J. OPERATOR THEORY
29(1993), 97–114



# AN INDEX THEOREM FOR TOEPLITZ OPERATORS

GERARD J. MURPHY

## 1. INTRODUCTION

The theory of Toeplitz operators on the unit circle $\mathbf{T}$ has been generalised in a number of different directions in the past two decades. One direction involves replacing the open unit disc and its boundary $\mathbf{T}$ by a suitable domain in $\mathbf{C}^n$ with a nice boundary, and considering Toeplitz operators with symbols defined on this boundary [2], [20], [21]. In another direction, the theory of Wiener-Hopf operators (these are unitarily equivalent to Toeplitz operators) has been generalised successfully [14]. A third approach takes as its starting point the key role played in the classical theory by the fact that the group $\mathbf{T}$ is connected and its dual $\mathbf{Z}$ is totally ordered. This point of view was taken by the author in [16], [17], and [18], where a generalised theory of Toeplitz operators is developed. For related material see [6], [7], [10], and [23].

Let $\varphi$ denote a continuous complex-valued function on $\mathbf{T}$, and let $T_\varphi$ denote the corresponding Toeplitz operator on the Hardy space $H^2$. The Gohberg-Krein index theorem ([9], [15]) asserts that $T_\varphi$ is a Fredholm operator if and only if $\varphi$ does not vanish, and that in this case the Fredholm index of $T_\varphi$ is equal to minus the winding number of $\varphi$ about the origin, in symbols,

$$\operatorname{ind}(T_\varphi) = -\operatorname{wn}(\varphi).$$

We prove an analogue of this result for our more general classes of Toeplitz operators.

A difficulty that arises in this context is that there may be no Toeplitz operators of continuous symbol that have non-zero Fredholm index. This, however, is not a defect of the Toeplitz theory but rather reflects the inability of ordinary Fredholm theory to provide an index suitable for all contexts, as has been found in many other situations. To surmount this difficulty we use a more powerful index theory due to

M. Breuer [3], [4]. This is defined relative to von Neumann algebras, and requires the construction of an appropriate representation for the Toeplitz operators. This involves the introduction of a suitable topological space, and a certain measure on it, to which we apply the Murray-von Neumann group measure construction.

The paper is organised as follows: In Section 2 we recall the basic definitions and results of Toeplitz operator theory over connected groups, and establish an index theorem in a special simple case. In Section 3 we introduce the constructions needed for our general index theorem. In Section 4 we demostrate it.

## 2. TOEPLITZ OPERATORS ON CONNECTED GROUPS

We begin by recalling some definitions and results from [16] which we shall need in the sequel. An *ordered group* is a pair $(G, \leqslant)$, where $G$ is a (discrete) abelian group, and $\leqslant$ is a linear (=total) order relation on $G$ which is *translation-invariant*, that is, if $x \leqslant y$, then $x + z \leqslant y + z$, for all $x, y, z \in G$. Obvious, and important, examples of ordered groups are the additive subgroups of $\mathbf{R}$, with the order induced from $\mathbf{R}$. If $G_1, G_2$ are ordered groups, so is the product $G_1 \times G_2$, when endowed with the *lexicographic order*: $(x_1, x_2) < (y_1, y_2)$ if $x_1 < y_1$, or if $x_1 = y_1$ and $x_2 < y_2$. Ordered groups exist in great abundance, for if $G$ is any abelian group, it admits an order relation making it an ordered group if and only if it is torsion-free [13]. It is well known that $G$ is torsion-free if and only if its Pontryagin dual $\hat{G}$ is connected [19, p. 47]. The fact that an ordered group has a connected dual plays an important role in the theory.

Let $G$ be an ordered group, and denote by $G^+$ its positive cone, that is, the set of all $x \in G$ such that $x \geqslant 0$. Denote by $m$ the normalised Haar measure of $\hat{G}$. If $x \in G$, the function

$$\varepsilon_x : \hat{G} \to \mathbf{T}, \quad \gamma \mapsto \gamma(x),$$

is, of course, a homomorphism, and it is well known that the family of elements $(\varepsilon_x)_{x \in G}$ forms an orthonormal basis for the Hilbert space $L^2(\hat{G}, m)$. The Hilbert subspace of $L^2(\hat{G}, m)$ having orthonormal basis $(\varepsilon_x)_{x \in G^+}$ is denoted by $H^2(G)$, and called the *Hardy space relative to* $G$. It exhibits analytic-type behaviour in the sense that if $\varphi$ and $\overline{\varphi}$ belong to $H^2(G)$, then $\varphi$ is constant almost everywhere. Denote by $P$ the orthogonal projection of $L^2(\hat{G}, m)$ onto $H^2(G)$. If $\varphi \in L^\infty(\hat{G}, m)$, the (bounded linear) operator $T_\varphi$ on $H^2(G)$ is defined by

$$T_\varphi(f) = P(\varphi f) \quad (f \in H^2(G)).$$

We say that $T_\varphi$ is a *Toeplitz operator (relative to G)*, with *symbol* $\varphi$.

Many results of the classical theory, where $G = \mathbb{Z}$, extend to this context, and some even to the context of partially ordered groups [16]. For instance, it is easily seen that $\|T_\varphi\| = \|\varphi\|_\infty$, and that $T_{\varphi\psi} = T_\varphi T_\psi$ if $\overline{\varphi}$ or $\psi$ belongs to $H^2(G)$. It is shown in [16] that

$$\sigma(\varphi) \subseteq \sigma(T_\varphi) \subseteq \overline{\mathrm{co}}\,\sigma(\varphi),$$

where $\sigma(\ )$ denotes the spectrum, and $\overline{\mathrm{co}}(\ )$ denotes the closed convex hull. These results generalise well-known theorems of Hartman and Wintner [11], and of Brown and Halmos [5].

As in [16], we shall derive our results using $C^*$-algebraic techniques. For this reason we need to introduce a number of $C^*$-algebras related to the Toeplitz operators. Specifically, let $A(G)$ be the $C^*$-subalgebra of the algebra $B(H^2(G))$ of all operators on $H^2(G)$ which is generated by Toeplitz operators with continuous symbols. This is called the *Toeplitz algebra* of $G$. Its commutator ideal is denoted by $K(G)$. In [16] it is shown that the canonical map

$$C(\hat{G}) \to A(G)/K(G), \quad \varphi \mapsto T_\varphi + K(G),$$

is a $*$-isomorphism. We shall use this on a number of occasions.

In the classical case, $K(\mathbb{Z})$ is the ideal of compact operators on $H^2$, but in the general case $K(G)$ is neither simple nor Type I (it is however primitive, as is $A(G)$ [16]). It turns out that another ideal, which we shall consider presently, plays the role that the ideal of compact operators plays in the classical case.

