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FLAT CONVEX HULLS IN THE PREDUAL OF AN OPERATOR ALGEBRA

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In the past dozen years the study of dual algebras of operators, initiated in [5], has made significant contributions to the structure theory of bounded linear operators on Hilbert space. Central has been the class $\mathbb{A}$ of contractions and its subclasses (definitions reviewed below), with [1], [9], [10], and [8] containing capstone results of some portions of the theory. More recently effort is turning to applications (see, e.g,. [6] and [13]) of these techniques to operators not in the class $\mathbb{A}$.

We present in this paper a dual algebra approach for contractions not in $\mathbb{A}$ and some applications to weighted shifts, yielding results weaker than but analoguous to those for the class $\mathbb{A}_{\aleph_0}$, the most restrictive of the subclasses of $\mathbb{A}$. For example, we obtain the following:

THEOREM. *Suppose S is a contractive, injective unilateral shift of multiplicity one whose spectral disk of radius r consists of eigenvalues for S^*. Suppose further that S/r is in the class C_{00}. Then S dilates, up to a unitary equivalence, both a diagonal normal operator with eigenvalues dense in $\{|z| < r\}$ and an infinite dimensional zero operator.*

There is a similar theorem for S a shift with rich left essential spectrum, without the C_{00} hypothesis but with certain technical assumptions. These results are substantively new if S is not polynomially (even power) bounded, and unify the theory in treating the polynomially bounded and unbounded cases together in any event.

The organizations of the paper is as follows: after preliminaries in Section 1 on dual algebras and weighted shifts, Section 2 presents an approach for solving a limited class of equations in the predual of an absolutely continuous contraction. In Section 3 are applications to unilateral weighted shifts whose adjoints have rich point spetrum, and in Section 4 to shifts with rich left essential spectrum. In Section 5 we make some remarks and raise some questions.

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1. PRELIMINARIES

Begin with some preliminaries on dual algebras (see [3] for more detail), starting with function spaces. Denote by $\mathbb{D}$ the open unit disk in the complex plane and by $\mathbb{T}$ the unit circle. Let $L^p = L^p(\mathbb{T})\,(1 \leqslant p \leqslant \infty)$ denote the usual (complex) Lebesgue spaces. It is well known that L^∞ is the Banach dual of L^1 under the pairing $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})g(e^{it})\,dt$ for $f \in L^\infty$ and $g \in L^1$. Let H_0^1 denote the subspace of L^1 consisting of functions whose non-positive Fourier coefficients vanish. Let $H^\infty = H^\infty(\mathbb{T})$ denote the subspace of L^∞ consisting of those functions whose negative Fourier coefficients vanish. From general facts it follows that H^∞ is the dual of L^1/H_0^1. The duality is given by $\langle f, [g]_{L^1/H_0^1}\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})g(e^{it})\,dt$ for $f \in H^\infty$ and $[g] \in L^1/H_0^1$ where the cosets in L^1/H_0^1 are denoted by $[\cdot] = [\cdot]_{L^1/H_0^1}$. It is easy to check that a coset $[g]_{L^1/H_0^1}$ is uniquely determined by the sequence of negative Fourier coefficients of any of its representatives. Denote by $\operatorname{Ball}(L^1/H_0^1)$ the unit ball in the quotient norm.

Let $\mathcal{H}$ denote a separable, infinite dimensional, complex Hilbert space with inner product $(\cdot,\cdot) = (\cdot,\cdot)_{\mathcal{H}}$ and $\mathcal{L}(\mathcal{H})$ the Banach algebra of bounded linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is a contraction if $\|T\| \leqslant 1$. A contraction T is absolutely continuous if in the decomposition $T = U \oplus T'$, where U is unitary (or absent) and T' is completely non-unitary, U is absent or has spectral measure absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$. Denote the spectrum, point spectrum, and left essential spectrum of T in $\mathcal{L}(\mathcal{H})$ by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{le}(T)$ respectively.

