Let K be the subgroup generated by H and $\{r_\delta : \delta \in \hat{B}\}$. Then $\overline{K} \subset N_{\mathrm{SO}(3)}(H)$. $\overline{K} \neq$ $\neq \mathrm{SO}(3)$, since otherwise H is normal in SO(3) and so $H = \{1\}$ which implies that $\overline{K}$ is abelian, and this is a contradiction. As in (a) above, $\theta_\delta(f) = f$ for $f \in L^\infty(\mathrm{SO}(3)/\overline{K})$ so

$$\Psi^{-1}(L^\infty(\mathrm{SO}(3)/\overline{K})) \subset Z(\mathcal{M}).$$

Thus $Z(\mathcal{M})$ contains more than the scalars and so $\mathcal{M}$ cannot be a factor. ∎

COROLLARY 1. *$\mathrm{SU}(2)\times B$ has no full multiplicity ergodic actions on a factor.*

Proof. Use Thereom 1(a) and [12, Thereom 5(c)]. ∎

5. FULL MULTIPLICITY ERGODIC ACTIONS OF U(2)

In this section we show that every full multiplicity ergodic action of U(2) is induced from a full multiplicity ergodic action of a maximal torus (Theorem 2). First we will prove the following lemma which appears in the introduction of [12]. Recall the definition of $m_{ij}(\pi)$ from section 2.

LEMMA 3. *If $\pi \in \hat{G}$, then $m_{ij}(\pi) \leqslant \dim\pi \min(c_i/c_j, c_j/c_i) \leqslant \dim\pi$. In particular, if π is a one dimensional representation, then $m_{ij}(\pi) = 1$ implies that $c_i = c_j$ and therefore $m_{ik}(\pi) = 0 = m_{rj}(\pi)$ for all $k \neq j$, $r \neq i$.*

Proof. By the Multiplicity Map Theorem [12, Theorem 17] we have that

$$\sum_k m_{ik}(\pi)c_k = \dim\pi c_i. \tag{1}$$

Therefore $m_{ij}(\pi) \leqslant \dim \pi(c_i/c_j)$. Similarly

$$\sum_k m_{jk}(\overline{\pi})c_k = \dim \overline{\pi} c_j, \tag{2}$$

so that $m_{ji}(\overline{\pi}) \leqslant \dim \pi(c_j/c_i)$. Hence $m_{ij}(\pi) \leqslant \dim \pi(c_j/c_i)$, and we have that

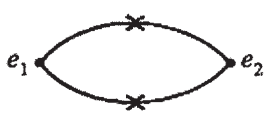
$$m_{ij}(\pi) \leqslant \dim \pi \min(c_i/c_j, c_j/c_i) \leqslant \dim \pi. \tag{3}$$

If π is a one dimensional representation, then by (3) above, $m_{ij}(\pi) = 0$ or 1. If $m_{ij}(\pi) = 1$, then by (3) above, $c_i = c_j$. Thus by (1) above, $m_{ik}(\pi) = 0$, for $k \neq j$. Also since $m_{ji}(\overline{\pi}) = 1$, we get by (2) above that $m_{jr}(\overline{\pi}) = 0$, for $r \neq i$, and so $m_{rj}(\pi) = 0$, for $r \neq i$. ∎

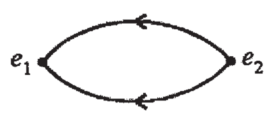
THEOREM 2. *Every full multiplicity ergodic action of* U(2) *is induced from a full multiplicity ergodic action of a maximal torus.*

Proof. Let $\varepsilon : \mathrm{U}(2) \to \mathrm{Aut}(\mathcal{M})$ be a full multiplicity ergodic action. There is a group homomorphism $\Lambda : \mathrm{SU}(2)\times\mathbf{T} \to \mathrm{U}(2)$ which takes (g, z) to zg. Define $\alpha = \varepsilon \circ \Lambda$. Then by [11, page 87] $\alpha : \mathrm{SU}(2)\times\mathbf{T} \to \mathrm{Aut}(\mathcal{M})$ is an ergodic action with spectral decomposition $\bigoplus_{m+k=\text{even}} (m+1)\gamma_{m,k}$.

