J. OPERATOR THEORY
29(1993),125–136

ON GENERATORS OF BIMODULES OF NEST ALGEBRAS

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0. INTRODUCTION

In this paper, a Hilbert space will be separable. We will use the same notation for a (self-adjoint) projection and for its range space. For any vectors x, y in $\mathcal{H}$, the rank-1 operator $\langle ., y\rangle x$ will be denoted by $x \otimes y$. The ideal of compact operators of $\mathcal{B}(\mathcal{H})$ will be denoted by $\mathcal{K}$.

A nest $\mathcal{N}$ in a Hilbert space $\mathcal{H}$ is a chain of (self-adjoint) projections ordered by inclusion of the corresponding range spaces. A nest is complete if it is closed in the strong operator topology. In this paper all nests will be complete. A nest $\mathcal{N}$ is of order type I if it is an infinite set, and exactly one of the following projections $E_\lambda = \vee\{N \in \mathcal{N} : \dim N < \infty\}$ and $E_\rho = \vee\{N^\perp : N \in \mathcal{N}$ and $\dim N^\perp < \infty\}$ is finite dimensional, and both E_λ and $E_\rho^\perp$ are limit points in $\mathcal{N}$. A nest $\mathcal{N}$ is of order type II if it is not of order type I. In [1] we proved that a nest $\mathcal{N}$ is of order type II is a necessary and sufficient condition for $\mathcal{B}(\mathcal{H})$, the set of bounded operators acting on a separable space $\mathcal{H}$, to be a norm-principal bimodule of $\operatorname{alg}\mathcal{N}$.

In this paper we continue our work. A nest $\mathcal{N}$ is said to be of order type II_1 if at least one of E_λ and E_ρ are infinite dimensional. A nest $\mathcal{N}$ of order type II is said to have order type II_2 if it is not of order type II_1. Many nests including finite nests, continuous nests and any nests order isomorphic to $\mathbb{Z}$ have order type II_2. Assume that a nest $\mathcal{N}$ has order type II_2. In this paper we will characterize conditions for an operator T in $\mathcal{B}(\mathcal{H})$ to be a generator for $\mathcal{B}(\mathcal{H})$ as a norm-principal bimodule of $\operatorname{alg}\mathcal{N}$.

This paper is organized as follows. In Section 1 we introduce a definition of an index of an operator T related to a nest $\mathcal{N}$. Then we state our Main Theorem. In Section 2 we prove some lemmas. In Section 3, we prove the Main Theorem and give some Corollaries. When a nest has order type II_2, we characterize the generators of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ as a norm-principal-bimodule of $\mathcal{A}(\mathcal{N})$, the image of $\operatorname{alg}\mathcal{N}$ under the Calkin homomorphism. We also characterize the generators of $\mathcal{B}(\mathcal{H})$ as a norm-principal bimodule of the quasi-triangular algebra $\operatorname{alg}\mathcal{N}+\mathcal{K}$. We prove that when $\mathcal{N}$ is of order type II_2, the set of generators is open and dense in $\mathcal{B}(\mathcal{H})/\mathcal{K}$. Also, the set of generators for $\mathcal{B}(\mathcal{H})$ as a norm-principal bimodule of the quasitriangular algebra is an open dense set in $\mathcal{B}(\mathcal{H})$.

1. PRELIMINARIES

Let $\mathcal{N}$ be a complete nest in a separable Hilbert space $\mathcal{H}$. Let x be a separating unit vector of $\mathcal{C}_{\mathcal{N}}$, the von Neumann algebra generated by $\mathcal{N}$. For $N \in \mathcal{N}$, we use $\omega(N) = (Nx, x) = \|Nx\|^2$ to index N. The index set $\Lambda = \omega(\mathcal{N})$ is a closed subset of [0,1] containing $\{0,1\}$. In this paper we use a fixed separating unit vector. The letter α will denote the smallest limit point of Λ and β will denote the largest limit point of Λ. Assume that $\dim\mathcal{H} = \infty$. We have $E_\lambda = N_{\alpha_0}, E_\rho^\perp = N_{\beta_0}$ for some $\alpha_0, \beta_0 \in \Lambda$. If $\mathcal{N}$ is an infinite nest, we have $0 \leqslant \alpha_0 \leqslant \alpha \leqslant \beta \leqslant \beta_0 \leqslant 1$. If Λ has no limit point, we assume $\alpha = \alpha_0$ and $\beta = \beta_0$. Let $\mathcal{N}$ be a nest of order type II_2. Let $\lambda \leqslant \mu$, $\lambda, \mu \in \Lambda$. Denote $P = N[\lambda, \mu] = N_\mu - N_\lambda$. The operator $N[\lambda, \mu]$ is called an $\mathcal{N}$-interval. If $\alpha_0 < \alpha$, then there is a λ in Λ such that $N[\alpha_0, \lambda]$ is an infinite dimensional minimal $\mathcal{N}$-interval (an atom). Similarly, if $\beta < \beta_0$ then there is a μ in Λ such that $N[\mu, \beta_0]$ is an infinite dimensional atom.

Let $\mathcal{N}$ be a nest in $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$. We define a seminorm $\Phi_{\mathcal{N}}(T)$ in $\mathcal{B}(\mathcal{H})$ as follows:

$$\Phi_{\mathcal{N}}(T) = \inf\{\|(N_{\beta_0} - N_\mu)T(N_\lambda - N_{\alpha_0})\|_c : \mu, \lambda \in \Lambda \ \text{ and } \ \mu < \beta_0, \lambda > \alpha_0\}$$

where $\|A\|_c = \inf\{\|A - K\| : K \in \mathcal{K}\}$ for $A \in \mathcal{B}(\mathcal{H})$.

We define a mapping $j_{\mathcal{N}} : \mathcal{B}(\mathcal{H}) \to \{0, 1\}$ as follows:

$$j_{\mathcal{N}}(T) = \inf\{\operatorname{sgn}\|N_\mu^\perp T N_\lambda\| : \mu < 1,\ \lambda > 0,\ \mu, \lambda \in \Lambda\}.$$

We have $j_{\mathcal{N}}(T) = 0$ if and only if there is an order pair (λ, μ) in $\Lambda \times \Lambda$, $\mu <$ < 1, $\lambda > 0$, $\mu, \lambda \in \Lambda$ such that $N_\mu^\perp T N_\lambda = 0$. In Lemma 4 $j_{\mathcal{N}}(T)$ gives a criterion for when the norm-bimodule generated by an operator T contains the compact operators $\mathcal{K}$.

DEFINITION 1. Let $\mathcal{N}$ be a nest of order type II_2 and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $\dim\mathcal{H} = \infty$. The index of T related to the nest $\mathcal{N}$, denoted by $i_{\mathcal{N}}(T)$ is given by $i_{\mathcal{N}}(T) = j_{\mathcal{N}}(T)\Phi_{\mathcal{N}}(T)$.