It is well known that $\mathcal{L}(\mathcal{H})$ is the Banach dual of the ideal τc of trace class operators on $\mathcal{H}$, with the action given by $\langle T, C\rangle = \operatorname{trace}(TC)$ for $T \in \mathcal{L}(\mathcal{H})$ and $C \in \tau c$. For T in $\mathcal{L}(\mathcal{H})$ denote by $\mathcal{A}_T$ the weak* closed unital algebra generated by T. It follows from general facts that $\mathcal{A}_T$ is the dual of the space $Q_T \equiv \tau c/{}^\perp\mathcal{A}_T$, where ${}^\perp\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in τc. This duality is given by

$$\langle S, [L]_T\rangle = \operatorname{trace}(SL), \quad S \in \mathcal{A}_T, L \in \tau c. \tag{1}$$

Denote by $\mathrm{Ball}(Q_T)$ the unit ball in Q_T.

In the case $T \in \mathcal{L}(\mathcal{H})$ is an absolutely continuous contraction there is (from [18]) the good Sz.-Nagy-Foiaş functional calculus $\Phi_T: H^\infty \to \mathcal{A}_T$. Further, there is a bounded, linear, one-to-one map $\varphi_T: Q_T \to L^1/H_0^1$ such that $\varphi_T^* = \Phi_T$. Some elements of L^1 inducing cosets important for our study are the P_λ ($\lambda \in \mathbb{D}$), where P_λ denotes the Poisson kernel defined by $P_\lambda(\mathrm{e}^{it}) = (1-|\lambda|^2)/|1-\overline{\lambda}\mathrm{e}^{it}|^2$ for $\mathrm{e}^{it} \in \mathbb{T}$. If $[P_\lambda]_{L^1/H_0^1}$ is in the range of φ_T define $[C_\lambda]_T$ by $[C_\lambda]_T = \varphi_T^{-1}([P_\lambda]_{L^1/H_0^1})$, and write $[C_\lambda]$ if no confusion will arise.

If x and y are vectors in $\mathcal{H}$ the operator $x \otimes y$ in $\mathcal{L}(\mathcal{H})$ is the usual rank one operator defined for u in $\mathcal{H}$ by $x \otimes y(u) = (u,y)x$. The cosets $[x \otimes y]_T$ are critical to the study of dual algebras; it is easy to check that their action is as follows:

$$\langle S, [x \otimes y]_T \rangle = (Sx, y)_{\mathcal{H}}, \quad S \in \mathcal{A}_T, x, y \in \mathcal{H}, \tag{2}$$

yielding negative Fourier coefficients $(c_0, c_{-1}, \ldots)$ for $[x \otimes y]_T$ given by

$$c_{-j} = (T^j x, y), \quad j = 0, 1, \ldots \; . \tag{3}$$

Much of the study of contractions via dual algebra trechniques has involved finding a rank one representative for some $[C_\lambda]_T$, usually called "solving the equation" $[x \otimes y]_T = [C_\lambda]_T$. Note for future use in such solutions that $[P_\lambda]$, and hence $[C_\lambda]$ if defined, has the sequence of negative Fourier coefficients

$$[P_\lambda] \sim 1, \lambda, \lambda^2, \ldots . \tag{4}$$

The following class and properties are fundamental (see [3]).

DEFINITION 1.1. The class $\mathbb{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the functional calculus $\Phi_T: H^\infty \to \mathcal{A}_T$ is an isometry.

DEFINITION 1.2. Let $\mathcal{M} \subseteq \mathcal{L}(\mathcal{H})$ be a weak* closed subspace, and let n be any cardinal number such that $1 \leqslant n \leqslant \aleph_0$. Then $\mathcal{M}$ will be said to have property $(\mathbb{A}_n)$ provided every $n \times n$ system of simultaneous equations of the form

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \leqslant i, j \leqslant n,$$

(where the $[L_{ij}]$ are arbitrary but fixed elements from $Q_{\mathcal{M}}$) has a solution $\{x_i\}_{0 \leqslant i<n}$, $\{y_j\}_{0 \leqslant j<n}$ consisting of pairs of sequences from $\mathcal{H}$. Further, we denote by $\mathbb{A}_n = \mathbb{A}_n(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property (A_n).