If $x \in G$, we define $|x| \in G^+$ in the obvious way. We say $x$ is *infinite* if there exists a positive element $y$ of $G$ such that $ny < |x|$, for all positive integers $n$. The set $F$ of finite elements of $G$ is a subgroup (it will play a critical role in the index theory). Recall that a subgroup $J$ of $G$ is said to be an *ideal* if the conditions $x \in G^+$, $y \in J$, and $x \leqslant y$ imply that $x \in J$. It is shown in [16] that $F$ is an ideal of $G$ that is contained in every non-zero ideal of $G$. Moreover, if $G$ is finitely-generated and non-zero, then $F$ also is non-zero (it is not true that $F$ is necessarily non-zero if $G$ is). Clearly, if $G$ is a subgroup of $\mathbb{R}$, then $F = G$.

If $x \in G$, set $V_x = T_{\varepsilon_x}$. An important result in the theory of Toeplitz algebras is the fact that the closed ideal $F(G)$ of $A(G)$ generated by the projections $1 - V_x V_x^*$ ($x \in F^+$) is simple [16]. Since $A(G)$ is primitive, it follows that $F(G)$ is contained in every non-zero closed ideal of $A(G)$. Note that $F(G)$ is non-zero if and only if $F$ is non-zero.

We shall use the following result of van Kampen [22], which extends a result of Bohr [1], to get an analogue of the winding number.

2.1. THEOREM. *Let $G$ be a connected compact group, and suppose that $\varphi$ is a continuous complex-valued function on $G$ which does not vanish anywhere. Then* *there exists a continuous homomorphism* $\chi$ *from* $G$ *to* $\mathbf{T}$, *and a continuous complex--valued function* $\psi$ *on* $G$, *such that* $\varphi = \chi e^{\psi}$.

Note that $G$ is not required to be abelian. As observed in [17], $\chi$ is unique.

Now let $G$ be any ordered group. Since its dual $\hat{G}$ is connected and compact, we may apply Theorem 2.1. Thus, if $\varphi$ is a continuous complex-valued function on $\hat{G}$ which does not vanish, there exists a unique element $x \in G$, and a continuous complex-valued function $\psi$ on $\hat{G}$, such that $\varphi = \varepsilon_x e^{\psi}$ (of course, this uses the fact that $G$ is the dual of $\hat{G}$, so that every continuous homomorphism of $\hat{G}$ is of the form $\varepsilon_x$). As in [17], we denote $x$ by $\omega(\varphi)$, and call it the *index* of $\varphi$. It is easily verified that $\omega(\varphi\varphi') = \omega(\varphi) + \omega(\varphi')$, and $\omega(\overline{\varphi}) = -\omega(\varphi)$. Obviously, $\omega(\varepsilon_x) = x$. Note that in the classical case the index is just the winding number about the origin.

The following result from [17] will be needed.

2.2. THEOREM. *Let* $G$ *be an ordered group. If* $\varphi \in C(\hat{G})$, *then* $T_\varphi$ *is invertible if and only if* $\varphi$ *does not vanish anywhere and* $\omega(\varphi) = 0$. *Equivalently,* $T_\varphi$ *is invertible if and only if* $\varphi$ *has a continuous logarithm.*

The following lemma gives an indication under what conditions it is possible to get a non-trivial index theorem involving classical Fredholm theory.

2.2. LEMMA. *Let* $G$ *be an ordered group whose subgroup* $F$ *of finite elements is non-zero. The following are equivalent conditions:*

(1) $A(G)$ *contains Fredholm operators of non-zero index.*

(2) $F(G)$ *is the ideal of compact operators on* $H^2(G)$.

(3) $F$ *is a cyclic group.*

(4) $G$ *admits a least positive element.*

*Proof.* Let $K$ denote the ideal of compact operators on $H^2(G)$. If $A(G)$ contains Fredholm operators of non-zero index, then the $*$-homomorphism

$$\pi : A(G) \to B(H^2(G))/K, \quad a \mapsto a + K,$$

has non-zero kernel. This follows the well-known Atkinson characterisation of the Fredholm operators as the operators which are invertible modulo the ideal of compact operators. The kernel of $\pi$ is equal to $K \cap A(G)$, so $A(G)$ contains $K$, because $A(G)$ acts irreducibly on $H^2(G)$. Hence, $F(G) = K$. Thus, Condition (1) implies Condition (2).

Now suppose that Condition (2) holds, and let $z$ be a positive element of $F$ such that the projection $Q_z = 1 - V_z V_z^*$ has minimum positive rank. Then $z$ is necessarily the least positive element of $F$, because $Q_x \leqslant Q_y$ if and only if $x \leqslant y$ $(x, y \in G^+)$. Hence, $F = \mathbf{Z}z$. Thus, Condition (2) implies Condition (3).

That Condition (3) implies Condition (4) follows from the observation that if $z$ is a positive element of $F$ such that $F = \mathbb{Z}z$, then $z$ is the least positive element of $F$, and therefore of $G$.

Finally, let us suppose that $G$ admits a least positive element $z$. Then the projection $Q_z = 1 - V_z V_z^*$ has range $\mathbb{C}\varepsilon_0$, and therefore $V_z$ is a Fredholm operator with index given by $\text{ind}(V_z) = \dim(\ker(V_z)) - \dim(\ker(V_z^*)) = 0 - \dim(\ker(V_z V_z^*)) =$ $= -\dim(Q_z(H^2(G))) = -1$. Thus, Condition (4) implies Condition (1), and the lemma is proved. ∎

2.4. Remark. Suppose that the ordered group $G$ has a least positive element $z$. If $\varphi$ is a continuous complex-valued function on $\hat{G}$ which does not vanish and has finite index $\omega(\varphi)$, then $\omega(\varphi) = nz$, for a unique integer $n$. We shall denote this integer by $\text{wn}(\varphi)$ (this is consistent with the notation used in the classical case).

2.5. Theorem. *Let $G$ be an ordered group admitting a least positive element. If $\varphi \in C(\hat{G})$, then $T_\varphi$ is Fredholm if and only if $\varphi$ does not vanish anywhere and has finite index. In this case,*

$$\text{ind}(T_\varphi) = -wn(\varphi).$$

*Proof.* Denote by $z$ the least positive element of $G$, and recall from the proof of Lemma 2.3 that $V_z$ is Fredholm of index minus one.

