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QUOTIENTS OF BOUNDED OPERATORS AND THEIR WEAK ADJOINTS

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Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

1. INTRODUCTION

For two bounded operators A and B on a Hilbert space H with the kernel condition $\ker A \subset \ker B$, we define the quotient $[B/A]$ by $Ax \to Bx,\ x \in H$ [7]. A quotient (of bounded operators) so defined is what was introduced by Dixmier [3] as a "single-valued J-operator" and also by Kaufman [12] as a "semiclosed operator", and several characterizations of it were given. It was proved in [12] that the family of all quotients contains all closed operators and is itself closed under addition and multiplication. Employing techniques due to Douglas [4] and Fillmore-Williams [5], we showed [7] explicit formulae for computing the quotients which correspond to the sum and the product of two given quotients, and also those for constructing the quotients which represent the adjoint and the closure of a given quotient if they exist.

In this paper we introduce the weak adjoint $[B/A]^\times$ for every quotient $[B/A]$, which coincides with the usual adjoint $[B/A]^*$ if the domain $AH := \{\, Ax : x \in H \,\}$ of $[B/A]$ is dense in H. We show some relations between quotients and their $^\times$-adjoints, which extends results known about densely defined closed operators and their adjoints.

Unless specially stated otherwise, all operators are assumed to be bounded linear, defined on a fixed Hilbert space H.

2. WEAK ADJOINTS OF QUOTIENTS

For a quotient $[B/A]$ (with the condition $\ker A \subset \ker B$) we define a set $G(A, B) := \{\, (Au, Bu);\ u \in H \,\}$ in the product Hilbert space $H \times H$. Then $G(A, B)$ is a graph, which just corresponds to the quotient $[B/A]$. To introduce a (weak) adjoint of $[B/A]$, we put

$$G(A,B)^{\times} = \{\,(x,y);\; B^{*}x = A^{*}y,\; y \in (\ker A^{*})^{\perp}\,\},$$

where $(\ker A^{*})^{\perp}$ means the orthogonal complement of $\ker A^{*}$ in H. The set $G(A,B)^{\times}$ is then a graph again, so that we define $[B/A]^{\times}$ as the corresponding mapping. When AH is dense in H, we see that $(x,y) \in G(A,B)^{\times}$ if and only if $\langle x, Bu\rangle = \langle y, Au\rangle$ for all $u \in H$ ($\langle\cdot,\cdot\rangle$ is the inner product of H), so that $[B/A]^{\times}$ coincides with the usual adjoint $[B/A]^{*}$ of $[B/A]$. By definition the domain of the weak adjoint $[B/A]^{\times}$ is hence $B^{*(-1)}(A^{*}H) := \{\,x \in H;\; B^{*}x \in A^{*}H\,\}$. To see that this set is represented as range of an operator, we employ techniques due to Douglas [4] and Fillmore-Williams [5]. Let

$$R = (\text{or } R_{A^{*},B^{*}} =)\ (A^{*}A + B^{*}B)^{1/2},$$

and consider the equations

$$XR = A \quad\text{and}\quad YR = B. \tag{2.1}$$

Then since $A^{*}H \subset RH$ and $B^{*}H \subset RH$ [5, Theorem 2.2], it follows from Douglas majorizations theorem [4, Theorem] ([5, Theorem 2.1]) that those equations (or the equivalent equations $RX^{*} = A^{*}$ and $RY^{*} = B^{*}$) have solutions. With the assumptions $\ker X \supset \ker R$ and $\ker Y \supset \ker R$, the solutions are unique [5, p.259, Remark], so that we then put $X = A_{\ell}$ and $Y = B_{\ell}$ respectively. On those operators we have the following facts which were shown in [7].

LEMMA 2.1 [7, pp. 430–431].
(1) *$A_{\ell}^{*}A_{\ell} + B_{\ell}^{*}B_{\ell} = P_R$, the orthogonal projection onto the closure $(RH)^{-}$ of RH.*
(2) *$B^{*(-1)}(A^{*}H) = (1 - B_{\ell}B_{\ell}^{*})^{1/2}H$.*

For convenience we write $A_{*} = (1 - B_{\ell}B_{\ell}^{*})^{1/2}$. Now consider the equation

$$A^{*}Z = B^{*}A_{*}. \tag{2.2}$$

Then since $B^{*}A_{*}H = A^{*}H \cap B^{*}H \subset A^{*}H$, we obtain, again from Douglas majorization theorem, a solution of (2.2), which is uniquely determined under the condition $\ker Z^{*} \supset \ker A^{*}$. Putting the unique operator $Z = B_{\times}$, we have the correspondence $x = A_{*}u \to B_{\times}u = y$, $u \in H$ for $(x,y) \in G(A,B)^{\times}$. Hence we have $[B/A]^{\times} = [B_{\times}/A_{*}]$. It is clear that $[B/A]^{\times} = [B/A]^{*}$ if AH is dense in H. On the $^{\times}$-adjoint of a quotient we have the following fact, which was shown in [7] with the restriction that AH is dense in H.