REMARK. The index $i_{\mathcal{N}}$ is not norm-continuous in general. When $\mathcal{N}$ is of order type II_2, the set $\{T \in \mathcal{B}(\mathcal{H}) : i_{\mathcal{N}}(T) = 0\}$ is a norm-closed $\mathrm{alg}\mathcal{N}$-bimodule in $\mathcal{B}(\mathcal{H})$, and the set $\mathcal{G} = \{T \in \mathcal{B}(\mathcal{H}) : i_{\mathcal{N}}(T) > 0\}$ is a norm-open set.

Now we state our main results.

MAIN THEOREM. *Let $\mathcal{N}$ be an order type II_2 nest in a separable Hilbert space $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$.*

1. *Assume that $\dim\mathcal{H} = \infty$. The operator T is a generator of $\mathcal{B}(\mathcal{H})$ as an $\mathrm{alg}\mathcal{N}$-norm-principal bimodule if and only if $i_{\mathcal{N}}(T) \neq 0$;*
2. *Assume that $\dim\mathcal{H} < \infty$. The operator is a generator of $\mathcal{B}(\mathcal{H})$ as a norm-principal bimodule of* $\mathrm{alg}\mathcal{N}$ *if and only if $j_{\mathcal{N}}(T) = 1$.*

NOTES. The proof for finite dimensional case is not difficult. We leave it to the reader.

2. LEMMAS

In this section we will prove some Lemmas and Propositions.

LEMMA 1. *Let $\mathcal{N}$ be a nest in $\mathcal{H}$ such that $\alpha = 0$ and $\beta = 1$ and let T be an operator in $\mathcal{B}(\mathcal{H})$ such that $\inf\{\|N_\mu^\perp T N_\lambda\| : \lambda, \mu \in \Lambda \cap (0,1)\} = 1$. Let $\varepsilon > 0$ and $\mu, \lambda \in \Lambda \cap (0,1), \lambda < \mu$. Then*

1. *There exist $\lambda_1, \lambda_2, \mu_1$ and μ_2 in Λ such that $0 < \lambda_1 < \lambda_2 < \lambda < \mu < \mu_1 <$ $< \mu_2 < 1$ and a unit vector $x \in N[\lambda_1, \lambda_2]$ such that*

$$1 - \varepsilon < \|N[\mu_1, \mu_2] T N[\lambda_1, \lambda_2]\| < 1 + \varepsilon,$$

and

$$1 - \varepsilon < \|N[\mu_1, \mu_2] T N[\lambda_1, \lambda_2] x\| < 1 + \varepsilon.$$

2. *If there is a strictly increasing sequence $\{\rho_n\}$ in Λ such that $\lim\rho_n = 1$ and* $\dim N[\rho_n, \rho_{n+1}] = \infty$ *for each n, then the numbers μ_1, μ_2 in (1) can be chosen such that* $\dim N[\mu_1, \mu_2] = \infty$.

Proof. The conclusion follows from the following facts:

- If $\mu \leqslant \mu'$ and $\lambda \geqslant \lambda'$, $\mu, \mu', \lambda, \lambda' \in \Lambda$, then $\|N_{\mu'}^\perp T N_{\lambda'}\| \leqslant \|N_\mu^\perp T N_\lambda\|$.
- If $\lim_t T_t = T \in \mathcal{B}(\mathcal{H})$ in sot, then $\|T\| \leqslant \sup_t \|T_t\| < +\infty$.

For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\mathcal{I}_{\mathcal{N}}(T) = [(\mathrm{alg}\mathcal{N})T(\mathrm{alg}\mathcal{N})]$, the norm-closed linear span of $(\mathrm{alg}\mathcal{N})T(\mathrm{alg}\mathcal{N})$. This is an $\mathrm{alg}\mathcal{N}$-norm-closed bimodule generated by T. Let $\mathcal{N}$ be a nest such that $\alpha = 0$ and $\beta = 1$. This is a nest of order type II_2. Then one of the following four cases must occur:

1. There exist a projection P in $\mathcal{N}$, an o.n. basis $\{e_{-k} : k \in \mathbb{N}\}$ for $P\mathcal{H}$ and a sequence of mutually orthogonal infinite dimensional projections $\{E_k : k \in \mathbb{N}\}$ in $P^{\perp}\mathcal{H}$ with $\vee\{E_k : k \in \mathbb{N}\} = I_{P^{\perp}\mathcal{H}}$ such that the interval nest $P\mathcal{N}$ is a subnest of $\mathcal{N}_1 = \{N_{-n} : n \in \mathbb{N}\} \cup \{0, I_{P\mathcal{H}}\}$ in $P\mathcal{H}$ where $N_{-n} = [e_{-k} : k \geqslant n]$ and the interval nest $P^{\perp}\mathcal{N}$ has a subnest $\mathcal{M}_1 = \{M_n : n \in \mathbb{N}\} \cup \{0, I_{P^{\perp}\mathcal{H}}\}$ in $P^{\perp}\mathcal{H}$ where $M_n = [E_k : 1 \leqslant k \leqslant n]$.

2. There exist a projection P in $\mathcal{N}$, a sequence of mutually orthogonal infinite dimensional projections $\{E_{-k} : k \in \mathbb{N}\}$ in $P\mathcal{H}$ with $\vee\{E_{-k} : k \in \mathbb{N}\} = I_{P\mathcal{H}}$ and an o.n. basis $\{e_k : k \in \mathbb{N}\}$ for $P^{\perp}\mathcal{H}$ such that the interval nest $P\mathcal{N}$ has a subnest $\mathcal{N}_1 = \{N_{-n} : n \in \mathbb{N}\} \cup \{0, I_{P\mathcal{H}}\}$ in $P\mathcal{H}$ where $N_{-n} = [E_{-k} : k \geqslant n]$ and the interval nest $P^{\perp}\mathcal{N}$ is a subnest of $\mathcal{M}_1 = \{M_n : n \in \mathbb{N}\} \cup \{0, I_{P^{\perp}\mathcal{H}}\}$ in $P^{\perp}\mathcal{H}$ where $M_n = [e_k : 1 \leqslant k \leqslant n]$.

3. There exist a projection P in $\mathcal{N}$, an o.n. basis $\{e_{-k} : k \in \mathbb{N}\}$ for $P\mathcal{H}$ and an o.n. basis $\{e_k : k \in \mathbb{N}\}$ for $P^{\perp}\mathcal{H}$ such that the interval nest $P\mathcal{N}$ is a subnest of $\mathcal{N}_1 = \{N_{-n} : n \in \mathbb{N}\} \cup \{0, I_{P\mathcal{H}}\}$ in $P\mathcal{H}$ where $N_{-n} = [e_{-k} : k \geqslant n]$ and the interval nest $P^{\perp}\mathcal{N}$ is a subnest of $\mathcal{M}_1 = \{M_n : n \in \mathbb{N}\} \cup \{0, I_{P^{\perp}\mathcal{H}}\}$ in $P^{\perp}\mathcal{H}$ where $M_n = [e_k : 1 \leqslant k \leqslant n]$.