Suppose that $T_\varphi$ is Fredholm. By Lemma 2.3, $F(G)$ is the ideal of compact operators on $H^2(G)$, so $T_\varphi$ is invertible modulo $F(G)$, by the Atkinson characterization. However, $F(G) \subseteq K(G)$, so $T_\varphi$ is also invertible modulo $K(G)$. Using the canonical $*$-isomorphism of $A(G)/K(G)$ with $C(\hat{G})$, we deduce that $\varphi$ is invertible in $C(\hat{G})$. Hence, $\varphi = \varepsilon_x e^\psi$, for $x = \omega(\varphi)$, and for some $\psi \in C(\hat{G})$. We shall now show that $x$ is finite, and to do this we may suppose that $x \geqslant 0$ (otherwise, replace $\varphi$ with $\overline{\varphi}$). Hence, $T_\varphi = T_{e^\psi} T_{\varepsilon_x} = T_{e^\psi} V_x$. By Theorem 2.2, $T_{e^\psi}$ is invertible. Hence, $V_x$ is Fredholm. If $x$ is not finite, then for all positive integers $n$, the elements $z_n = x - nz$ of $G$ are positive. Hence, $V_x = V_{nz} V_{z_n}$, and since $V_x$ and $V_{nz} = V_z^n$ are Fredholm, therefore $V_{z_n}$ is Fredholm for all $n$. Now, $\text{ind}(V_x) = \text{ind}(V_{nz}) + \text{ind}(V_{z_n})$, and $\text{ind}(V_{z_n}) \leqslant 0$ because $V_{z_n}$ is an isometry, so $\text{ind}(V_x) \leqslant \text{ind}(V_{nz}) = \text{ind}(V_z^n) = -n$, for all positive integers $n$. This is impossible, so $x$ cannot be infinite. Thus, we have shown that if a Toeplitz operator is Fredholm, its symbol does not vanish anywhere and has finite index.

Conversely, suppose now that $\varphi$ is invertible with finite index. Then $\omega(\varphi) = nz$, where $n = \text{wn}(\varphi)$. We shall show that $T_\varphi$ is Fredholm, and that $\text{ind}(T_\varphi) = -\text{wn}(\varphi)$, and to do this we may suppose that $\text{wn}(\varphi) \geqslant 0$ (replacing $\varphi$ by $\overline{\varphi}$, if necessary). We may write $\varphi = \varepsilon_{nz} e^{\psi}$, where $\psi$ is some element of $C(\hat{G})$. Since $T_{e^\psi}$ is invertible, since $V_z$ is Fredholm, and since $T_\varphi = T_{e^\psi} V_z^n$, therefore $T_\varphi$ is Fredholm. Also, $\text{ind}(T_\varphi) = = \text{ind}(T_{e^\psi}) + \text{ind}(V_z^n) = 0 + n(-1) = -n = -\text{wn}(\varphi)$. ■

It is clear from Lemma 2.3 that any extension of Theorem 2.5 will have to involve an index more general that the Fredholm index. As we mentioned earlier, we shall use Breuer's index, for which we shall have to set up some facilitating constructions. We do this in the following section.

## 3. THE INVARIANT MEASURE

We are going to associate with each ordered group admitting non-zero finite elements a certain topological space which reflects both the group and the order structure. The role that this space plays *vis-à-vis* the group bears some resemblance to the role that the real line $\mathbb{R}$ plays *vis-à-vis* a dense subgroup in the theory developed in [6], but the analogy is not close, and the justification for introducing this space rests on its succes in the index theory. There is reason to believe that this space may play a role in the difficult problem of analysing the structure of the Toeplitz algebra of the group, but this subject will not be pursued here.

We begin by recalling a well-known technique for constructing a factor. The details will be pertinent for our considerations in the sequel, so we set them out here, not however in greatest generality, but in a form sufficient for our purpose. (For these results, see [12, Section 8.6].)

Suppose given a measure space $(\Omega, \mu)$, where $\mu$ is non-zero, and a countable group $G$ consisting of bijections of $\Omega$ onto itself. The group operation is function composition, and the unit of $G$ is denoted by $e$. We assume that the following four conditions are satisfied:

(1) There exists a sequence of non-empty measurable sets $E_n$ of $\Omega$, of finite measure, which separate $\Omega$ in the sense that whenever we are given distinct points $\omega, \omega'$ of $\Omega$, there exists an integer $n$ such that $\omega \in E_n$, and $\omega' \in \Omega \setminus E_n$;

(2) For each element $x$ of $G$, and each subset $E$ of $\Omega$, the image $x(E)$ is measurable if and only if $E$ is measurable, and in this case, $\mu(x(E)) = \mu(E)$;

(3) $G$ acts freely on $\Omega$; that is, if $x\omega = \omega$, for some point $\omega$, then $x = e$;

(4) For each measurable set $E$ of $\Omega$ such that $\mu(x(E) \setminus E) = 0$ for all $x$ in $G$, either $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$.

Set $H = L^2(\Omega, \mu)$, and let $A$ be the maximal abelian von Neumann algebra on $H$ consisting of the multiplication operators $M_\varphi$, where $\varphi \in L^\infty(\Omega, \mu)$. For each $x$ in $G$, let $U_x$ be the unitary on $H$ defined by $(U_x f)(\omega) = f(x^{-1}(\omega))$ (for almost all $\omega$).

The map $U : G \to B(H)$, $x \mapsto U_x$, which we shall refer to as *canonical*, is a unitary representation of $G$.

Denote by $K$ the Hilbert space $\ell^2(G,H)$, and if $x \in G$, and $a \in A$, define operators $W_x$ and $\pi(a)$ on $K$ by

$$(W_x f)(y) = U_x f(x^{-1}y)$$

and

$$(\pi(a)f)(y) = af(y),$$

where $f \in K$ and $y \in G$. The map $W : G \to B(K)$, $x \mapsto W_x$, is a unitary representation of $G$, and the map $\pi : A \to B(K)$, $a \mapsto \pi(a)$, is an injective $*$-homomorphism. The von Neumann algebra $R$ on $K$ generated by all the elements $W_x$ and $\pi(a)$ is a factor. We shall call it the *factor associated to* $(\Omega, \mu)$ and $G$, and we shall call $W$ and $\pi$ the *canonical* maps.

The factor $R$ is Type I if and only if $\mu(\omega) > 0$ for some $\omega \in \Omega$. If $\mu(\omega) = 0$ for all $\omega$, then $R$ is Type II; $R$ is Type $\mathrm{II}_1$ if $\Omega$ is of finite measure, and $R$ is Type $\mathrm{II}_\infty$ if $\Omega$ is of infinite measure.

We now begin our construction and analysis of the topological space associated with an ordered group.

Suppose $G$ is an ordered group admitting non-zero finite elements. Denote by $\Omega = \Omega(G)$ the set of decreasing functions $\omega : G \to \mathbb{R}$ having range $\{0,1\}$, and which are such that $\omega(x) \neq \omega(y)$ for some elements $x, y \in G$ whose difference $x - y$ is finite. The set $\Omega$ is non-empty, for if $x$ is an element of $G$, we can define an element $\overline{x}$ of $\Omega$ by setting

$$\overline{x}(y) = \begin{cases} 1, & \text{if } y \leqslant x; \\ 0, & \text{if } y > x. \end{cases}$$

In general, $\Omega$ admits many more elements. (We shall return to this question below.) A linear order relation is defined on $\Omega$ by setting $\omega \leqslant \omega'$ when $\omega(x) \leqslant \omega'(x)$ for all $x \in G$. The map $G \to \Omega$, $x \mapsto \overline{x}$, is strictly increasing. Observe that if $x \in G$ and $\omega \in \Omega$, then $\overline{x} \leqslant \omega$ if and only if $\omega(x) = 1$, and hence, $\omega < \overline{x}$ if and only if $\omega(x) = 0$.