LEMMA 2.2. (cf. [7, Theorem 4.1]). *Let V_{ℓ} be the partial isometry obtained from the polar decomposition $A_{\ell} = V_{\ell}(A_{\ell}^{*}A_{\ell})^{1/2}$ of A_{ℓ}. Then*

(1) $B_\times = V_\ell B_\ell^\star$, *so that*
(2) $[B/A]^\times (= [B_\times/A_\star]) = [V_\ell B_\ell^\star/(1-B_\ell B_\ell^\star)^{1/2}]$.

Proof. It suffices to show (1). Since $RA_\ell^\star = A^\star$, $RB_\ell^\star = B^\star$ and $B_\ell^\star = B_\ell^\star P_R$, we have

$$A^\star B_\times = B^\star A_\star = B^\star(1-B_\ell B_\ell^\star)^{1/2} = RB_\ell^\star(P_R - B_\ell B_\ell^\star)^{1/2} = \\ = R(P_R - B_\ell^\star B_\ell)^{1/2} B_\ell^\star = R(A_\ell^\star A_\ell)^{1/2} B_\ell^\star = RA_\ell^\star V_\ell B_\ell^\star = A^\star V_\ell B_\ell^\star.$$

Hence $A^\star(B_\times - V_\ell B_\ell^\star) = 0$, that is, $(B_\times - V_\ell B_\ell^\star)H \subset \ker A^\star$. On the other hand, by definition we have $B_\times H \subset (\ker A^\star)^\perp$, and clearly $V_\ell B_\ell^\star H \subset (\ker A^\star)^\perp$, so that $(B_\times - V_\ell B_\ell^\star)H \subset (\ker A^\star)^\perp$. Hence $B_\times - V_\ell B_\ell^\star = 0$. ■

Denoting by $P_{A_\star}$ the orthogonal projection onto $(A_\star H)^-$, we proved the following fact about the operator in [8].

LEMMA 2.3. [8, Lemma 2.2 (1)].

$$P_{A_\star} = A_\star^2 + B_\times^\star B_\times = 1 - B_\ell B_\ell^\star + B_\ell V_\ell^\star V_\ell B_\ell^\star.$$

Now concerning the successive $^\times$-adjoints of a quotient we have

THEOREM 2.4. *Let* $[B/A]$ *be a quotient. Then*
(1) $[B/A]^{\times\times} = [B_\times^\star/(1-B_\times B_\times^\star)^{1/2}]$.
(2) $[B/A]^{\times\times\times} = [B_\times^\star/(1-B_\times B_\times^\star)^{1/2}]^\star = [B_\times/(1-B_\times^\star B_\times)^{1/2}]$.
(3) $[B/A]^{\times\times\times\times} = [B/A]^{\times\times}$.

Proof. (1) Since $[B/A]^{\times\times} = [B_\times/A_\star]^\times$, we have to show the identity $(A_\star)_\star = (1-B_\times B_\times^\star)^{1/2}$ and $(B_\times)_\times = B_\times^\star$. From Lemma 2.3 $R_{A_\star, B_\times^\star} = A_\star^2 + B_\times^\star B_\times = P_{A_\star}$. Hence we can see that the unique solutions of the equations

$$XR_{A_\star,B_\times^\star} = A_\star \quad \text{and} \quad YR_{A_\star,B_\times^\star} = B_\times$$

with $\ker X \supset \ker A_\star$ and $\ker Y \supset \ker A_\star$ are $X = A_\star$ and $Y = B_\times$, respectively. Hence we obtain the desired identites.

(2) Let $x \in \ker(1-B_\times B_\times^\star)^{1/2}$. Then by Lemma 2.3

$$A_\star B_\times^\star x = (P_{A_\star} - B_\times^\star B_\times)^{1/2} B_\times^\star x = B_\times^\star(1-B_\times B_\times^\star)^{1/2} x = 0.$$

Hence $B_\times^\star x \in \ker A_\star$. However, since $\ker A_\star \subset \ker B_\times$, we also have $B_\times^\star x \in (\ker A_\star)^\perp$, so that $B_\times^\star x = 0$. Hence $x = (1-B_\times B_\times^\star)x + B_\times B_\times^\star x = 0$. This implies that $(1-B_\times B_\times^\star)^{1/2}H$ is dense in H. Hence we have $[B/A]^{\times\times\times} = ([B/A]^{\times\times})^\star$, which is the first identity. The second identity can be obtained by a similar argument adopted as in (1).

(3) From (2) we have

$$[B/A]^{\times\times\times\times} = [B_\times/(1 - B_\times^\star B_\times)^{1/2}].$$

By the definition of the $^\times$-adjoint of a quotient we can see that $[B_\times/(1-B_\times^\star B_\times)^{1/2}]^\times = [B_\times^\star/(1 - B_\times B_\times^\star)^{1/2}]$. Hence by (1) we have the desired identity. ∎

In [7] we introduced between two quotients $[B/A]$ and $[D/C]$ the relation $[B/A] \subset [D/C]$ when $G(A, B) \subset G(C, D)$, and then said that $[D/C]$ is an extension of $[B/A]$. Note that a graph $G(E, F)$ of a quotient $[F/E]$ is the range of the operator (matrix) $\begin{pmatrix} E & 0 \\ F & 0 \end{pmatrix}$ on $H' := H \times H$. Hence by the Douglas majorization theorem we can see the following

Lemma 2.5. *$[B/A] \subset [D/C]$ if and only if there exists an operator X (on H) such that $A = CX$ and $B = DX$.*