4. There exist a projection P in $\mathcal{N}$, a sequence of mutually orthogonal infinite dimensional projection $\{E_{-k} : k \in \mathbb{N}\}$ in $P\mathcal{N}$ with $\vee\{E_{-k} : k \in \mathbb{N}\} = I_{P\mathcal{H}}$ and a sequence of mutually orthogonal infinite dimensional projections $\{E_k : k \in \mathbb{N}\}$ in $P^{\perp}\mathcal{N}$ with $\vee\{E_k : k \in \mathbb{N}\} = I_{P^{\perp}\mathcal{H}}$ such that the interval nest $P\mathcal{N}$ has a subnest $\mathcal{N}_1 = \{N_{-n} : n \in \mathbb{N}\} \cup \{0, I_{P\mathcal{H}}\}$ in $P\mathcal{H}$ where $N_{-n} = [E_{-k} : k \geqslant n]$, and the interval nest $P^{\perp}\mathcal{N}$ has a subnest $\mathcal{M}_1 = \{M_n : n \in \mathbb{N}\} \cup \{0, I_{P^{\perp}\mathcal{H}}\}$ in $P^{\perp}\mathcal{H}$ where $M_n = [E_k : 1 \leqslant k \leqslant n]$.

In the following Proposition we will discuss the above four cases.

PROPOSITION 1. *Let $\mathcal{N}$ be a nest in $\mathcal{H}$ such that $\alpha = 0$ and $\beta = 1$ and let T be an operator in $\mathcal{B}(\mathcal{H})$ such that $\Phi_{\mathcal{N}}(T) > 0$. Then*

1. *If the nest $\mathcal{N}$ is in case (1), then there exist a sequence of mutually orthogonal $\mathcal{N}$-intervals $\{F_k : k \in \mathbb{N}\}$ in $P^{\perp}\mathcal{H}$ with $\vee\{F_k : k \in \mathbb{N}\} = I_{P^{\perp}\mathcal{H}}$, and an o.n. sequence $\{f_k\}$, $f_k \in F_k$ for each k, such that $S_0 = \sum f_k \otimes e_{-k} \in \mathcal{I}_{\mathcal{N}}(T)$;*

2. *If the nest $\mathcal{N}$ is in case (2), then there exist a sequence of mutually orthogonal $\mathcal{N}$-intervals $\{F_{-k} : k \in \mathbb{N}\}$ in $P\mathcal{H}$, $\vee\{F_{-k} : k \in \mathbb{N}\} = I_{P\mathcal{H}}$, and an o.n. sequence $\{f_{-k}\}$, $f_{-k} \in F_{-k}$ for each $k \in \mathbb{N}$, such that $S_0 = \sum e_k \otimes f_{-k} \in \mathcal{I}_{\mathcal{N}}(T)$;*

3. *If the nest $\mathcal{N}$ is in case (3), then the operator $S_0 = \sum e_k \otimes e_{-k} \in \mathcal{I}_{\mathcal{N}}(T)$;*

4. *If the nest $\mathcal{N}$ is in case (4), then there exist a sequence of mutually orthogonal $\mathcal{N}$-intervals $\{F_{-k} : k \in \mathbb{N}\}$ in $P\mathcal{H}$ with $\vee\{F_{-k} : k \in \mathbb{N}\} = I_{P\mathcal{H}}$, an o.n. sequence $\{f_{-k}\}$, $f_{-k} \in F_{-k}$ for each $k \in \mathbb{N}$, a sequence of mutually orthogonal projections*

$\{F_k : k \in \mathbf{N}\}$ in $P^\perp\mathcal{H}$ with $\vee\{F_k : k \in \mathbf{N}\} = I_{P^\perp\mathcal{H}}$, and an o.n. sequence $\{f_k\}$, $f_k \in F_k$ for each $k \in N$, such that $S_0 = \sum f_k \otimes f_{-k} \in \mathcal{I}_\mathcal{N}(T)$.

REMARK. As discussed before Proposition 1, we have four cases. The operator S_0 we obtained in Proposition 1 in each case is a generator for $\mathcal{B}(\mathcal{H})$ as an alg$\mathcal{N}$-norm -bimodule. (See the proof of Proposition 3 in [1] for details.)

COROLLARY 1. *Let $\mathcal{N}$ be a nest in a separable Hilbert space $\mathcal{H}$ with $\alpha = 0$ and $\beta = 1$. Assume that $T \in \mathcal{B}(\mathcal{H})$ and that $i_\mathcal{N}(T) > 0$. Then $\mathcal{I}_\mathcal{N}(T) = \mathcal{B}(\mathcal{H})$.*

In order to prove Proposition 1, we need the following Lemma 2.

LEMMA 2. *Let $\mathcal{N}$ be a nest in the case (1) and $T \in \mathcal{B}(\mathcal{H})$. Let $\varepsilon_{i;j}$, $i \neq j$, $i, j \in \mathbf{N}$, be given positive numbers such that $\sum_{i\neq j} \varepsilon_{i;j} < \frac{1}{2}$. Assume that $\Phi_\mathcal{N}(T) = 1$. Then there exists a strictly increasing sequence $\{\lambda_n : n \in \mathbf{Z}\}$ in Λ such that the set $\mathcal{N}_0 = \{N_{\lambda_n} : n \in \mathbf{Z}\} \cup \{0, I\}$ is a subset of $\mathcal{N}$. There also exists an o.n. sequence $\{e'_n : n \in \mathbf{Z} \setminus \{0\}\}$ in $\mathcal{B}(\mathcal{H})$, $e'_n \in N[\lambda_n, \lambda_{n+1}]$ for $n < 0$, and $e'_n \in N[\lambda_{n-1}, \lambda_n]$ for $n > 0$ such that*

1. *$\dim N[\lambda_n, \lambda_{n+1}] = \infty$ for each $n \geqslant 0$;*
2. *$|\langle Te'_{-2k}, e'_{2j}\rangle| < e_{k;j}$, $k \neq j$, $k, j \geqslant 1$; and*
3. *$\frac{1}{2} < |\langle Te'_{-2k}, e'_{2k}\rangle| < \frac{3}{2}$, $k \geqslant 1$.*

Proof. Since $\Phi_\mathcal{N}(T) = 1$, $E_\lambda = E_\rho = 0$, we have $\inf\{\|N_\mu^\perp T N_\lambda\| : \lambda, \mu \in \Lambda \cap (0,1)\} = 1$. Let P be the projection in case (1), then there is a $\lambda_0 \in \Lambda$ such that $N_{\lambda_0} = P$. We will only construct λ_n for $-4 \leqslant n \leqslant 4$, and e'_n for $-4 \leqslant n \leqslant 4$. Using induction we can complete the construction. Let $\{\rho_n : n \in \mathbf{Z}\}$ be a sequence in Λ such that $\lim_{n\to-\infty} \rho_n = 0$ and $\lim_{n\to\infty} \rho_n = 1$.