If $x, y \in G$, put

$$[x, y) = \{\omega \in \Omega | \overline{x} \leqslant \omega < \overline{y}\}.$$

A set of this form will be called an *interval* of $\Omega$. We define the *length* of $[x,y)$ as follows: If $x < y$, we set $\mathrm{length}([x,y)) = y - x$, and if $x \geqslant y$ (equivalently, $[x,y) = \emptyset$), we set $\mathrm{length}([x,y)) = 0$.

The intervals, and their lengths, have properties analogous to the similarly named objects of elementary analysis. For example, the intersection of finitely many intervals is an interval. The length function is monotone increasing, and if $E$ is an interval which is the union of the intervals $E_1, \dots, E_n$, then length$(E) \leqslant$ length$(E_1) + \cdots +$ +length$(E_n)$, with equality if $E_1, \dots, E_n$ are pairwise disjoint. Another useful result: The set difference of two intervals is the disjoint union of two intervals. The proofs of these observations are elementary.

The set $\Omega$ has not only an order structure, but more importantly, a natural topology which is intimately related to the order. The topology is defined to be the weakest one making continuous the functions

$$\Omega \to \mathbf{R}, \quad \omega \mapsto \omega(x), \qquad (x \in G).$$

3.1. PROPOSITION. *Let $G$ be an ordered group admitting non-zero finite elements. Then $\Omega(G)$ is a locally compact Hausdorff space. A base for its topology is given by the intervals of finite length. Every interval is a clopen set, and is compact if and only if it is of finite length.*

*Proof.* The product space $\Omega'$ of all functions from $G$ to $\{0,1\}$ is compact and Hausdorff, and clearly $\Omega$ is a subspace. Hence, $\Omega$ is Hausdorff. Also, since $[x,y) = \{\omega \in \Omega | \omega(x) = 1 \text{ and } \omega(y) = 0\}$, and since $\{0\}$, $\{1\}$ are clopen sets in $\{0,1\}$, therefore $[x,y)$ is a clopen subset of $\Omega$. Moreover, if $y-x$ is finite, then $[x,y)$ is closed as a subset of $\Omega'$, and therefore compact.

To see that $\Omega$ is locally compact, let $\omega$ be an arbitrary point, and note that by the definition of $\Omega$, there exists $x, y \in G$ such that $x-y$ is finite and $\omega(x) \neq \omega(y)$. By symmetry, we may suppose that $x < y$. Then $\omega(x) > \omega(y)$, so $\omega(x) = 1$, and $\omega(y) = 0$. Hence $\omega$ belongs to the compact open set $[x,y)$.

Suppose now that $\omega$ is a point of an open set $U$ of $\Omega$. By definition of the topology of $\Omega$, there exist elements $\varepsilon_1, \dots, \varepsilon_n$ of $\{0,1\}$, and elements $x_1, \dots, x_n$ of $G$, such that the intersection of the sets

$$E_i = \{\omega' \in \Omega | \omega'(x_i) = \varepsilon_i\}$$

contains $\omega$, and is contained in $U$. Choose an interval $E_0$ of finite length containing $\omega$. Then the intersection $E_0 \cap \cdots \cap E_n$ is an interval of finite length containing $\omega$ and contained in $U$. Thus the finite-length intervals (which are clearly closed under the operation of taking finite intersections) form a base for the topology of $\Omega$.

To complete the proof, we show that if $x, y \in G$, $x < y$, and $[x,y)$ is compact, then $y-x$ is finite. Since $[x,y)$ is open, we may write it as the union of intervals of finite length, and since it is compact, finitely many such intervals, $E_1, \dots, E_n$ say, suffice. Hence, $y - x = \text{length}([x,y)) \leq \text{length}(E_1) + \cdots + \text{length}(E_n)$, and this sum is a finite element of $G$, so $y-x$ is finite. ■

3.2. REMARK. Since the intervals of finite length form a base for the topology of $\Omega(G)$, the set of all elements $\overline{x}$ ($x \in G$) is dense in $\Omega(G)$. Let us explicitly note also that $\Omega(G)$ is totally disconnected, because it has a base of clopen sets.

Of course, the space $\Omega(G)$ might be discrete. This turns out to be the case precisely when $G$ has a least positive element.

3.3. PROPOSITION. *If $G$ is an ordered group having non-zero finite elements, then the following conditions are equivalent:*

(1) *$\Omega(G)$ is discrete.*

(2) *$\Omega(G) = \{\overline{x} | x \in G\}$.*

(3) *$G$ has a least positive element.*

*Proof.* Suppose that $\Omega$ is discrete, and choose a finite positive element $x$ of $G$. The interval $[0, x)$ is compact and discrete, and is, therefore, a finite set. Hence its intersection with $\overline{G} = \{\overline{y} \,|\, y \in G\}$ is finite, so we may write $[0, x) \cap \overline{G} = \{\overline{x}_1 \ldots, \overline{x}_n\}$, where $0 = x_1 < \cdots < x_n$. An easy argument shows that $x_2$ is the least positive element of $G$. Thus, Condition (1) implies Condition (3).

Now assume Condition (3) holds, that is, $G$ has a least positive element, $z$ say. Then $\mathbf{Z}z$ is the subgroup of finite elements of $G$. If $\omega$ is an element of $\Omega$, some finite interval $[x, y)$ contains it, and $y - x = nz$, for some positive integer $n$. Thus, $\omega(x + nz) = \omega(y) = 0$. Let $k$ be the least positive integer such that $\omega(x + kz) = 0$, and set $t = x + (k - 1)z$. Then the interval $[t, t + z)$ contains $\omega$, and from the fact that $z$ is the least positive element of $G$, it follows easily that this interval contains nothing else. Thus, $\omega = \overline{t}$. This shows that Condition (3) implies Conditions (1) and (2).

Finally, let us suppose that $G$ contains no least positive element. Define the function $\omega : G \to \mathbf{R}$ by

$$\omega(x) = \begin{cases} 1, & \text{if } x < 0; \\ 0, & \text{if } x \geqslant 0. \end{cases}$$

Then $\omega$ is an element of $\Omega$, and we claim that $\omega \notin \overline{G}$. Suppose otherwise, so that $\omega = \overline{x}$, say. Then, for any element $y \in G$, $y \leqslant x$ if and only if $\omega(y) = 1$ if and only if $y < 0$. In particular, $x < 0$. Since $G$ has no least positive element, there exists a positive element $y$ such that $y < -x$. Hence, $-y < 0$, so $\omega(-y) = 1$, and therefore $-y \leqslant x < -y$, which is impossible. Thus, we have shown that Condition (2) implies Condition (3). ∎

If $G$ is an orderd group admitting non-zero finite elements, then we can define an action of $G$ on $\Omega = \Omega(G)$ by setting $(x\omega)(y) = \omega(y - x)$, for all $x, y \in G$, and all $\omega \in \Omega$. For each $x$, the map $\omega \mapsto x\omega$ is a homeomorphism of $\Omega$. Hence, if $E$ is a subset of $\Omega$, it is a Borel set if and only if its image $xE = \{x\omega | \omega \in E\}$ is a Borel set.