Applying the above lemma, we have

Lemma 2.6. *$[D/C] \subset [B/A]^\times$ if and only if $B^\star C = A^\star D$ and $DH \subset (AH)^-$.*

Proof. Let $[D/C] \subset [B_\times/A_\star]$. Then we have $C = A_\star X$ and $D = B_\times X$ for some operator X. Hence $B^\star C = B^\star A_\star X = A^\star B_\times X = A^\star D$, and $DH \subset B_\times H \subset (AH)^-$. Conversely, if $B^\star C = A^\star D$ and $DH \subset (AH)^-$ then $CH \subset B^{\star(-1)}(A^\star H) = A_\star H$, so that $C = A_\star Y$ for an operator Y. Hence $A^\star D = B^\star C = B^\star A_\star Y = A^\star B_\times Y$, or $A^\star(D - B_\times Y) = 0$. This shows that $(D - B_\times Y)H \subset \ker A^\star$. By the assumption $DH \subset (AH)^-$, we can also have $(D - B_\times Y)H \subset (\ker A^\star)^\perp$. Hence $D - B_\times Y = 0$, which implies $[D/C] \subset [B_\times/A_\star]$. ∎

Remarks

(1) Let us call $[B/A]$ $^\times$-symmetric if $[B/A] \subset [B/A]^\times$. From the preceding lemma we see that $[B/A]$ is $^\times$-symmetric if and only if $B^\star A = A^\star B$ and $BH \subset (AH)^-$.

(2) The relation $[B/A] \subset [D/C]$ does not imply $[D/C]^\times \subset [B/A]^\times$. We can deduce the relation on the $^\times$-adjoint if we add the assumption $CH \subset (AH)^-$.

A quotient $[B/A]$ is closed if its graph $G(A, B)$ is closed in $H \times H$. An (equivalent) condition for closedness of $[B/A]$ is that $A^\star H + B^\star H$ ($= R_{A^\star,B^\star}H$ [5, Theorem 2.2]) is closed in H [11, Theorem 1].

Recall that $A = A_\ell R$ and $B = B_\ell R$ for $R = R_{A^\star,B^\star}$. On a closed quotient we have

Theorem 2.7. *Let $[B/A]$ be a closed quotient. Then $\ker A_\ell \subset \ker B_\ell$, and*

(1). $$[B/A] = [B_\ell/A_\ell] = [B_\times^\star/(P_A - B_\times B_\times^\star)^{1/2}]$$

Here P_A is the orthogonal projection onto $(AH)^-$.

(2) $$[B/A]^\times = [B_\times/(1 - B_\times^\star B_\times)^{1/2}] = [B/A]^{\times\times\times}$$

Proof. Since the operator $R = R_{A^*,B^*}$ has closed range, there exists an operator $R^\dagger$ called a generalized inverse of R [1, p. 321, Theorem 3] ([6, p. 48]) such that $A_\ell = AR^\dagger$ and $B_\ell = BR^\dagger$. Hence from the kernel condition $\ker A \subset \ker B$ of $[B/A]$ we obtain $\ker A_\ell \subset \ker B_\ell$.

For (1), note that $\ker A_\ell = (1 - V_\ell^* V_\ell)H$. Hence we have $B_\ell(1 - V_\ell^* V_\ell) = 0$, that is,

$$B_\ell V_\ell^* V_\ell = B_\ell. \tag{2.3}$$

Since $A_\ell V_\ell^* V_\ell = A_\ell$, we then easily have

$$[B/A] = [B_\ell/A_\ell] = [B_\ell V_\ell^*/A_\ell V_\ell^*].$$

Further we can obtain the identities

$$A_\ell V_\ell^* = (P_A - V_\ell B_\ell^* B_\ell V_\ell^*)^{1/2} = (P_A - B_\times B_\times^*)^{1/2}$$

by a simple computation. Hence we have $[B_\ell V_\ell^*/A_\ell V_\ell^*] = [B_\times^*/(P_A - B_\times B_\times^*)^{1/2}]$.

For (2), from the definition of the $^\times$-adjoint we can see the identity $[B_\times^*/(P_A - B_\times B_\times^*)^{1/2}]^\times = [B_\times/(1 - B_\times^* B_\times)^{1/2}]$, which is the first identity. The second identity is clear by Theorem 2.4. ∎

COROLLARY 2.8. *A quotient $[B/A]$ is densely defined and closed if and only if*

$$[B/A] = [B_\times^*/(1 - B_\times B_\times^*)^{1/2}] \ (= [B/A]^{\times\times}). \tag{2.4}$$

REMARK. Put $C = B_\times^*$ in the above identity (2.4). Then we have Kaufman's representation of a densely defined closed quotient $[B/A]$ (cf. [11; Theorem 2]): $[B/A] = [C/(1 - C^*C)^{1/2}]$ with a pure contraction C, that is, a contraction C satisfying $\ker(1 - C^*C) = \{0\}$. (see Proof of Theorem 2.4 (2).)