We will show that the set of nodes Γ of the multiplicity diagrams [12, Section 10] consists of only two nodes of equal valency. The multiplicity diagrams for $\gamma_{1,0}$ and $\gamma_{0,1}$ will be as follows



$\gamma_{1,0}$ diagram



$\gamma_{0,1}$ diagram

Recall the definition of $\alpha' : \mathrm{SU}(2)\times\mathbf{T} \to \mathrm{Aut}(\mathcal{M}_1)$ from section 2. From the remark after the Multiplicity Map Theorem in section 2 we see that $m_{ii}(\gamma_{1,1}) = 2$ for some node $i \in \Gamma$ (since $\gamma_{1,1}$ appears with multiplicity 2 in α). For convenience of notation we denote this node by 1 (i.e. we let $i = 1$). Therefore by the Multiplicity Map Theorem [12, Theorem 17], since $\gamma_{0,1} \otimes \gamma_{1,0} = \gamma_{1,1}$, we get

$$\sum_{k\in\Gamma} m_{1k}(\gamma_{0,1})m_{k1}(\gamma_{1,0}) = 2.$$

Using Lemma 3, let 2 be the unique node in Γ such that $m_{12}(\gamma_{0,1}) = 1$, and so $c_2 = c_1$. Notice that nodes 1 and 2 are not the same node since $m_{11}(\gamma_{0,1}) \neq 1$, (since $\gamma_{0,1}$ does not appear in α). We have $m_{21}(\gamma_{1,0}) = 2 = m_{12}(\gamma_{1,0})$, and $m_{1k}(\gamma_{1,0}) = 0 = m_{k1}(\gamma_{1,0})$, for $k \neq 2$, by the Multiplicity Map Theorem [12, Theorem 17]. We also have

$$\sum_{k \in \Gamma} m_{1k}(\gamma_{1,0}) m_{k1}(\gamma_{0,1}) = 2,$$

so that $m_{21}(\gamma_{0,1}) = 1$ and $m_{k1}(\gamma_{0,1}) = 0$ for $k \neq 2$. Consequently, if we let $\tau = \gamma_{1,0} \oplus \gamma_{0,1}$, then $m_{1k}(\tau) = 0 = m_{k1}(\tau)$, for $k \neq 2$, and $m_{2k}(\tau) = 0 = m_{k2}(\tau)$, for $k \neq 1$. Hence by the τ-connectedness of the multiplicity diagram for τ [12, Section 10], we see that Γ consists only of the nodes 1 and 2 as claimed.

Let ρ be the permutation (12) so that $\rho(1) = 2$ and $\rho(2) = 1$. We denote the pairwise orthogonal minimal projections in $\mathcal{M}_1^{\alpha'}$ by e_1, e_2, so that $\mathcal{M}_1^{\alpha'} = \mathbf{C}e_1 + \mathbf{C}e_2$. As in [14, Case VIII], we can find a $\gamma_{1,0}$ unitary eigenmatrix [12, Section 6] A in $M_2(\mathcal{M}_1)$ such that

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix},$$

and $ae_i = e_{\rho(i)}a$, $be_i = e_{\rho(i)}b$. Also we have $\gamma_{0,1}$ eigenvectors $u_i \in e_i \mathcal{M}_1 e_{\rho(i)}$, such that $u_i u_i^* = e_i = u_{\rho(i)}^* u_{\rho(i)}$. Let $u = u_1 + u_2$. Then u is a $\gamma_{0,1}$ unitary eigenvector in $\mathcal{M}_1$ such that $ue_i = e_{\rho(i)}u$. Notice that by using the stable duality of α' [12, page 299] and [12, Theorem 14], the elements a, b, u and $\mathcal{M}_1^{\alpha'}$ generate $\mathcal{M}_1$. We have

$$(u \otimes 1) A (u^* \otimes 1) = LA, \tag{1}$$

and the commutator

$$(u \otimes 1) A (u^* \otimes 1) A^* = L,$$

for some $L \in M_2(\mathcal{M}_1^{\alpha'})$. Let

$$L = \begin{pmatrix} k & \eta \\ \mu & \nu \end{pmatrix}.$$

Set $k = k_1 e_1 + k_2 e_2$ (where the k_i are scalars) and similarly for η, μ, ν. Using the four equations in (1) above, we get $uau^* = ka - \eta b^*$, $ubu^* = kb + \eta a^*$, and $\nu_i = \overline{k_{\rho(i)}}$, $-\mu_i = \overline{\eta_{\rho(i)}}$ (which we denote as $\nu = \rho(k^*)$, $\mu = -\rho(\eta^*)$), since