1. By Lemma 1, we can find $\lambda_{-2}, \lambda_{-1}, \lambda_1, \lambda_2 \in \Lambda$, such that $\lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2$, and $\lambda_{-2} < \rho_{-2}$, $\lambda_{-1} < \rho_{-1}$, $\lambda_1 > \rho_1$, $\lambda_2 > \rho_2$, and such that $\dim N[\lambda_1, \lambda_2] = \infty$, and there exists a unit vector e'_{-2} in $N[\lambda_{-2}, \lambda_{-1}]$ such that

$$\frac{4}{5} < \|N[\lambda_2, \lambda_1]Te'_{-2}\| < \frac{6}{5};$$

We write $y_1 = N[\lambda_1, \lambda_2]Te'_{-2}$ and define $e'_2 = \frac{y_1}{\|y_1\|}$. We have $e'_2 \in N[\lambda_1, \lambda_2]$ and $\frac{1}{2} < |\langle N[\lambda_1, \lambda_2]Te'_{-2}, e'_2\rangle| < \frac{3}{2}$. Let e'_{-1} be a unit vector in $N[\lambda_{-1}, \lambda_0]$ and e'_1 be a unit vector in $N[\lambda_0, \lambda_1]$.

2. Since $\lim_{\mu\to 1, \mu\in\Lambda} N_\mu^\perp y = 0$ and $\lim_{\mu\to 0, \mu\in\Lambda} N_\mu y = 0$ for any element y in $\mathcal{H}$, there exist $\mu_{-1}, \mu_1 \in \Lambda$, $0 < \mu_{-1} < \mu_1 < 1$, $\mu_{-1} < \lambda_{-2}$, $\mu_1 > \lambda_2$, such that

$$\|N_{\mu_{-1}} T^* e'_2\| < \varepsilon_{2;1} \quad \text{and} \quad \|N_{\mu_1}^\perp Te'_{-2}\| < \frac{\varepsilon_{1;2}}{2\|T\|}.$$

By Lemma 1 again, there exist $\lambda_{-4}, \lambda_{-3}, \lambda_3$, and λ_4 in Λ, such that $\lambda_{-3} < \min\{\rho_{-3}, \mu_{-1}, \lambda_{-2}\}$, $\lambda_{-4} < \min\{\rho_{-4}, \lambda_{-3}\}$, $\lambda_3 > \max\{\rho_3, \lambda_2, \mu_1\}$, $\lambda_4 > \max\{\rho_4, \lambda_4\}$, and there exists an element e'_{-4} in $N[\lambda_{-4}, \lambda_{-3}]$ such that $\dim[\lambda_3, \lambda_4] = \infty$ and $\frac{1}{2} < \|N[\lambda_3, \lambda_4]Te'_{-4}\| < \frac{3}{2}$.

We write $y_2 = N[\lambda_3, \lambda_4]Te'_{-4}$. Define $e'_4 = \frac{y_2}{\|y_2\|}$. We have

$$|\langle Te'_{-4}, e'_2\rangle| = |\langle e'_{-4}, T^*e'_2\rangle| = |\langle N_{\mu_{-1}}e'_{-4}, T^*e'_2\rangle| =$$
$$= |\langle e'_{-4}, N_{\mu_{-1}}T^*e'_2\rangle| \leqslant \|N_{\mu_{-1}}T^*e'_2\| < \varepsilon_{2;1};$$
$$|\langle Te'_{-2}, e'_4\rangle| = \left|\left\langle Te'_{-2}, \frac{y_2}{\|y_2\|}\right\rangle\right| = \frac{1}{\|y_2\|}|\langle N[\lambda_3, \lambda_4]Te'_{-4}, Te'_{-2}\rangle| \leqslant$$
$$\leqslant \frac{1}{\|y_2\|}|\langle N^{\perp}_{\mu_1}Te'_{-4}, Te'_{-2}\rangle| \leqslant \frac{1}{\|y_2\|}|\langle Te'_{-4}, N^{\perp}_{\mu_1}Te'_{-2}\rangle| \leqslant$$
$$\leqslant \frac{1}{\|y_2\|}\|T\|\,\|N^{\perp}_{\mu_1}Te'_{-2}\| < 2\|T\|\frac{\varepsilon_{1;2}}{2\|T\|} = \varepsilon_{1;2}.$$

Let e'_{-3} be an arbitrary element in $N[\lambda_{-3}, \lambda_{-2}]$ and let e'_3 be an arbitrary element in $N[\lambda_2, \lambda_3]$. Lemma 2 is proven.

Proof of Proposition 1. We will prove case (1). For the proofs of other cases, first we have to establish a lemma similar to Lemma 2 in each case. Then we can use the following proof for case (1). We leave these to the reader.

Proof of case (1). Since $\Phi_{\mathcal{N}}(T) > 0$, we have

$$\inf\{\|N^{\perp}_{\mu}TN_{\lambda}\| : \mu\lambda \in \Lambda \cap (0,1)\} \geqslant$$
$$\geqslant \inf\{\|N^{\perp}_{\mu}TN_{\lambda}\|_e : \mu, \lambda \in \Lambda \cap (0,1)\} = \Phi_{\mathcal{N}}(T) > 0.$$

Notice that for any given non-zero complex number α, we have $\Phi_{\mathcal{N}}(\alpha T) = |\alpha|\Phi_{\mathcal{N}}(T)$ and $\mathcal{I}_{\mathcal{N}}(\alpha T) = \mathcal{I}_{\mathcal{N}}(T)$. Without loss of generality, we can assume that $\inf\{\|N^{\perp}_{\mu}TN_{\lambda}\| : \mu, \lambda \in \Lambda \cap (0,1)\} = 1$. The nest $\mathcal{N}$ satisfies the conditions in Lemma 2. Let $\{e'_n : n \in \mathbf{Z} \setminus \{0\}\}$ and $\{\lambda_n : n \in \mathbf{Z}\}$ be those as in Lemma 2.

Define $S_1 = \sum_{k=1}^{\infty} e'_{2k-1} \otimes e'_{2k}$ and $S_2 = \sum_{k=1}^{\infty} e'_{-2k} \otimes e'_{-2k+1}$. Then we have $S_1 \in \operatorname{alg}\mathcal{N}$ and $S_2 \in \operatorname{alg}\mathcal{N}$. Hence $S_1TS_2 \in \mathcal{I}(T)$.