A Borel measure $\mu$ on $\Omega$ is *invariant* if $\mu(xE) = \mu(E)$, for all Borel sets $E$ of $\Omega$, and for all $x \in G$.

In order to consider the question of existence and "uniquenes" of such a measure, we need to consider the existence and "uniqueness" of a related homomorphism on the group of finite elements of $G$. First, recall that a homomorphism of ordered groups is *positive* if it is increasing. In this case its kernel is an ideal of the domain. It is elementary that if $\tau$ is a non-zero real-valued positive homomorphism on a subgroup of $\mathbf{R}$, then there exists a positive number $c$ such that $\tau(x) = cx$ for all elements $x$ in the subgroup. Moreover, it is well known that if an ordered group is *archimedean*, that is, all its elements are finite, then it is isomorphic as an ordered group to a subgroup of $\mathbf{R}$ [19, p. 194]. Putting these two facts together, we get the following result: If $G$ is any ordered group whose subgroup $F$ of finite elements is non-zero, then there is a non-zero positive homomorphism $\tau : F \to \mathbf{R}$ which is unique up to multiplication by a positive constant. We shall call any such homomorphism $\tau$ a *trace* of $G$.

The following is the principal result of this section.

3.4. THEOREM. *Let $G$ be a countable ordered group having non-zero finite elements. If $\tau$ is a trace of $G$, then there is a unique Borel measure $\mu$ on $\Omega(G)$ such that*

$$\mu(E) = \tau(\text{length}(E))$$

*for each finite-length interval $E$ of $\Omega(G)$. Moreover, $\mu$ is non-zero and invariant. Each non-zero invariant measure of $G$ is associated with a unique trace of $G$ in this fashion. Any two non-zero invariant measures of $\Omega(G)$ are proportional.*

*Proof.* Let $\tau$ be a trace of $G$. If $E_0, E_1, \ldots$ are intervals of $\Omega$ of finite length, and $E_0 \subseteq \bigcup_{n=1}^{\infty} E_n$, then

$$\tau(\text{length}(E_0)) \leqslant \sum_{n=1}^{\infty} \tau(\text{length}(E_n)).$$

For, by compactness, $E_0 \subseteq E_1 \cup \cdots \cup E_N$ for some $N$, from which $\text{length}(E_0) \leqslant \sum_{n=1}^{N} \text{length}(E_n)$, and hence, $\tau(\text{length}(E_0)) \leqslant \sum_{n=1}^{N} \tau(\text{length}(E_n))$, because $\tau$ is a positive homomorphism.

If $E$ is a subset of $\Omega$, define $\mu^*(E)$ to be the infimum of all sums $\sum_{n=1}^{\infty} \tau(\text{length}(E_n))$, where $(E_n)$ is a sequence of finite-length intervals whose union contains $E$. Using the sub-additivity condition proved above, it is routine to show that $\mu^* : E \mapsto \mu^*(E)$ is an outer measure on $\Omega$. Likewise, using arguments similar to those employed in the elementary theory of the Lebesgue measure on the real line, each finite-length interval

$E$ is easily seen to be measurable with respect to $\mu^*$, and $\mu^*(E) = \tau(\text{length}(E))$. Since $G$ is countable, and the finite-length intervals form a base for the topology of $\Omega$, each open set is a countable union of such intervals, and therefore measurable with respect to $\mu^*$. Consequently, each Borel set is $\mu^*$-measurable, and the restriction $\mu$ of $\mu^*$ to the $\sigma$-algebra of Borel sets is a measure. Since $\tau$ is non-zero, $\mu$ is non-zero.

Let $E$ be a finite-length interval, say $E = [x, y)$. Then, for any $z \in G$, we have $zE = [x + z, y + z)$, so $\text{length}(zE) = \text{length}(E)$, and therefore $\mu(zE) = \mu(E)$. It follows easily from this and the definition of $\mu^*$ that $\mu(zE) = \mu(E)$ for any Borel set $E$ of $\Omega$, that is, $\mu$ is invariant.

Suppose that $\mu'$ is another Borel measure of $\Omega$ such that $\mu'(E) = \tau(\text{length}(E))$ for each finite-length interval $E$ of $\Omega$. We shall show that $\mu' = \mu$. Since $G$ is countable, $\Omega$ is second countable. Therefore, all Borel measures on $\Omega$ are regular. Hence, to show equality of $\mu'$ and $\mu$ we need only show that if $U$ is an open set of $\Omega$, then $\mu'(U) = \mu(U)$. We may write $U = \bigcup_{n=1}^{\infty} E_n$, where the $E_n$ are finite-length intervals. Set $E_1' = E_1$, and for $n > 1$, set $E_n' = E_n \setminus (E_1 \cup \cdots \cup E_{n-1}) = (E_n \setminus E_1) \cap \cdots \cap (E_n \setminus E_{n-1})$. Since each difference $E_n \setminus E_i$ is a disjoint union of two finite-length intervals, each $E_n'$ is a union of finitely many pairwise-disjoint finite-length intervals. Using this, and the fact that the sets $E_1', E_2', \ldots$ are pairwise disjoint and have union $U$, it follows that $U$ is the union of countably many pairwise-disjoint finite-length intervals, $E_1'', E_2'', \ldots$ say. Hence, $\mu'(U) = \sum_{n=1}^{\infty} \mu'(E_n'') = \sum_{n=1}^{\infty} \mu(E_n'') = \mu(U)$. This proves the claim that $\mu' = \mu$.

Now let us suppose that $\mu'$ is an arbitrary non-zero invariant measure on $\Omega$, and we shall show that it is associated with a trace. For each positive finite element $x$ of $G$, set $\tau'(x) = \mu'([0, x))$. Since $[0, x)$ is compact, $\tau'(x)$ is finite, that is, $\tau'(x) \in \mathbf{R}^+$. If $x, y$ are positive finite elements of $G$, then $\tau'(x + y) = \tau'(x) + \tau'(y)$, because

$$\begin{aligned}\tau'(x) + \tau'(y) &= \mu'([0, x)) + \mu'([0, y)) = \\ &= \mu'([0, x)) + \mu'([x, x + y)) = \quad \text{(by invariance of } \mu'\text{)} \\ &= \mu'([0, x + y)) = \tau'(x + y).\end{aligned}$$

It follows that we can uniquely extend $\tau'$ to a positive homomorphism from the group of finite elements of $G$ to $\mathbf{R}$. Moreover, by invariance of $\mu'$, it is clear that $\mu'(E) = \tau'(\text{length}(E))$ for each finite-length interval $E$ of $\Omega$. Since $\mu'$ is non-zero, $\mu'(E)$ is non-zero for some finite-length interval $E$, from which it is clear that $\tau'$ is non-zero. Thus, $\tau'$ is a trace of $G$.