$$\{e_i a, e_i a^*, e_i b, e_i b^*, \quad i = 1, 2\}$$

spans the eight dimensional subspace

$$\mathcal{M}_{1_{\gamma_{1,0}}} = e_1 \mathcal{M}_{1_{\gamma_{1,0}}} e_2 \oplus e_2 \mathcal{M}_{1_{\gamma_{1,0}}} e_1,$$

and so is linearly independent. Thus

$$L = \begin{pmatrix} k & \eta \\ -\rho(\eta^*) & \rho(k^*) \end{pmatrix},$$

and $L = L_1(e_1 \otimes 1) + L_2(e_2 \otimes 1)$, where

$$L_1 = \begin{pmatrix} k_1 & \eta_1 \\ -\overline{\eta_2} & \overline{k_2} \end{pmatrix} = \begin{pmatrix} k_1 & \eta_1 \\ -\overline{\rho(\eta_1)} & \overline{\rho(k_1)} \end{pmatrix}.$$

Similarly for L_2. The commutation relations between a and $\mathcal{M}_1^{\alpha'}$ (and between b and $\mathcal{M}_1^{\alpha'}$) are captured in the following definition;

$$\Phi(x) = A(x \otimes 1)A^*, \quad x \in \mathcal{M}_1^{\alpha'}.$$

Notice that $\Phi : \mathcal{M}_1^{\alpha'} \to M_2(\mathcal{M}_1^{\alpha'})$ and $\Phi(e_i) = e_{\rho(i)} \otimes 1$.

The strategy will be as follows: We have some freedom in the choice of A. We want to transform A by multiplying A on the left by some unitary $V \in M_2(\mathcal{M}_1^{\alpha'})$ so that the following three conditions are satisfied.

STEP (1): $\Phi' = \Phi$, where $\Phi'(x) = A'(x \otimes 1)A'^*$, $x \in \mathcal{M}_1^{\alpha'}$, and $A' = VA$.

STEP (2): A' is a $\gamma_{1,0}$ unitary eigenmatrix of the same form as A, i.e.

$$A' = \begin{pmatrix} a' & b' \\ -b'^* & a'^* \end{pmatrix}.$$

STEP (3): L_1' is diagonal, where $L' = (u \otimes 1)A'(u^* \otimes 1)A'^*$ and $L' = L_1'(e_1 \otimes 1) + L_2'(e_2 \otimes 1)$. We denote L' by

$$L' = \begin{pmatrix} k' & \eta' \\ \mu' & \nu' \end{pmatrix},$$

where $k' = k_1'e_1 + k_2'e_2$, etc. as before.

We will then complete the proof of the Theorem by the following step:

STEP (4): We will show that Step (3) implies that $a'a'^*, b'b'^*, a'b'^*$ etc. are non-scalar central elements in $\mathcal{M}_1$. We will use these central elements to exhibit an equivariant copy of $L^\infty(\mathrm{SU}(2) \times \mathbf{T}/\mathbf{T}^2)$ inside $Z(\mathcal{M}_1)$, and the statement of the Theorem will then follow from [10, Theorem 10.5].

We now tackle Step (1).

STEP (1): $\Phi'(x) = VA(x \otimes 1)A^*V^* = V\Phi(x)V^* = VV^*\Phi(x)$ (since $\mathcal{M}_1^{\alpha'}$ is abelian and $\Phi(x)$ is diagonal)$= \Phi(x)$.

STEP (2): A' is clearly a $\gamma_{1,0}$ unitary eigenmatrix. In order that A' be of the same form as A, one can check that we need V to be a unitary of the form

$$V = \begin{pmatrix} v_{11} & v_{12} \\ -\rho(v_{12}^*) & \rho(v_{11}^*) \end{pmatrix},$$

where $v_{11} = v_{11,1}e_1 + v_{11,2}e_2$ and $\rho(v_{11}) = v_{11,2}e_1 + v_{11,1}e_2$ etc. Let $V = V_1(e_1 \otimes 1) + +V_2(e_2 \otimes 1)$, so that

$$V_1 = \begin{pmatrix} v_{11,1} & v_{12,1} \\ -\overline{v_{12,2}} & \overline{v_{11,2}} \end{pmatrix} = \begin{pmatrix} v_{11,1} & v_{12,1} \\ -\overline{\rho(v_{12,1})} & \overline{\rho(v_{11,1})} \end{pmatrix}.$$