Let $P_0 = \operatorname{proj}[e'_{2k} : k \in \mathbf{N}] = \sum_{k=1}^{\infty} e'_{2k} \otimes e'_{2k}$. We have

$$S_1TS_2 = \left(\sum_{k=1}^{\infty} e'_{2k-1} \otimes e'_{2k}\right) T \left(\sum_{k=1}^{\infty} e'_{-2k} \otimes e'_{-2k+1}\right) =$$
$$= \left(\sum_{k=1}^{\infty} e'_{2k-1} \otimes e'_{2k}\right)\left(\sum_{k=1}^{\infty} e'_{2k} \otimes e'_{2k}\right)\left(\sum_{k=1}^{\infty} Te'_{-2k} \otimes e'_{-2k+1}\right) =$$

$$= \sum_{k=1}^{\infty} \langle Te'_{-2k}, e'_{2k}\rangle e'_{2k-1} \otimes e'_{-2k+1} + T_0,$$

where $T_0 = \sum_{k\neq j} \langle Te'_{-2j}, e'_{2k}\rangle e'_{2k-1} \otimes e'_{-2j+1}$. Since $\sum_{k\neq j} |\langle Te'_{-2j}, e'_{2k}\rangle| < \sum_{k\neq j} \varepsilon_{j;k} < \frac{1}{2}$, the operator T_0 is compact. We will show that T_0 is in $\mathcal{I}_{\mathcal{N}}(T)$. This is immediate from the fact that $e'_{2k-1}\otimes e'_{2k}$ and $e'_{-2j}\otimes e'_{-2j+1}$ are in alg$\mathcal{N}$ and $(e'_{2k-1}\otimes e'_{2k})T(e'_{-2j}\otimes e'_{-2j+1}) = \langle Te'_{-2j}, e'_{2k}\rangle e'_{2k-1} \otimes e'_{-2j+1}$. So, the operator $\sum_{k=1}^{\infty} \alpha_k e'_{2k-1} \otimes e'_{-2k+1} = S_1TS_2 - T_0 \in \mathcal{I}_{\mathcal{N}}(T)$ where $\alpha_k = \langle Te'_{-2k}, e'_{2k}\rangle$. Recall that we also have $\frac{1}{2} < |\alpha_k| < \frac{3}{2}$. Let $\beta_k = \alpha_k^{-1}$ and define $S_3 = \sum_{k=2}^{\infty} \beta_k e'_{2k-3} \otimes e'_{2k-1}$. Then $S_3 \in$ alg$\mathcal{N}$ and $S_3S_1TS_2 =$

$$= \left(\sum_{k=2}^{\infty} \beta_k e'_{2k-3} \otimes e'_{2k-1}\right)\left(\sum_{k=1}^{\infty} \alpha_k e'_{2k-1} \otimes e'_{-2k+1}\right) = \sum_{k=2}^{\infty} e'_{2k-3} \otimes e'_{-2k+1} \in \mathcal{I}_{\mathcal{N}}(T).$$

Let $\{e_{-k}\}$ be the o.n. basis in case (1). We will prove that there exists a strictly increasing sequence $\{n_k : k \in \mathbb{N}\}$ such that the operator $\sum_{k=1}^{\infty} e'_{-2n_k+1} \otimes e_{-k}$ is in $\mathcal{I}_{\mathcal{N}}(T)$. Let n_1 be the smallest natural number n such that $e'_{-2n+1} \otimes e_{-1} \in$ alg$\mathcal{N}$ and $n > 1$. Assume that $\{n_i : i = 1, 2, \ldots, k-1\}$ are chosen. We define n_k be the smallest natural number n such that $n > \max\{n_i : i = 1, 2, \ldots, k-1\}$ and $e'_{-2n+1} \otimes e_{-k} \in$ alg$\mathcal{N}$.

Let $T_1 = \sum_{k=1}^{\infty} e'_{-2n_k+1} \otimes e'_{-k}$. Then $T_1 \in$ alg$\mathcal{N}$. Hence $\left(\sum_{k=2}^{\infty} e'_{2k-3} \otimes e'_{-2k+1}\right) T_1 = \sum_{k=1}^{\infty} e'_{2n_k-3} \otimes e_{-k} \in \mathcal{I}_{\mathcal{N}}(T)$.

We denote $f_k = e'_{2n_k-3}$, $F_1 = N[\lambda_0, \lambda_{2n_1-3}]$ and $F_k = N[\lambda_{2n_{k-1}-3}, \lambda_{2n_k-3}]$, for $k > 1$. The sequences $\{f_k\}$ and $\{F_k\}$ and operator $S_0 = \sum_{k=1}^{\infty} f_k \otimes e_{-k}$ are what we needed. The proposition is proven.

LEMMA 3. *Let $\mathcal{N}$ be a nest of order type II_2 in a separable Hilbert space $\mathcal{H}$. Then there exists an operator S_0 in $\mathcal{B}(\mathcal{H})$ such that $i_{\mathcal{N}}(S_0) = 1$.*

Proof. If $\alpha_0 = \alpha$ and $\beta_0 = \beta$, we can find a strictly increasing sequence $\{\lambda_n : n \in \mathbb{Z}\}$ in Λ such that $\lim_{n\to\infty} N[0, \lambda_n] = N_\beta$ and $\lim_{n\to-\infty} N[0, \lambda_n] = N_\alpha$. For each $n \in \mathbb{N}$, let e_{-n} and f_n be unit vectors such that $e_{-n} \in N[\lambda_{-n-1}, \lambda_{-n}]$, and $f_n \in N[\lambda_n, \lambda_{n+1}]$. Let $S_1 = \sum f_n \otimes e_{-n}$.

If $\alpha_0 < \alpha$, we can choose e_{-n} to be an o.n. sequence in the smallest $\mathcal{N}$-interval $N[\alpha_0, \lambda]$ which is infinite dimensional; in case $\beta < \beta_0$ we can choose f_n to be an o.n. sequence in the smallest $\mathcal{N}$-interval $N[\mu, \beta_0]$ which is infinite dimensional. For such an operator S_1, we have $\Phi_{\mathcal{N}}(S_1) = 1$.

Let e be a separating vector of $\mathcal{C}_{\mathcal{N}}$, the abelian von Neumann generated by $\mathcal{N}$. Then we have $\Phi_{\mathcal{N}}(S_1 + e \otimes e) = \Phi_{\mathcal{N}}(S_1) = 1$ and $j_{\mathcal{N}}(S_1 + e \otimes e) = 1$. The operator $S_0 = S_1 + e \otimes e$ satisfies the condition. Lemma 3 has proven.

LEMMA 4. *Let $\mathcal{N}$ be an order type II_2 nest in a separable Hilbert space $\mathcal{H}$. Assume that T is an operator in $\mathcal{B}(\mathcal{H})$ such that $j_{\mathcal{N}}(T) > 0$. Then $\mathcal{K} \subseteq \mathcal{I}_{\mathcal{N}}(T)$.*

REMARK. The lemma is true for nests of any type. For proof, we need discuss in cases.