Following the definition of a closable operator [10, p. 165] we call a quotient $[B/A]$ closable if

$$Au_n \to 0, \ Bu_n \to v \text{ for a sequence } \{u_n\} \text{ in } H \text{ imply } v = 0.$$

In [8] we showed several equivalent conditions for a quotient to be closable, one of which was the condition [8, Lemma 2.3]

$$\ker A_\ell \subset \ker B_\ell. \tag{2.5}$$

We also proved that then the closure $[B/A]^-$ of $[B/A]$ coincides with $[B_\ell/A_\ell]$. Recall that $A_* = (1 - B_\times^* B_\times)^{1/2} P_{A_*}$ and $B_\times = B_\times P_{A_*}$ (e.g. Proof of Theorem 2.4 (1)). Hence $[B_\times/A_*] \subset [B_\times/(1 - B_\times^* B_\times)^{1/2}]$, so that we have

$$[B/A]^\times \subset [B_\times/(1 - B_\times^* B_\times)^{1/2}] = [B/A]^{\times\times\times}.$$

Now we show equivalent conditions for closability of a quotient related to its $^\times$-adjoints.

THEOREM 2.9 (cf. [8, Lemma 2.3]). *Let $[B/A]$ be a quotient. Then the following conditions are equivalent.*

(1) $[B/A]$ is closable.

(2) $[B/A] \subset [B/A]^{\times\times}$.

(3) $[B/A]^\times = [B/A]^{\times\times\times}$ $(= [B_\times/(1 - B_\times^* B_\times)^{1/2}])$.

Proof. (1)⇔(2): Let $[B/A]$ be closable. Then from [8, Lemma 2.3] we have (2.5), that is, $\ker A_\ell \subset \ker B_\ell$, as an equivalent condition. Hence we have the identity $B_\ell V_\ell^* V_\ell = B_\ell$ (cf. (2.3), Proof of Theorem 2.7 (1)), so that we have $[B_\ell/A_\ell] = [B_\times^*/(P_A - B_\times B_\times^*)^{1/2}]$. Hence

$$[B/A] \subset [B/A]^- = [B_\ell/A_\ell] \subset [B_\times^*/(1 - B_\times B_\times^*)^{1/2}] = [B/A]^{\times\times}.$$

Conversely, if (2) is assumed then there exists an operator X such that $A = (1 - B_\times B_\times^*)^{1/2} X$ and $B = B_\times^* X$. Hence if $Au_n \to 0$, $Bu_n \to v$ for a sequence $\{u_n\}$ in H, then

$$\begin{aligned}(1 - B_\times^* B_\times)^{1/2} v &= \lim_{n\to\infty} (1 - B_\times^* B_\times)^{1/2} B_\times^* u_n = \\ &= \lim_{n\to\infty} B_\times^* (1 - B_\times B_\times^*)^{1/2} X u_n = \lim_{n\to\infty} B_\times^* A u_n = 0.\end{aligned}$$

Since $B_\times$ is a pure contraction (cf. Proof of Theorem 2.4 (2)), we then have $v = 0$.

(1)⇔(3): If we assume (1), then from (2.3) (and (2.5)) we can see (as an equivalent condition)

$$B_\times^* B_\times = B_\ell B_\ell^*. \tag{2.6}$$

Hence by Theorem 2.4 (2) we obtain the identity (3). If we assume (3), then $A_* H = (1 - B_\times^* B_\times)^{1/2} H$, so that $A_* H$ is dense in H, or $P_{A_*} = 1$. Hence by Lemma 2.3 we have (2.6), which implies (1). ■

Let us call a quotient $[B/A]$ singular if $P_{A_*} B = 0$ (or $P_{A_*}^\perp B = B$) [7, p. 434] ([8, p. 202]). Every quotient $[B/A]$ is then decomposed as the sum [8, p. 203]([9, p. 285])

$$[B/A] = [P_{A_*} B/A] + [P_{A_*}^\perp B/A],$$

where $[P_{A_*}B/A]$ is its closable part and $[P_{A_*}^{\perp}B/A]$ is its singular part. For the $^{\times}$-adjoints of those parts we have

PROPOSITION 2.10.
(1) $[P_{A_*}B/A]^{\times} = [B/A]^{\times\times\times}$.
(2) $[P_{A_*}^{\perp}B/A]^{\times} = [0/P_{A_*}]$, *i.e., the restriction of* 0 *to* $P_{A_*}H$.

Proof. (1) The closure $[P_{A_*}B/A]^{-}$ of $[P_{A_*}B/A]$ is $[B_\ell V_\ell^* V_\ell/A_\ell]$ [8, Proposition 2.7]. Hence we have

$$[P_{A_*}B/A]^{\times} = ([P_{A_*}B/A]^{-})^{\times} = [B_\ell V_\ell^* V_\ell/A_\ell]^{\times} = [B_\ell V_\ell^*/A_\ell V_\ell^*]^{\times} =$$
$$= [B_{\times}^*/(P_A - B_{\times}B_{\times}^*)^{1/2}]^{\times} = [B_{\times}/(1 - B_{\times}^* B_{\times})^{1/2}] = [B/A]^{\times\times\times}.$$

(cf. Proof of Theorem 2.7 (1).)