Because the trace of $G$ is esentially unique, there is a positive constant $c$ such that $\tau' = c\tau$. Hence, $\mu'(E) = c\mu(E)$ for each finite-length interval $E$ of $\Omega$. Each open set $U$ of $\Omega$ is a countable union of pairwise-disjoint finite-length intervals, so $\mu'(U) = c\mu(U)$. Hence, by regularity, $\mu' = c\mu$. This proves the theorem. ∎

3.5. COROLLARY. *Let $G$ be a countable ordered group with non-zero finite elements, and let $\mu$ be a non-zero invariant measure on $\Omega(G)$. If $E$ is a Borel set of $\Omega(G)$ such that $\mu(xE \setminus E) = 0$ for all $x \in G$, then $\mu(E) = 0$ or $\mu(\Omega(G) \setminus E) = 0$.*

*Proof.* This is immediate from Theorem 3.4 and the observation that the Borel measure defined by $E' \mapsto \mu(E \cap E')$ is invariant. ∎

3.6. REMARK. A non-zero invariant measure of $\Omega(G)$ is infinite, since its range contains the positive part of the range of the associated trace.

The following result is relevant in determining the type of the factor we shall construct from $\Omega(G)$ and $\mu$.

3.7. PROPOSITION. *Let $G$ be a countable ordered group with non-zero finite elements, and let $\mu$ be a non-zero invariant measure on $\Omega(G)$. The following are equivalent conditions:*

(1) *$\mu(\{\omega\}) = 0$ for all points $\omega$ of $\Omega(G)$.*

(2) *$G$ has no least positive element.*

*Proof.* If $G$ has a least positive element, then $\Omega = \{\overline{x} | x \in G\}$, by Proposition 3.3, from which it is immediate that $\mu(\{\overline{x}\}) > 0$, for some $x$ in $G$. Thus, Condition (1) implies Condition (2).

Now suppose that $G$ has no least positive element, and let $\omega$ be an arbitrary point of $\Omega$. Choose a positive finite element $z$ of $G$. There exists a finite-length interval $[x, y)$ containing $\omega$, and by finiteness of $y - x$, there exists a positive integer $n$ such that $y - x < nz$. Hence, $\omega < \overline{x} + n\overline{z}$. We may suppose that $n$ is the least positive integer for which this inequality holds. Therefore, $\overline{x} + (n-1)\overline{z} \leqslant \omega$, so $\omega \in [x + (n-1)z, x + nz)$. Consequently, $\mu(\{\omega\}) \leqslant \mu([x + (n-1)z, x + nz)) = \tau(z)$, where $\tau$ is the trace associated to $\mu$. Thus, $\mu(\{\omega\}) \leqslant \tau(z)$ for all positive finite elements $z$ of $G$. Since $F$ has no least positive element, $\tau(F)$ has none either, which implies the density of $\tau(F)$ in $\mathbf{R}$. It follows that $\mu(\{\omega\}) = 0$. We have shown that Condition (2) implies condition (1). ∎

We have now assembled the material required to construct a suitable representation of the Toeplitz algebra with which to derive an index theorem. We do this in the following section.

## 4. THE INDEX THEOREM

Throughout this section $G$ is a countable ordered group admitting non-zero finite elements, $\tau$ is a trace of $G$, and $\mu$ is the associated invariant measure on $\Omega = \Omega(G)$.

If $x \in G$, we may identify it with the corresponding map $\omega \mapsto x\omega$ on $\Omega$, and therefore we may regard $G$ as a group of homeomorphisms of $\Omega$. We claim that $(\Omega, \mu)$ and $G$ satisfy Conditions (1)–(4) of Section 3. Condition (4) is given by Corollary 3.5, Condition (2) is clear, and Condition (1) is easily checked. To show Condition (3) holds, that is, that $G$ acts freely, observe that the stabiliser group of any point $\omega$ of $\Omega$ is an ordered subgroup of $G$. If it is non-zero, it contains all the finite elements. Hence, for any two elements $x, y$ of $G$ whose difference is finite, we have $x\omega = y\omega$, which implies that $\omega(x) = \omega(y)$. This is impossible, so the stabiliser group of $\omega$ is trivial.

Let $A$ be the von Neumann algebra on $H = L^2(\Omega, \mu)$ consisting of all multiplication operators $M_\varphi$ ($\varphi \in L^\infty(\Omega, \mu)$), and denote by $R$ the factor on $K = \ell^2(G, H)$ associated to $(\Omega, \mu)$ and $G$. Observe that $R$ is of Type I if and only if $\mu(\{\omega\}) > 0$ for some $\omega$ in $\Omega$, so by Proposition 3.7, $R$ is Type I if and only if $G$ has a least positive element. If $R$ is Type I, it is easily seen to be Type $\mathrm{I}_\infty$. When $G$ admits no least positive element, $R$ is of Type $\mathrm{II}_\infty$.

Let $U : G \to B(H)$, $W : G \to R$, and $\pi : A \to R$ be the canonical maps.

We shall need a formula for the trace on $R$, and for this and other reasons, we shall introduce a certain positive linear map from $R$ to $A$. First, define an isometry $v : H \to K$ by setting

$$(vf)(x) = \begin{cases} f, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0, \end{cases}$$

where $f \in H$ and $x \in G$. Clearly, the map

$$e : B(K) \to B(H), \quad a \mapsto v^* a v,$$

is linear, norm-decreasing, positive, and strongly continuous. It is easy to show that if $a \in A$ and $x \in G$, then

$$(*) \qquad e(\pi(a)W_x) = \begin{cases} a, & \text{if } x = 0; \\ 0, & \text{if } x \neq 0, \end{cases}$$

and therefore $e(R) = A$. Note also that the function

$$t : A^+ \to [0, \infty], \quad M_\varphi \mapsto \int \varphi \, d\mu,$$

is a faithful, normal, semifinite trace on $A$. Since $t(U_x a U_x^*) = t(a)$ for all $a \in A^+$ and all $x \in G$ (because $\mu$ is invariant), the function

$$\mathrm{tr} : R^+ \to [0, \infty], \quad a \mapsto t(e(a)),$$

is a faithful, normal, semifinite trace on $R$ [8, Proposition 9.2.1].

If $E$ is a Borel subset of $\Omega$, we shall denote by $\chi_E$ the multiplier operator on $H$ corresponding to the characteristic function of $E$. If $x \in G$, let $[x, \infty)$ be the set of all $\omega \in \Omega$ such that $\omega \geqslant \overline{x}$. This set is closed in $\Omega$. Denote by $P_x$ the projection $\pi(\chi_{[x,\infty)})$ in $R$. Note that $P_x = \pi(U_x\chi_{[0,\infty)}U_x^*) = W_x\pi(\chi_{[0,\infty)})W_x^* = W_xP_0W_x^*$. If $x \leqslant y$, then $P_x - P_y = \pi(\chi_{[x,y)})$, so $P_y \leqslant P_x$. Also $\mathrm{tr}(P_x - P_y) = t(\chi_{[x,y)}) = \mu([x, y))$, so $P_x - P_y$ is a finite projection if and only if $\mu([x, y)) < \infty$ if and only if $y - x$ is finite.