Proof. It suffices to prove that any rank-1 operator in $\mathcal{B}(\mathcal{H})$ is in $\mathcal{I}_{\mathcal{N}}(T)$. Let $x, y \in \mathcal{B}(\mathcal{H})$. Since the nest $\mathcal{N}$ is of order type II_2, the projections E_λ and E_ρ are finite dimensional. We denote $P = 0_+$ and $Q = I_-$. Since $j_{\mathcal{N}}(T) \neq 0$, there is a unit vector $u \in P$ such that $Q^\perp T P u \neq 0$. Define $v = \dfrac{Q^\perp T P u}{\|Q^\perp T P u\|}$. We have $u \otimes x \in \mathrm{alg}\mathcal{N}$ and $y \otimes v \in \mathrm{alg}\mathcal{N}$. Therefore we have $y \otimes x = \|Q^\perp T P u\|(y \otimes v)T(u \otimes x) \in \mathcal{I}_{\mathcal{N}}(T)$.

3. PROOF OF MAIN THEOREM AND COROLLARIES

In this section we will complete the proof of the Main Theorem. Some properties of the generating set are stated and proven. Also, we will characterize the generators of $\mathcal{B}(\mathcal{H})/\mathcal{K}$ as an $\mathcal{A}(\mathcal{N})$-norm-closed bimodule and characterize the generators of $\mathcal{B}(\mathcal{H})$ as a norm-bimodules of $\mathrm{alg}\mathcal{N} + \mathcal{K}$. We start with some special cases.

THEOREM 1. *Let $\mathcal{N}$ be a nest of order type II_2 in a separable Hilbert space $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. Assume that the nest and the operator T satisfying the following conditions*

1. *The interval nest $N[\alpha_0, \beta_0]\mathcal{N}$ is in one of the four cases in Proposition 1.*
2. *$\Phi_{\mathcal{N}}(T) \neq 0$.*

Then we have $\mathcal{I}_{\mathcal{N}}(T) + \mathcal{K} = \mathcal{B}(\mathcal{H})$.

Proof. Denote $\mathcal{M} = N[\alpha_0, \beta_0]\mathcal{N}$. The nest $\mathcal{M}$ in $N[\alpha_0, \beta_0]$ satisfies the condition in Proposition 1. It is easy to verify that $\Phi_{N[\alpha_0,\beta_0]\mathcal{N}}(N[\alpha_0, \beta_0]) = \Phi_{\mathcal{N}}(T) \neq 0$. By Proposition 1 we have

$$\mathcal{I}_{\mathcal{N}}(T) = [\mathrm{alg}\mathcal{N} T \mathrm{alg}\mathcal{N}] \supseteq$$

$$\supseteq [(N[\alpha_0, \beta_0]\mathrm{alg}\mathcal{N} N[\alpha_0, \beta_0])T(N[\alpha_0, \beta_0]\mathrm{alg}\mathcal{N} N[\alpha_0, \beta_0])] \supseteq$$

$$\supseteq N[\alpha_0, \beta_0]\mathcal{B}(\mathcal{H})N[\alpha_0, \beta_0].$$

The conclusions follows from the following fact:

$$\{T \in \mathcal{B}(\mathcal{H}) : N[\alpha_0, \beta_0]TN[\alpha_0, \beta_0] = 0\} \subseteq \mathcal{K},$$

since $N[0,\alpha_0]$ and $N[\beta_0,1]$ are compact.

THEOREM 2. *Let $\mathcal{N}$ be an arbitrary nest of order* II_2 *in $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$ such that $\Phi_{\mathcal{N}}(T) = a > 0$. Then there exists a refinement nest $\mathcal{M}$ of $\mathcal{N}$ satisfying the following conditions:*

1. *The nest $\mathcal{M}$ satisfies the condition in Theorem 1.*
2. $\Phi_{\mathcal{M}}(T) = \Phi_{\mathcal{N}}(T)$.

Proof. Since the nest has order type II_2, there are only 3 possible cases:

1. $\alpha_0 = \alpha$ and $\beta < \beta_0$.
2. $\alpha_0 < \alpha$ and $\beta = \beta_0$.
3. $\alpha_0 < \alpha$ and $\beta < \beta_0$.

1. Assume $\alpha_0 = \alpha$ and $\beta < \beta_0$ and $\Phi_{\mathcal{N}}(T) = a > 0$. Then $N[\beta,\beta_0]$ is an infinite dimensional subspace. Let $\{f_n : n \in \mathbb{N}\}$ be an orthonormal basis for $N[\beta,\beta_0]$. Define a sequence of projections $\{Q_n : n \in \mathbb{N}\}$ such that $Q_n = N_\beta + [f_k : 0 \leqslant k \leqslant n]$. Then $\lim\limits_{n\to\infty} Q_n = N_{\beta_0}$. So $\mathcal{M} = \mathcal{N} \cup \{Q_n : n \in \mathbb{N}\}$ is a refinement of $\mathcal{N}$. The new nest satisfies the condition in Theorem 1 since β_0 is a new limit point. We must prove that $\Phi_{\mathcal{M}}(T) = \Phi_{\mathcal{N}}(T)$.

It is obvious that $\Phi_{\mathcal{M}}(T) \leqslant \Phi_{\mathcal{N}}(T) = a$ since $\mathcal{M} \supset \mathcal{N}$. We must show that $\Phi_{\mathcal{M}}(T) \geqslant a$. Recall that for a compact operator K we have $\|K\|_c = 0$ and that $\|\cdot\|_c$ is a seminorm. Notice that $Q_n - N_\beta = [f_k : 1 \leqslant k \leqslant n]$ is compact. Let N_λ be any projection in $\mathcal{M}$ such that $N_\mu > N_{\alpha_0}$ and $n > 0$. Then

$$\begin{aligned}
&\|(N_{\beta_0} - Q_n)T(N_\lambda - N_{\alpha_0})\|_c \geqslant \\
&\geqslant \|(N_{\beta_0} - N_\beta)T(N_\lambda - N_{\alpha_0})\|_c - \|(Q_n - N_\beta)T(N_\lambda - N_{\alpha_0})\|_c = \\
&= \|(N_{\beta_0} - N_\beta)T(N_\lambda - N_{\alpha_0})\|_c \geqslant a,
\end{aligned}$$

for any n and for any $N_\lambda \in \mathcal{M}$.

Therefore $\Phi_{\mathcal{M}}(T) \geqslant a$. Proofs of case 2 and case 3 are similar to this. Theorem 2 is proven.