(2) The domain of $[P_{A_*}^{\perp}B/A]^{\times}$ is $(P_{A_*}^{\perp}B)^{*(-1)}(A^*H)$, and we can see that the set is identical to $(A_*H)^{-} = P_{A_*}H$. The unique solution of $A^*Z = (P_{A_*}^{\perp}B)^* P_{A_*}$ with $\ker Z^* \supset \ker A^*$ is $Z = 0$. Hence we obtain the desired identity. ■

3. THE PRODUCTS OF QUOTIENTS AND THEIR $^{\times}$-ADJOINTS

For two quotients $[B/A]$ and $[D/C]$ the product $[B/A][D/C]$ is defined by

$$Cx \to Dx = Ay \to By.$$

Here the element x runs through $D^{-1}(AH)$, and y is an element such that $Dx = Ay$. Hence the domain of the product is $CD^{-1}(AH)$. With a similar argument as in constructing the operator A_* in the preceding section we can obtain an operator M such that $MH = D^{-1}(AH)$. We then have $DMH = DD^{-1}(AH) = DH \cap AH$, so that the equation

$$AX = DM$$

has a solution X. With the restriction $\ker X^* \supset \ker A$ we write $X = N$ the operator which is the unique solution of the equation. We now have the composition

$$CMu \to DMu = ANu \to BNu, \quad u \in H.$$

Hence $[B/A][D/C] = [BN/CM]$ [7, Theorem 3.2]. (We remark that the operators M and N defined as above can be replaced by any operators M' and N' respectively which satisfy $M'H = D^{-1}(AH)$ and $AN' = DM'$ [7, p. 430, Remark].)

Let us call a quotient $[B/A]$ $^{\times}$-positive if $B^*A = A^*B \geqslant 0$. (For an operator C, $C \geqslant 0$ means $\langle Cx, x\rangle \geqslant 0$ for $x \in H$.) On the product of a quotient and its $^{\times}$-adjoint we have

THEOREM 3.1. *The products* $[B/A]^{\times}[B/A]$ *and* $[B/A][B/A]^{\times}$ *are* $^{\times}$*-positive.*

Proof. Let $[B_{\times}/A_{*}][B/A] = [B_{\times}N/AM]$, where M and N are operators satisfying $MH = B^{-1}(A_{*}H)$ and $BM = A_{*}N$. Then

$$(B_{\times}N)^{*}(AM) = N^{*}B_{\times}^{*}AM = N^{*}A_{*}BM = N^{*}A_{*}^{2}N \geqslant 0.$$

Hence $[B/A]^{\times}[B/A]$ is $^{\times}$-positive. Similarly we can see that another product is $^{\times}$--positive. ■

On the $^{\times}$-adjoint of the product of two quotients we have

THEOREM 3.2. *Let* $[B/A]$ *and* $[D/C]$ *be quotients, and let* $D^{-1}(AH)$ *be dense in* H. *Then*

$$([B/A][D/C])^{\times} \supset [D/C]^{\times}[B/A]^{\times}.$$

Proof. Let M and N be operators satisfying $MH = D^{-1}(AH)$ and $DM = AN$. Then by the product formula we have $[B/A][D/C] = [BN/CM]$. Similarly, let U and V be operators such that $UH = B_{\times}^{-1}(C_{*}H)$ and $B_{\times}U = C_{*}V$. Then

$$[D/C]^{\times}[B/A]^{\times} = [D_{\times}/C_{*}][B_{\times}/A_{*}] = [D_{\times}V/A_{*}U].$$

Now what we have to show is the relation $[BN/CM]^{\times} \supset [D_{\times}V/A_{*}U]$. From Lemma 2.6 it suffices to prove the following two facts.

(1) $(BN)^{*}(A_{*}U) = (CM)^{*}(D_{\times}V)$.

(2) $D_{\times}VH \subset (CMH)^{-}$.

For (1), since $B^{*}A_{*} = A^{*}B_{\times}$ and $D^{*}C_{*} = C^{*}D_{\times}$, we have

$$(BN)^{*}(A_{*}U) = N^{*}B^{*}A_{*}U = N^{*}A^{*}B_{\times}U = (AN)^{*}(B_{\times}U) =$$
$$= (DM)^{*}(C_{*}V) = M^{*}D^{*}C_{*}V = M^{*}C^{*}D_{\times}V = (CM)^{*}(D_{\times}V).$$

For (2), note that $MH = D^{-1}(AH)$ is dense in H, and that $D_{\times}H \subset (CH)^{-}$. Hence $D_{\times}VH \subset D_{\times}H \subset (CH)^{-} = (CMH)^{-}$. ■

When does the equality sign hold in the above theorem? We shall answer this question afterwards in Theorem 4.3.

The next theorem gives conditions for a quotient to be $^{\times}$-normal, that is, for the $^{\times}$-adjoint of a quotient to be normal.

THEOREM 3.3. *Let* $[B/A]$ *be closable. Then the following conditions are equivalent.*

(1) $([B/A]^{\times})^{*}[B/A]^{\times} = [B/A]^{\times}([B/A]^{\times})^{*}$.

(2) $B_{\times}^{*}B_{\times} = B_{\times}B_{\times}^{*}$, *i.e.,* $B_{\times}$ *is normal.*

(3) $A_\ell A_\ell^* + B_\ell B_\ell^* = P_A$.