Set $R_\tau = P_0RP_0$, and regard this as a von Neumann algebra on $P_0(K)$ in the usual manner. SInce $R$ is a factor, so is $R_\tau$, and it is clear that tr, when restricted to $R_\tau$, is again a faithful, normal, semifinite trace, which we shall also denote by tr.

If $x \in G^+$, then $P_x \in R_\tau$. Also, since $P_0P_x = P_x$, that is $P_0W_xP_0W_x^* = W_xP_0W_x^*$, therefore $P_0W_xP_0 = W_xP_0$. Hence, in the terminology of [16], the map

$$G^+ \to R_\tau, \quad x \mapsto P_0W_xP_0,$$

is a semigroup of isometries. Moreover, if $P_0W_xP_0$ is a unitary of $R_\tau$, then $P_x = (W_xP_0)(W_xP_0)^* = P_0$, so $x = 0$. It follows from [16] that there is a unique $*$-homomorphism $\theta$ from $A(G)$ to $R_\tau$ such that $\theta(V_x) = P_0W_xP_0$ for all $x \in G^+$, and that $\theta$ is injective. We shall identify each element $a$ in $A(G)$ with its image $\theta(a)$ in $R_\tau$, and thereby identify $A(G)$ with the $C^*$-subalgebra $\theta(A(G))$ of $R_\tau$.

Observe that $\pi e(R_\tau) \subseteq R_\tau$. We denote by $\tilde{e}$ the restriction of $\pi e$ to $R_\tau$, so $\tilde{e}$ is a faithful, positive, norm-decreasing operator on $R_\tau$.

4.1 LEMMA. *If $a$ is a positive element of $A(G)$, then $a \in F(G)$ if and only if $\tilde{e}(a) \in F(G)$.*

*Proof.* We shall need an alternative description of the restriction of $\tilde{e}$ to $A(G)$. If $\gamma \in \hat{G}$, then the map

$$G^+ \to A(G), \quad x \mapsto \gamma(x)V_x,$$

is a non-unitary semigroup of isometries. Hence, it induces a $*$-isomorphism $\delta_\gamma$ of $A(G)$ onto itself [16]. It is easy to verify that $\delta_{\gamma\gamma'} = \delta_\gamma\delta_{\gamma'}$, and that for each $a \in A(G)$ the map $\gamma \mapsto \delta_\gamma(a)$ is continuous. Set $e'(a) = \int \delta_\gamma(a)\mathrm{d}m\gamma$, where $m$ denotes normalised Haar measure on $\hat{G}$. This defines a faithful, positive, norm-decreasing, idempotent linear map $e'$ from $A(G)$ to itself. Moreover, it is straightforward to show that $A(G)$ is the closed linear span of the elements of the form $V_xV_x^*V_y$ ($x \in G^+$, $y \in G$), so to show that $\tilde{e}$ is equal to $e'$ on $A(G)$, it suffices to show $\tilde{e}$ is equal to $e'$ on such elements. However, $e'(V_xV_x^*V_y) = \int \delta_\gamma(V_xV_x^*V_y)\mathrm{d}m\gamma = \int \gamma(y)\mathrm{d}m\gamma V_xV_x^*V_y$, so

$$e'(V_xV_x^*V_y) = \begin{cases} V_xV_x^*, & \text{if } y = 0; \\ 0, & \text{if } y \neq 0. \end{cases}$$

Comparing this with Equation $(*)$, and noting that $V_x V_x^* V_y = P_x W_y P_0 = P_x W_y P_0 \cdot \cdot W_y^* W_y = P_x P_y W_y = \pi(\chi_{[x,\infty)}\chi_{[y,\infty)})W_y$, it is clear that $\tilde{e}$ is equal to $e'$ on the elements $V_x V_x^* V_y$, as required.

Since $F(G)$ is invariant under the automorphism $\delta_\gamma$, we get an induced automorphism $\Delta_\gamma$ on the quotient algebra $A(G)/F(G)$, and hence we get a faithful, positive, norm-decreasing, linear operator $e''$ on $A(G)/F(G)$ by setting

$$e''(a + F(G)) = \int \Delta_\gamma(a + F(G))\mathrm{d}m\gamma.$$

Clearly, $e''(a + F(G)) = \left(\int \delta_\gamma(a)\mathrm{d}m\gamma\right) + F(G) = e'(a) + F(G)$, so if $a$ is a positive element of $A(G)$ such that $e'(a) \in F(G)$, then $e''(a + F(G)) = 0$, and therefore $a + F(G) = 0$, that is, $a \in F(G)$. The lemma is proved. ∎

We need to make few remarks now on Breuer's theory. The *null projection* of an element $a$ of $R_\tau$ is the greatest projection $p$ of $R_\tau$ such that $ap = 0$, and is denoted by $\mathrm{nul}(a)$. Let $M$ be the closed ideal of $R_\tau$ generated by the finite projections. An element $a$ of $R_\tau$ is *Fredholm relative to* $R_\tau$ if it is invertible modulo $M$. In this case the null projections of $a$ and $a^*$ are finite, and the *Breuer index* of $a$ is defined by

$$\mathrm{ind}_{R_\tau}(a) = \mathrm{tr}(\mathrm{nul}(a)) - \mathrm{tr}(\mathrm{nul}(a^*)).$$

The theory developed by Breuer has many of the features of the classical Fredholm theory, see [3], [4] for details.

Let $N$ be the ideal of $R_\tau$ whose positive part is $N^+ = \{a \in R_\tau^+ | \mathrm{tr}(a) < \infty\}$. Clearly, $N^+$ contains all finite projections, so $\overline{N}$ contains $M$. If $a$ is a norm-one positive element of $R_\tau$, then $a = \sum_{n=1}^{\infty} \frac{p_n}{2^n}$ for a sequence of projections $(p_n)$ of $R_\tau$ [15, Theorem 4.1.13], so if $a \in N$, all the projections $p_n$ belong to $\overline{N}$, by the hereditary property of closed ideals. Consequently, $p_n \in N$, and therefore $p_n$ is finite, so $p_n \in M$. Thus, $N^+ \subseteq M$, and since $N^+$ linearly spans $N$, we have $\overline{N} \subseteq M$. Hence, $\overline{N} = M$. Clearly, $N^+$ is invariant for $\tilde{e}$, and therefore so is $M$, a result we shall use in the following theorem.