COROLLARY 1. *Let $\mathcal{N}$ be an order type* II_2 *nest in a separable Hilbert space $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $\Phi_{\mathcal{N}}(T) \neq 0$. Then $\mathcal{I}_{\mathcal{N}}(T) + \mathcal{K} = \mathcal{B}(\mathcal{H})$.*

Proof. Let $\mathcal{M}$ be a refinement of $\mathcal{N}$ as in Theorem 2. Then we have $\Phi_{\mathcal{M}}(T) \neq 0$. By Theorem 1 we have

$$[\operatorname{alg}\mathcal{M} T \operatorname{alg}\mathcal{M}] + \mathcal{K} = \mathcal{B}(\mathcal{H}).$$

The conclusion of Corollary 1 follows the fact that $\operatorname{alg}\mathcal{N} \supseteq \operatorname{alg}\mathcal{M}$.

COROLLARY 2. *Let $\mathcal{N}$ be an order type* II_2 *nest in a separable Hilbert space $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $\Phi_{\mathcal{N}}(T) \neq 0$. Then $\mathcal{I}_{\mathcal{N}}(T) \subseteq N[\alpha_0,\beta_0]\mathcal{B}(\mathcal{H})N[\alpha_0,\beta_0]$.*

Now the following Theorem 3 is in hand.

THEOREM 3. *Let $\mathcal{N}$ be an order type II_2 nest in a separable Hilbert space $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$. Assume that $i_{\mathcal{N}}(T) \neq 0$. Then $\mathcal{I}_{\mathcal{N}}(T) = \mathcal{B}(\mathcal{H})$.*

Proof. This is from Lemma 4 and Corollary 2 of Theorem 2.

Proof of Main Theorem.

Theorem 3 shows that the condition is sufficient.

Let $\mathcal{N}$ be a nest of order type II_2. If T is an operator such that $i_{\mathcal{N}}(T) = 0$, then $j_{\mathcal{N}}(T) = 0$ or $\Phi_{\mathcal{N}}(T) = 0$. We have the following two cases.

1) $j_{\mathcal{N}}(T) = 0$. There are $\lambda, \mu \in \Lambda \setminus \{0,1\}$ such that $N_\mu^\perp T N_\lambda = 0$. So for any A and B in alg$\mathcal{N}$, we have $N_\mu^\perp ATBN_\lambda = N_\mu^\perp A N_\mu^\perp T N_\lambda B N_\lambda = 0$. This implies that $j_{\mathcal{N}}(S) = 0$ for any $S \in \mathcal{I}_{\mathcal{N}}(T)$.

2) $\Phi_{\mathcal{N}}(T) = 0$. We will show that for any S in $\mathcal{I}_{\mathcal{N}}(T)$, we have $i_{\mathcal{N}}(S) = 0$. It sufficies to prove that for any operators A and B in alg$\mathcal{N}$ we have $\Phi_{\mathcal{N}}(ATB) = 0$. Since $\mathcal{N}$ has order type II_2. Both of $N[0,\alpha_0]$ and $N[\beta_0, 1]$ are finite dimensional. So we have $T = (N_{\beta_0} + N[\beta_0, 1])T(N_{\alpha_0}^\perp + N[0,\alpha_0]) = N_{\beta_0} T N_{\alpha_0}^\perp + K$ for some compact operator K. Recall that for any operator K in $\mathcal{B}(\mathcal{H})$, we have $\Phi_{\mathcal{N}}(K) = 0$. Let A and B be any operators in alg$\mathcal{N}$. We have

$$
\begin{gathered}
\Phi_{\mathcal{N}}(ATB) \leqslant \Phi_{\mathcal{N}}(N_{\beta_0} T N_{\alpha_0}^\perp) + \Phi_{\mathcal{N}}(K) = \Phi_{\mathcal{N}}(N_{\beta_0} T N_{\alpha_0}^\perp) = \\
= \inf\{\|(N_{\beta_0} - N_\mu) A N_{\beta_0} T N_{\alpha_0}^\perp B (N_\lambda - N_{\alpha_0})\|_c : \mu, \lambda \in \Lambda \text{ and } \mu < \beta_0, \lambda > \alpha_0\} = \\
= \inf\{\|(N_{\beta_0} - N_\mu) A (N_{\beta_0} - N_\mu) T (N_\lambda - N_{\alpha_0}) B (N_\lambda - N_{\alpha_0})\|_c : \\
: \mu, \lambda \in \Lambda \text{ and } \mu < \beta_0, \lambda > \alpha_0\} \leqslant \\
\leqslant \|A\|\,\|B\| \inf\{\|(N_{\beta_0} - N_\mu) T (N_\lambda - N_{\alpha_0})\|_c : \mu, \lambda \in \Lambda \text{ and } \mu < \beta_0, \lambda > \alpha_0\} = \\
= \|A\|\,\|B\| \Phi_{\mathcal{N}}(T) = 0.
\end{gathered}
$$

Since any S in $\mathcal{I}_{\mathcal{N}}(T)$ is a norm-limit of finite combinations of operators in form $A_i T B_i$ for $i \in \mathbb{N}$ and $A_i, B_i \in$ alg$\mathcal{N}$, and the seminorm $\|\cdot\|_c$ is norm continuous, we proved that $\Phi_{\mathcal{N}}(S) = 0$. By Lemma 3 there exists an operator S_0 in $\mathcal{B}(\mathcal{H})$ with $i_{\mathcal{N}}(S_0) = 1$. Hence $S_0 \notin \mathcal{I}_{\mathcal{N}}(T)$. So any operator T with $i_{\mathcal{N}}(T) = 0$ can not be a generator.

The proof is complete.

COROLLARY 1. *Let $\mathcal{N}$ be a continuous nest in $\mathcal{H}$. Then the property that $\Phi_{\mathcal{N}}(T) \neq 0$ is a necessary and sufficient condition for $T \in \mathcal{B}(\mathcal{H})$ to be a generator of $\mathcal{B}(\mathcal{H})$ as a norm-closed-principal bimodule of alg$\mathcal{N}$.*

Proof. In this case we have $i_{\mathcal{N}}(T) = \Phi_{\mathcal{N}}(T)$.