Proof. Since $A_\star H$ is dense in H, we see that (1) is equivalent to

$$(1') \qquad [B/A]^{\times\times}[B/A]^{\times} = [B/A]^{\times}[B/A]^{\times\times}.$$

By Theorems 2.4, 2.9 and the product formula we have

$$[B/A]^{\times\times}[B/A]^{\times} = [B_\times^*/(1-B_\times B_\times^*)^{1/2}][B_\times/(1-B_\times^* B_\times)^{1/2}] = [B_\times^* B_\times/(1-B_\times^* B_\times)].$$

Similarly we have $[B/A]^{\times}[B/A]^{\times\times} = [B_\times B_\times^*/(1-B_\times B_\times^*)^{1/2}]$. Hence we can easily see that (1') is equivalent to (2). For the equivalence between (2) and (3), we first note that from Lemma 2.1 (1)

$$A_\ell A_\ell^* + B_\times B_\times^* = V_\ell A_\ell^* A_\ell V_\ell^* + V_\ell B_\ell^* B_\ell V_\ell^* = V_\ell P_R V_\ell^* = V_\ell V_\ell^* = P_A.$$

Hence (3) is equivalent to the identity $B_\times B_\times^* = B_\ell B_\ell^*$. Next since $[B/A]$ is closable, we also have $B_\times^* B_\times = B_\ell B_\ell^*$ ((2.6)). Hence we obtain (2) as an equivalent condition. ∎

We call $[B/A]$ $^\times$-normal if it is closable and one of (1)-(3) in the above theorem holds. If $[B/A]$ is densely defined and closed, then since $[B/A]^{\times\times} = [B/A]$ (by Corollary 2.8) we see that $[B/A]$ is normal whenever it is $^\times$-normal. Likewise we want to call $[B/A]$ $^\times$-self-adjoint if it is closable and satisfies $[B/A]^{\times\times} = [B/A]^{\times}$. From Theorem 2.9 a closable quotient $[B/A]$ is $^\times$-self-adjoint if $B_\times$ is self-adjoint.

To deal with a quotient defined by two commuting self-adjoint (or normal) operators, we prepare the following lemma.

LEMMA 3.4. *Let S and T be commuting self-adjoint (resp. normal) operators with $SH \subset TH$. Then the unique solution X of the equation*

$$TX = S, \ \ker X^* \supset \ker T \ (= \ker T^*)$$

is a self-adjoint (resp. normal) operator commuting with S and T.

Proof. It suffices to show the case for normal operators, because we can show the case for self-adjoint operators, the more restrictive case, by an almost similar argument. Let S and T be commuting normal operators. Then since $T(XT-TX) = $ $= ST - TS = 0$, we see that $(XT-TX)H \subset \ker T$. By assumption we also have $XH \subset (\ker X^*)^\perp \subset (\ker T)^\perp$, so that $(XT-TX)H \subset XH + TH \subset (\ker T)^\perp$. Hence $XT = TX$. We also have $XS = SX$ from $S = TX$. To see that X is normal, first note that X^* is defined as the natural extension of $T^*u \to S^*u$ ($u \in H$), so that $X^*H \subset (S^*H)^- = (SH)^- \subset (TH)^- = (\ker T)^\perp$. Next $T(X^*X - XX^*) =$

$= X^*TX - TXX^* = X^*S - SX^* = 0$ by the Fuglede-Putnam theorem applied to the identity $XS = SX$. Hence $(X^*X - XX^*)H \subset \ker T$. Further, since both XH and X^*H are subspaces of $(\ker T)^\perp$ we conclude that $X^*X - XX^* = 0$, which implies normality of X. ■

THEOREM 3.5. *Let $[B/A]$ be a quotient, and suppose that A and B are commuting self-adjoint (resp. normal) operators. Then $[B/A]$ is closable and $^\times$-self-adjoint (resp. $^\times$-normal).*

Proof. We prove the case for normal operators A and B only. Since A, B and $R_{A^*,\,B^*}(= (A^*A + B^*B)^{1/2})$ commute with each other, we see from the above lemma that both A_ℓ and B_ℓ are normal operators commuting with A, B and $R_{A^*,\,B^*}$. Using those facts and the condition $\ker A \subset \ker B$, we can see that $\ker A_\ell \subset \ker B_\ell$. Hence $[B/A]$ is closable. For the $^\times$-normality of $[B/A]$ it suffices to observe that A^* and $B^*A_\star$ are commuting normal operators and that $B_\times$ is normal as the unique solution Z of $A^*Z = B^*A_\star$ with $\ker Z^* \supset \ker A$. ■

4. SOME APPLICATIONS

In this section we show some extensions of known results, using $^\times$-adjoints of quotients. The first application concerns the self-adjoint extension of a $^\times$-positive quotient, which gives a quotient version of a theorem due to Ando-Nishio [1].