Finally, recall, essentially from [4], that a subset Λ of a disk E contained in the complex plane is said to be *dominating* for E if almost every point (Lebesgue measure on the boudary of E) is a non-tangential limit of points in Λ.

We next recall some basic facts about unilateral weighted shifts from [15] and [17], and use the notation of the latter. We study only injective unilateral weighted shifts with all weights positive (but see [17]). A unilateral shift S with spectral radius $r = r(S) > 0$ has as its spectrum the disk $\sigma(S) = \overline{r\mathbf{D}}$. There is a pair of other radii $r_1 = r_1(S)$ and $r_2 = r_2(S)$ satisfying $0 \leqslant r_1(S) \leqslant r_2(S) \leqslant r(S)$, and also (from [17] and [15] respectively):

(5) $$\{0\} \cup \{|\lambda| < r_2(S)\} \subseteq \sigma_p(S^*) \subseteq \{|\lambda| \leqslant r_2(S)\}, \text{ and}$$

(6) $$\sigma_{\mathrm{le}}(S) = \{r_1(S) \leqslant |\lambda| \leqslant r(S)\}.$$

2. AN EQUATION SOLVING PROCEDURE

The procedure is founded upon some subsets of L^1/H_0^1 associated with subsets (specifically subdisks) of $\mathbf{D}$. Consider the following properties, one for each r satisfying $0 < r < 1$:

PROPERTY C(r): If $[f]$ is any coset of L^1/H_0^1 whose representatives have the sequence of negative Fourier coefficients $f_0, f_{-1}, f_{-2}, \ldots$ then we say $[f]$ has Property C(r) if the sequence $f_0, \frac{f_{-1}}{r}, \frac{f_{-2}}{r^2}, \ldots$ is the sequence of negative Fourier coefficients of some g in $L^1(\mathbf{T})$.

The collection of those $[f]$ with Property C(r) is denoted by K_r $(0 < r < 1)$, and denote by $D_{1/r}[f]$ the coset $[g]$ above. An example useful later is, with P_λ the Poisson kernel at λ and using (4), that

(7) $$D_{1/r}[P_\lambda] = [P_{\lambda/r}], \quad |\lambda| < r.$$

For g in $L^1(\mathbf{T})$ consider the harmonic extension $\hat{g}$ of g to $\mathbf{D}$, $\hat{g}(z) \equiv \sum_{-\infty}^{\infty} g_n \rho^{|n|} e^{in\theta}$ for $z = \rho e^{i\theta}$. Define using $\hat{g}$ the function $\hat{g}_r(z) = \hat{g}(rz)$; then $g_r \in L^1(\mathbf{T})$. Indeed, a computation using the expression of $\hat{g}_r$ as a convolution with the Poisson kernel shows that $\|g_r\|_{L^1} \leqslant \|g\|_{L^1}$. It is equally clear that $D_{1/r}[g_r] = [g]$. Now for $[f] \in K_r$,

(8) $$\begin{aligned} \|D_{1/r}[f]\|_{L^1/H_0^1} &= \inf_{g \in D_{1/r}[f]} \|g\|_{L^1} \geqslant \inf_{g \in D_{1/r}[f]} \|g_r\|_{L^1} = \\ &= \inf_{g_r \in [f]} \|g_r\|_{L^1} \geqslant \inf_{h \in [f]} \|h\|_{L^1} = \|[f]\|_{L^1/H_0^1}. \end{aligned}$$

Note in passing that (7) shows that strict inequality need not hold in (8).

It is convenient to introduce a norm $\|\cdot\|_r$ on K_r by $\|[f]\|_r \equiv \|D_{1/r}[f]\|_{L^1/H_0^1}$ for $[f] \in K_r$. From (8) we have that $\|\cdot\|_r \geqslant \|\cdot\|_{L_1/H_0^1}$ on K_r. Denote by $\mathrm{Ball}^r(K_r)$ the $\|\cdot\|_r$-unit ball of K_r; then $\mathrm{Ball}^r(K_r) \subseteq \mathrm{Ball}(L^1/H_0^1)$.