Notice that V_1 can be any 2×2 unitary matrix, and that $V_2 = J\overline{V_1}J$ for $J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

STEP (3):

$$L' = (u \otimes 1)VA(u^* \otimes 1)A^*V^* = \rho(V)(u \otimes 1)A(u^* \otimes 1)A^*V^* = \rho(V)LV^*.$$

This gives the relation between L' and L, and we get

$$L_1' = V_2L_1V_1^* = J\overline{V_1}JL_1V_1^*.$$

Using the fact that $J^2 = 1$, we get

$$JL_1' = \overline{V_1}(JL_1)\overline{V_1}^t.$$

Notice that L_1' is diagonal if and only if JL_1' is of the form $\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$. Notice that $J, \overline{V_1}, L_1'$ and L_1 are all unitary. Consequently we can make L_1' diagonal if we can prove the following claim:

CLAIM: *If $R \in \mathrm{U}(2)$, then there is a $Y \in \mathrm{U}(2)$ such that YRY^t has zeros on the diagonal.*

Proof of Claim: We start by observing that the matrix YRY^t will have a zero in the first diagonal entry if and only if

$$r_{11}y_1^2 + (r_{12} + r_{21})y_1y_2 + r_{22}y_2^2 = 0,$$

where $R = (r_{ij})$, and (y_1, y_2) is the first row of Y. This quadratic equation always admits a solution with $|y_1|^2 + |y_2|^2 = 1$. This determines the first row of Y and the second row of Y is then uniquely specified up to a scalar multiple by the unitarity condition on Y. We have thus found the required unitary Y and this completes the proof of the claim.

This completes the proof of Step (3). We now move on to Step (4).

STEP (4): L_1' is diagonal and so $\eta' = 0$. Therefore $ua'u^* = k'a'$ and $ub'u^* = k'b'$, where $k'k'^* = 1$. Consequently, $a'a'^*$ commutes with u. However a', b', u and $\mathcal{M}_1^{\alpha'}$ generate $\mathcal{M}_1$, and so $a'a'^* \in Z(\mathcal{M}_1)$, since $a'a'^*$ commutes with all these generators.

For convenience of notation we will drop the $'$ and from now on we will write a instead of a', etc. Notice that aa^* is not a scalar by [14, Theorem 3], since A' is a $\gamma_{1,0}$ unitary eigenmatrix. Indeed if we let X be the spectrum of the unital abelian C^*-algebra generated by a and b, so that we may regard a and b as functions on X, then the map

$$X \to \mathrm{SU}(2), \quad x \mapsto \begin{pmatrix} a(x) & b(x) \\ -b^*(x) & a^*(x) \end{pmatrix}$$

is a homeomorphism and the von Neumann algebra generated by a and b is equivariantly isomorphic to $L^\infty(\mathrm{SU}(2))$. Similarly bb^*, ab^* and a^*b are non-scalar central elements in $\mathcal{M}_1$. Define $x = ab^* + a^*b$, $y = i(ab^* - a^*b)$, $z = aa^* - bb^*$. Let N be the von Neumann algebra generated by the self-adjoint elements x, y, z. Notice that $N \subset \mathcal{M}_1^{\{1,-1\}}$, so that SO(3) acts on N. We see that (x, y, z) is a commuting self-adjoint basis for π_2 in N, with $x^2 + y^2 + z^2 = 1$, and so by [14, Lemma 13]

$$(\alpha', \mathrm{SO}(3), N) \overset{\Psi}{\simeq} (\lambda, \mathrm{SO}(3), L^\infty(\mathrm{SO}(3)/\mathrm{T})).$$

Therefore

$$(\alpha', \mathrm{SU}(2), N) \sim (\lambda, \mathrm{SU}(2), L^\infty(\mathrm{SU}(2)/\mathrm{T})),$$

and so we have an equivariant copy of $L^\infty(\mathrm{SU}(2)\times\mathrm{T}/\mathrm{T}^2)$ inside $Z(\mathcal{M}_1)$. Thus by [10, Theorem 10.5] α' is induced from an action of the maximal torus T^2, and so is α. Therefore ε is induced from an action of a maximal torus of U(2). Using [12, Theorem 5(b) and Theorem 15], it follows that ε is induced from a full multiplicity ergodic action of a maximal torus of U(2). ∎

COROLLARY 2. *U(2) has no full multiplicity ergodic actions on a factor.*

Proof. Use Theorem 2 and [12, Theorem 5 (c)]. ∎

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