THEOREM 4.1 (cf. [1, Theorem 1]). *Let $[B/A]$ be a $^\times$-positive quotient. Suppose that $\ker B^*A \subset \ker B$ and the quotient $[B/(B^*A)^{1/2}]$ is closable. Then there exists a densely defined $^\times$-positive self-adjoint extension of $[B/A]$.*

Proof. Let $R = (B^*A + B^*B)^{1/2}$, and let X, Y be the unique solutions of the equations

$$XR = (B^*A)^{1/2}, \ \ker X \supset \ker R,$$

and

$$YR = B, \ \ker Y \supset \ker R,$$

respectively. Then (by [8, Lemma 2.3])

$$[B/(B^*A)^{1/2}] \subset [B/(B^*A)^{1/2}]^- = [Y/X].$$

Letting $E = X^*(B^*A)^{1/2} - Y^*A$, we have $RE = (B^*A)^{1/2}(B^*A)^{1/2} - B^*A = 0$, so that $EH \subset \ker R$. Also, since both X^*H and Y^*H are contained in $(\ker R)^\perp$, we have $EH \subset (\ker R)^\perp$. Hence $E = 0$, that is, $X^*(B^*A)^{1/2} = Y^*A$. Further since $(B^*A)^{1/2}H \subset (XH)^-$, we have, from Lemma 2.6, $[(B^*A)^{1/2}/A] \subset [Y/X]^\times$. Hence

$$[B/A] = [B/(B^*A)^{1/2}][(B^*A)^{1/2}/A] \subset [Y/X][Y/X]^\times = \\ = [Y/X][Y^*/(1-YY^*)^{1/2}] = [YY^*/(1-YY^*)].$$

We can see that $[YY^*/(1-YY^*)]$ is densely defined and $^\times$-positive, and that it is self-adjoint (e.g., by Theorem 3.5). ■

Ôta [13] proved that if a densely defined closable operator is nilpotent (resp. idempotent) then its adjoint is again nilpotent (resp. idempotent). We show a similar fact on quotients. We say that a quotient $[B/A]$ is nilpotent (resp. idempotent) if

(4.1) $$BH \subset AH \text{ and } [B/A]^2 \subset [0/1] \text{ (resp. } [B/A]^2 \subset [B/A]).$$

It is easy to see that (4.1) is equivalent to

$$AX = B \text{ and } X^2 = 0 \text{ (resp. } X^2 = X) \text{ for some operator } X.$$

THEOREM 4.2 (cf. [13, Proposition 3.4]). *Let $[B/A]$ be nilpotent (resp. idempotent). Then $[B/A]^\times$ is again nilpotent (resp. idempotent).*

Proof. Let $[B/A]$ be nilpotent, and let $AX = B$, $X^2 = 0$ for an operator X. Then

$$B^*B_\times = (AX)^*B_\times = X^*A^*B_\times = X^*B^*A_* = (X^2)^*A^*A_* = 0.$$

Hence by Lemma 2.6, $[0/B_\times] \subset [B/A]^\times = [B_\times/A_*]$. Hence there exists an operator Y such that $B_\times = A_*Y$ and $0 = B_\times Y$ (Lemma 2.5). We can assume that Y is the (unique) solution of $B_\times = A_*Y$ with $\ker Y^* \supset \ker A_*$.

Then we can see $Y^2 = 0$, so that $[B_\times/A_*]$ is nilpotent.

Next, let $[B/A]$ be idempotent, and let $AZ = B$, $Z^2 = Z$ for an operator Z. Then by a similar argument as taken in the nilpotent case we have $B^*B_\times = A^*B_\times$. Hence with the fact $B_\times H \subset (AH)^-$ we have $[B_\times/B_\times] \subset [B/A]^\times = [B_\times/A_*]$. Hence there exists an operator W such that $B_\times = A_*W$ and $B_\times = B_\times W$. We can assume that $\ker W^* \supset \ker A_*$. Then we get $W^2 = W$, which implies that $[B_\times/A_*]$ is idempotent. ■

Schechter [14] proved that if S and T are densely defined closed operators on a Hilbert space then

$$(ST)^* = T^*S^*,$$

whenever the range $\mathrm{ran}T$ of T is closed and its codimension $\dim(\mathrm{ran}T)^\perp$ is finite. Our last application is to show an extension of this result.

THEOREM 4.3 (cf. [14, Corollary 1]). *Let* $[B/A]$ *and* $[D/C]$ *be quotients and let* $D^{-1}(AH)$ *be dense in* H. *If* DH *is closed and* $\dim(DH)^{\perp} < \infty$, *then*

$$([B/A][D/C])^{\times} = [D/C]^{\times}[B/A]^{\times}.$$

Proof. From Theorem 3.2 we have only to show the relation $([B/A][D/C])^{\times} \subset \subset [D/C]^{\times}[B/A]^{\times}$. With the same notations as used in the proof of Theorem 3.2 the relation is equivalent to

$$(4.2) \qquad (BN)^{*(-1)}((CM)^{*}H) \subset A_{*}UH \ (= A_{*}B_{\times}^{-1}(C_{*}H)).$$

Before we show (4.2) we want to prove the weaker relation

$$(4.3) \qquad (BN)^{*(-1)}((CM)^{*}H) \subset A_{*}H.$$

Let $w \in (BN)^{*(-1)}((CM)^{*}H)$ or $(BN)^{*}w \in (CM)^{*}H$. Then for (4.3) we have to show $B^{*}w \in A^{*}H$, or equivalently (e.g. [5, p. 259, Corollary 2]),

$$(4.4) \qquad \sup_{x\in H, Ax\neq 0} \frac{\|\langle Bx, w\rangle\|}{\|Ax\|} < \infty$$