4.2. THEOREM. *Let $G$ be a countable ordered group admitting non-zero finite elements, and $\tau$ a trace of $G$. Let $\varphi \in C(\hat{G})$. The following conditions are equivalent:*

*(1) $T_\varphi$ is Fredholm relative to $R_\tau$.*

*(2) $T_\varphi$ is invertible modulo the ideal $F(G)$.*

*(3) $\varphi$ does not vanish anywhere and its index $\omega(\varphi)$ is finite.*

*If $T_\varphi$ is Fredholm relative to $R_\tau$, then its Breuer index is given by*

$$\mathrm{ind}_{R_\tau}(T_\varphi) = -\tau(\omega(\varphi)).$$

*Proof.* We retain the notation introduced above, and denote the unit of $R_\tau$ by 1. If $x$ is a finite positive element of $G$, then the projection $1 - V_x V_x^* = P_0 - P_x$ is finite in $R_\tau$, and is therefore an element of $M$. Hence, $F(G) \subseteq M \cap A(G)$. To show that Condition (1) implies Condition (2), we need only show the reverse inclusion holds, that is, $M \cap A(G) \subseteq F(G)$. Thus, it suffices to show injectivity of the $*$-homomorphism

$$\rho : A(G)/F(G) \to R_\tau/M, \quad a + F(G) \mapsto a + M.$$

Let $Z_0$ be the linear span of the projections $V_x V_x^*$ ($x \in G^+$). Then its closure $Z$ is a $C^*$-subalgebra of $A(G)$, and it follows easily from the proof of the preceding lemma that $\tilde{e}(A(G)) = Z$. Let $e_l$ be the obvious linear map from $A(G)/F(G)$ to $(Z + F(G))/F(G)$ induced by $\tilde{e}$, and likewise, let $e_r$ be the obvious linear map from $R_\tau/M$ to itself induced by $\tilde{e}$. Denote by $\rho'$ the restriction of $\rho$ to $(Z + F(G))/F(G)$. The diagram

$$\begin{array}{ccc} A(G)/F(G) & \xrightarrow{\quad\rho\quad} & R_\tau/M \\ {\scriptstyle e_l}\downarrow & & \downarrow{\scriptstyle e_r} \\ (Z + F(G))/F(G) & \xrightarrow{\quad\rho'\quad} & R_\tau/M \end{array}$$

commutes, and all the maps involved are positive, so to show $\rho$ is injective, that is, faithful, it suffices to show that $e_l$ and $\rho'$ are faithful. It is immediate from Lemma 4.1 that $e_l$ is faithful. To show $\rho'$ is faithful, it suffices to show that $Z \cap M \subseteq F(G)$ (this is clear from the canonical isomorphism of $(Z + F(G))/F(G)$ with $Z/(Z \cap F(G))$). Now $Z_0$ is the union of an increasing sequence of finite-dimensional $C^*$-subalgebras, so $Z$ is an AF-algebra. By a well-known result, if $I$ is a closed ideal of $Z$, it is the closure of $Z_0 \cap I$. In particular, $Z \cap M$ is the closure of $Z_0 \cap M$, so we are reduced to showing that $Z_0 \cap M \subseteq F(G)$. We do this by showing that if $a$ is a positive element of $Z_0 \cap M$, then $a$ belongs to $F(G)$. We may write $a$ as a linear combination of projections, $Q_1, \dots, Q_n$, where $Q_i = V_{x_i} V_{x_i}^*$. Since $G$ is totally ordered, we may suppose that $x_1 < \dots < x_n$, and therefore $Q_1 > \dots > Q_n$. Hence, we may write $a = \sum_{i=1}^{n-1} \lambda_i (Q_i - Q_{i+1}) + \lambda_n Q_n$, where $\lambda_i \in \mathbf{R}^+$. If $\lambda_n \neq 0$, then $Q_n \leqslant \frac{a}{\lambda_n}$, so $V_{x_n} \in M$, and therefore $1 \in M$, which is impossible. Hence, $\lambda_n = 0$. If $i < n$, and $\lambda_i \neq 0$, then $Q_i - Q_{i+1} \leqslant \frac{a}{\lambda_i}$, so $Q_i - Q_{i+1} \in M$, and therefore $\operatorname{tr}(Q_i - Q_{i+1}) = \operatorname{tr}(P_{x_i} - P_{x_{i+1}}) = \mu([x_i, x_{i+1}))$ is finite, so $y = x_{i+1} - x_i$ is finite, and hence $Q_i - Q_{i+1} = V_{x_i}(1 - V_y V_y^*) V_{x_i}^*$ belongs to $F(G)$. Thus, $a \in F(G)$. We have therefore shown that Condition (1) implies Condition (2).

Suppose now that $T_\varphi$ is invertible modulo the ideal $F(G)$. Since $F(G) \subseteq K(G)$, therefore $T_\varphi$ is invertible modulo $K(G)$. Using the canonical isomorphism of $A(G)/K(G)$ with $C(\hat{G})$, we deduce that the symbol $\varphi$ is invertible. We may write $\varphi = \varepsilon_x e^\psi$, where $x = \omega(\varphi)$, and $\psi$ is some element of $C(\hat{G})$. To show that $x$ is finite, we may suppose $x \geqslant 0$ (replacing $\varphi$ by $\overline{\varphi}$ if necessary). Then $T_\varphi = T_{e^\psi} V_x$, and since $T_{e^\psi}$ is invertible, $V_x$ is invertible modulo $F(G)$. Hence, the projection $1 - V_x V_x^*$ belongs to $F(G)$, and therefore to $M$. Consequently, $\mathrm{tr}(1 - V_x V_x^*) = \mu([0, x))$ is finite, so $x$ is finite. Thus, Condition (2) implies Condition (3).

Finally, suppose that $\varphi$ does not vanish anywhere, and that $x = \omega(\varphi)$ is finite. Write $\varphi = \varepsilon_x e^\psi$ for some $\psi \in C(\hat{G})$. To show $T_\varphi$ is Fredholm relative to $R_\tau$, and that $\mathrm{ind}_{R_\tau}(T_\varphi) = -\tau(\omega(\varphi))$, we may suppose $x \geqslant 0$ (as usual, replace $\varphi$ by $\overline{\varphi}$ if necessary). Clearly, $V_x$ is invertible modulo $F(G)$, and therefore modulo $M$, so $V_x$ is Fredholm relative to $R_\tau$. Moreover, $\mathrm{nul}(V_x) = 0$, and $\mathrm{nul}(V_x^*) = 1 - V_x V_x^*$. Hence, $\mathrm{ind}_{R_\tau}(V_x) = -\mathrm{tr}(1 - V_x V_x^*) = -\mu([0, x)) = -\tau(x)$. Thus, $T_\varphi = T_{e^\psi} V_x$ is the product of an invertible operator and an operator which is Fredholm relative to $R_\tau$, so it also is Fredholm relative to $R_\tau$, and its index is given by $\mathrm{ind}_{R_\tau}(T_\varphi) = \mathrm{ind}_{R_\tau}(T_{e^\psi}) + \mathrm{ind}_{R_\tau}(V_x) = 0 - \tau(x) = -\tau(\omega(\varphi))$. This shows that Condition (3) implies Condition (1), and proves the theorem. ■

The special case of the preceding theorem, where $G$ is an ordered subgroup of **R**, was obtained in [6].

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*GERARD J. MURPHY*
*Department of Mathematics,*
*University College, Cork,*
*Ireland.*

Received September 24, 1990; revised January 10, 1991.