THEOREM 4. *Let $\mathcal{N}$ be a nest of order type* II_2 *in a separable Hilbert space $\mathcal{H}$ and let $\mathcal{G}$ be the set of all single generators of $\mathcal{B}(\mathcal{H})$ as an* $\mathrm{alg}\mathcal{N}$*-norm-principal--bimodule, then $\mathcal{G}$ possesses the following properties:*

1. *The set $\mathcal{G}$ is open dense in $\mathcal{B}(\mathcal{H})$ and its complement $\mathcal{B}(\mathcal{H}) \setminus \mathcal{G}$ is a closed subset in $\mathcal{B}(\mathcal{H})$.*

2. *Both of $\mathcal{G}$ and $\mathcal{B}(\mathcal{H}) \setminus \mathcal{G}$ are star shaped set in $\mathcal{B}(\mathcal{H})$. In other words, for any non-zero complex number a we have $a\mathcal{G} = \mathcal{G}$ and $a(\mathcal{B}(\mathcal{H}) \setminus \mathcal{G}) = \mathcal{B}(\mathcal{H}) \setminus \mathcal{G}$.*

3. *If $E_\lambda = E_\rho = 0$ and let K be any compact operator in $\mathcal{B}(\mathcal{H})$, then we have $\mathcal{G} + K = \mathcal{G}$. That is any compact perturbation of a generator is a generator.*

Proof. Under the condition of theorem, by the Main Theorem, it clear that $\mathcal{G}$ is an open set. Let T be an arbitrary operator in $\mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. We will show that there exists an operator T' in the ε-neighborhood of T such that T' is in $\mathcal{G}$. Assume that $T \notin \mathcal{G}$. Then $j_{\mathcal{N}}(T) = 0$ or $\Phi_{\mathcal{N}}(T) = 0$. Let S_1 and S_2 be the operators in the proof of Lemma 3. We define $S_\varepsilon = \frac{\varepsilon}{2}(S_1 + S_2)$, let $T' = T + S_\varepsilon$. The operator T' is in $\mathcal{G}$ and $\|T' - T\| = \|S_\varepsilon\| < \varepsilon$.

The proofs of other two properties are left to the reader.

COROLLARY 1. *Let $\mathcal{N}$ be a continuous nest in $\mathcal{H}$ and let G be a generator for $\mathcal{B}(\mathcal{H})$ as a norm-closed singly generated bimodule of* $\mathrm{alg}\mathcal{N}$ *and let K be an arbitrary compact operator in $\mathcal{B}(\mathcal{H})$. Then $G + K$ is also a generator.*

In the following Theorem we characterize generators of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ as a norm-bimodule of $\mathcal{A}(\mathcal{N})$, the image of $\mathrm{alg}\mathcal{N}$ under the Calkin homomorphism and we will characterize the generators of $\mathcal{B}(\mathcal{H})$ as a norm-bimodule of $\mathrm{alg}\mathcal{N} + \mathcal{K}$, the quasi-triangular algebra.

THEOREM 5. *Let $\mathcal{N}$ be a type* II_2 *nest in an infinite dimensional separable Hilbert space $\mathcal{H}$ and T an operator in $\mathcal{B}(\mathcal{H})$. Then*

1. *$T + \mathcal{K}$ is a generator of $\mathcal{B}(\mathcal{H})/\mathcal{K}$ as a norm-principal bimodule of $\mathcal{A}(\mathcal{N})$ if and only if $\Phi_{\mathcal{N}}(T) > 0$. Furthermore, the set of generators for $\mathcal{B}(\mathcal{H})/\mathcal{K}$ is an open dense subset in $\mathcal{B}(\mathcal{H})/\mathcal{K}$.*

2. *The operator T is a generator of $\mathcal{B}(\mathcal{H})$ as a norm-principal bimodule of* $\mathrm{alg}\mathcal{N} + \mathcal{K}$ *if and only if $\Phi_{\mathcal{N}}(T) > 0$. The set of generators for $\mathcal{B}(\mathcal{H})$ is an open dense subset in $\mathcal{B}(\mathcal{H})$.*

Proof. 1) If $T \in \mathcal{B}(\mathcal{H})$ with $\Phi_{\mathcal{N}}(T) > 0$. Then $\mathcal{I}_{\mathcal{N}}(T) = [(\mathrm{alg}\mathcal{N})T(\mathrm{alg}\mathcal{N})]$ contains $N[\mu, \beta_0]\mathcal{B}(\mathcal{H})N[\alpha_0, \lambda]$. Since the nest $\mathcal{N}$ has order type II_2, the projections $N[0, \alpha_0]$ and $N[\beta_0, 1]$ are finite dimensional hence compact. So we have $[(\mathrm{alg}\mathcal{N})\{T + +K : K \in \mathcal{K}\}(\mathrm{alg}\mathcal{N})] = \mathcal{B}(\mathcal{H})$. This implies that the element $T + \mathcal{K}$ generates $\mathcal{B}(\mathcal{H})/\mathcal{K}$ as an $\mathcal{A}(\mathcal{N})$ norm-principal-bimodule.

Let π be the Calkin homomorphism from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}$. Let $S \in \mathcal{B}(\mathcal{H})$ and $\pi(S) \in \pi(A)\pi(T)\pi(B)$, then $S = ATB + K$ for some $K \in \mathcal{K}$. If $\Phi_{\mathcal{N}}(T) = 0$, then $\Phi_{\mathcal{N}}(S) = \Phi_{\mathcal{N}}(ATB + K) = 0$. If $\mathcal{S}$ is the norm limit of cosets $\mathcal{S}_n = S_n + \mathcal{K} = = \sum_1^{m_n} \pi(A_{i;n})\pi(T)\pi(B_{i;n})$, where $A_{i;n}$ and $B_{i;n}$ are in alg$\mathcal{N}$ and m_n is some natural number. Then S_n is in form of finite linear combination of A_iTB_i for some $A_i, B_i \in \in \text{alg}\mathcal{N}$. So there exist $K_n \in \mathcal{K}$ such that $\lim \|S - (S_n + K_n)\| = 0$. Since $\Phi_{\mathcal{N}}$ is continuous, this implies that $\Phi_{\mathcal{N}}(S) = 0$ for any S for which $\pi(S) \in [\mathcal{A}(\mathcal{N})(T+\mathcal{K})\mathcal{A}(\mathcal{N})]$. Since there exists an element T in $\mathcal{B}(\mathcal{H})$ such that $\Phi_{\mathcal{N}}(T) = 1$. So $[\mathcal{A}(\mathcal{N})(S+\mathcal{K})\mathcal{A}(\mathcal{N})]$ is a proper subset of $\mathcal{B}(\mathcal{H})/\mathcal{K}$ and the set $\{T \in \mathcal{B}(\mathcal{H}) : \Phi_{\mathcal{N}}(T) > 0\}$ is an open subset in $\mathcal{B}(\mathcal{H})$ (in the norm topology). Therefore the set $\{T + \mathcal{K} \in \mathcal{B}(\mathcal{H})/\mathcal{K} : \Phi_{\mathcal{N}}(T) > 0\}$ is an open subset in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$. The generator set is dense in $\mathcal{B}(\mathcal{H})$. This follows from Theorem 3 and the continuity of π.

2) For this part we just point out that if $T \neq 0$ then $\mathcal{K}$ is a subset of $[\text{alg}\mathcal{N} + +\mathcal{K})T(\text{alg}\mathcal{N} + \mathcal{K})]$. The rest part of proof is essentialy the same as 1).

The author thanks Professor David R. Larson for his comments.

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Received December 12, 1990; revised January 19, 1992.