Now to make the above inequality more tractable, we put $K = A^{-1}(DH)$. Then since DH is closed, we see that K is a closed subspace of H and $H = K + K^{\perp}$. From the assumption $\dim(DH)^{\perp} < \infty$, we see that $\dim K^{\perp} < \infty$; if we denote by P_D the orthogonal projection onto DH and define an operator L from $K^{\perp}$ into $(DH)^{\perp}$ by $Lv = (1 - P_D)Av$, $v \in K^{\perp}$, then L is one to one, so that $\dim K^{\perp} \leqslant \dim(DH)^{\perp}$. Note that $AK = AH \cap DH = ANH$. Hence

$$AH = AK + AK^{\perp} = ANH + AK^{\perp} = \{A(Nu + v);\ u \in H, v \in K^{\perp}\}.$$

From the condition $\ker A \subset \ker B$, we can also see that

$$BH = \{\, B(Nu + v);\ u \in H, v \in K^{\perp}\,\}.$$

Hence the ineqality (4.4) turns into the following

$$(4.5) \qquad \sup_{A(Nu+v)\neq 0} \frac{|\langle (BNu + Bv), w\rangle|}{\|ANu + Av\|} < \infty \ (u \in H, v \in K^{\perp}).$$

Since $AK^{\perp}$ is finite-dimensional, we shall obtain (4.5) if we prove the following two inequalities.

$$(4.6) \qquad |\ \langle BNu, w\rangle\ | \leqslant \lambda_1 \|ANu\| \text{ for } u \in H.$$

(4.7) $$| \langle Bv, w\rangle | \leqslant \lambda_2 \|Av\| \text{ for } v \in K^{\perp}.$$

Here λ_1 and λ_2 are positive constants independent of u and v, respectively.

For (4.6), we first prove the relation

(4.6′) $$(BN)^{\star(-1)}((CM)^{\star}H) \subset (BN)^{\star(-1)}((CD^{\dagger}DM)^{\star}H).$$

Here $D^{\dagger}$ is the generalized inverse of D such that $DD^{\dagger}D = D$ and $D^{\dagger}D$ is the orthogonal projection onto $(\ker D)^{\perp}$ (e.g. [6, p. 47-p. 48]). Since $D^{\dagger}DMH \subset MH$, we see that $CD^{\dagger}DMH \subset CMH$. Hence there exists an operator X such that $CD^{\dagger}DM = CMX$. From the condition $\ker C \subset \ker D$ and the fact $C(D^{\dagger}DM - MX) = 0$, we then have $D(D^{\dagger}DM - MX) = 0$, that is, $DM = DMX$. Hence we also have $AN = ANX$. From the condition $\ker A \subset \ker B$ we further have $BN = BNX$. Hence (since $[BN/CM]^{\times} = [(BN)_{\times}/(CM)_{\star}]$), we have

$$(BN)^{\star}(CM)_{\star} = (BNX)^{\star}(CM)_{\star} = X^{\star}(BN)^{\star}(CM)_{\star} = \\ = X^{\star}(CM)^{\star}(BN)_{\times} = (CMX)^{\star}(BN)_{\times} = (CD^{\dagger}DM)^{\star}(BN)_{\times}.$$

Hence $(CM)_{\star}H \subset (BN)^{\star(-1)}((CD^{\dagger}DM)^{\star}H)$, which implies (4.6'). Now from (4.6') we can find a vector $z \in H$ such that $(BN)^{\star}w = (CD^{\dagger}DM)^{\star}z$, and we have

$$| \langle BNu, w\rangle | = | \langle u, (BN)^{\star}w\rangle | = | \langle u, (CD^{\dagger}DM)^{\star}z\rangle | = \\ = | \langle (CD^{\dagger}DM)u, z\rangle | \leqslant \| CD^{\dagger} \| \cdot \| DMu \| \cdot \| z \| .$$

Putting $\lambda_1 = \| CD^{\dagger} \| \cdot \| z \|$, we have (4.6).

For (4.7), since $K^{\perp}$ is finite-dimensional we can see that the mapping $Av \longrightarrow Bv$, $v \in K^{\perp}$ is bounded. Hence there exists a positive constant λ_2 satisfying (4.7).

Finally we deduce (4.2) from (4.3). Let $w \in (BN)^{\star(-1)}((CM)^{\star}H)$, so that, let $w = A_{\star}h$ for some $h \in H$. Then

$$(BN)^{\star}w = (BN)^{\star}A_{\star}h = N^{\star}B^{\star}A_{\star}h = (AN)^{\star}B_{\times}h = \\ = (DM)^{\star}B_{\times}h = M^{\star}D^{\star}B_{\times}h.$$

By assumption we have $(BN)^{\star}w \in (CM)^{\star}H$, so that there exists an element $k \in H$ satisfying $(BN)^{\star}w = (CM)^{\star}k$. Hence we have $M^{\star}D^{\star}B_{\times}h = M^{\star}C^{\star}k$. Since we can assume $M^{\star} = M$, and since MH is dense in H, we then have $D^{\star}B_{\times}h = C^{\star}k$. Hence $B_{\times}h \in D^{\star(-1)}(C^{\star}H) = C_{\star}H$, so that $h \in B_{\times}^{-1}(C_{\star}H)$. Hence we have $w = A_{\star}h \in A_{\star}B_{\times}^{-1}(C_{\star}H)$, which implies (4.2). ■

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