This notation in hand, we consider some further subsets of the unit ball of L^1/H_0^1 related to those above. Denote by $\overline{\text{aco}}(G)$ the closure of the absolutely convex hull of a set G. It is well known ([4]) that if $\Lambda \subseteq \mathbb{D}$ is a set dominating for $\mathbb{D}$ then $\overline{\text{aco}}\{[P_\lambda]: \lambda \in \Lambda\} = \text{Ball}(L^1/H_0^1)$, and as well that if $\Lambda \subseteq \mathbb{D}$ is not dominating then $\overline{\text{aco}}\{[P_\lambda]: \lambda \in \Lambda\}$ contains no (quotient norm) ball of positive radius. The next lemma, whose proof is a computation from the definitions and therefore omitted, identifies some other absolutely convex hulls which, although they contain no quotient norm open ball (hence "flat"), are nevertheless large enough to be useful.

LEMMA 2.1. *Let $0 < r < 1$ and suppose $\Lambda \subseteq r\mathbb{D}$ is dominating for $r\mathbb{D}$. Then* $\overline{\text{aco}}^r\{[P_\lambda]: \lambda \in \Lambda\} = \text{Ball}^r(K_r)$.

The preliminaries so far have concerned L^1/H_0^1 but the goal is in fact to "solve equations" in the predual Q_S of the ultraweakly closed algebra $\mathcal{A}_S$ generated by an absolutely continuous contraction S. Since in general $S \notin \mathbb{A}$, neither Φ_S nor φ_S is surjective and Q_S is not well related to L^1/H_0^1 (see [3]). We turn next to a technical device for avoiding this difficulty.

It is well known that if $T \in \mathbb{A}(\mathcal{K})$ and S is an absolutely continuous contraction then $S \oplus T \in \mathbb{A}(\mathcal{H} \oplus \mathcal{K})$. Choose (for definiteness) the unweighted bilateral shift B in $\mathbb{A}(\mathcal{K})$ and set some notation fixed for the rest of the paper (the results are easily seen to be independent of the choice of B as representative for the class $\mathbb{A}$).

NOTATION. Denote by S in $\mathcal{L}(\mathcal{H})$ an absolutely continuous contraction, and by $\hat{S}$ the operator $S \oplus B$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. Write vectors in $\mathcal{H}$ as $x, y, \ldots$ and denote by $\hat{x}, \hat{y}, \ldots$ the vectors $x \oplus 0, y \oplus 0, \ldots$ in $\mathcal{H} \oplus \mathcal{K}$.

The technique to "solve equations" is as follows: given an element $[f]$ from L^1/H_0^1 (or perhaps from a smaller set such as some K_r) we seek vectors x and y in $\mathcal{H}$ satisfying $[x \oplus y]_{Q_S} = \varphi_S^{-1}([f])$ (hoping, of course, that $[f]$ is in the range φ_S). One may compute that it is enough to find instead $\hat{x} = x \oplus 0$ and $\hat{y} = y \oplus 0$ in $\mathcal{H} \oplus \mathcal{K}$ satisfying $[\hat{x} \oplus \hat{y}]_{Q_{\hat{S}}} = \varphi_{\hat{S}}^{-1}([f])$ and use x and y.

Since L^1/H_0^1 is isometrically isomorphic with $Q_{\hat{S}}$ we may import to $Q_{\hat{S}}$ the special subsets of L^1/H_0^1 defined previously. With a slight abuse of notation we may define $K_r \subseteq Q_{\hat{S}}$, and for $[L]$ in $Q_{\hat{S}}$, $D_{1/r}[L]$ and $|||[L]|||_r$ in the obvious way.

The next definition is fundamental to our technique for "solving equations" using the set K_r. Let $\mathbb{N}$ denote the set of positive integers.

DEFINITION 2.2. Let S denote a fixed absolutely continuous contraction and $\hat{S} = S \oplus B$ as usual. Let $r > 0$ be given. We say $\mathcal{A}_S$ has property $X_{0,1}^r$ if there exist a collection $\{[L_\beta]\}_{\beta \in \mathbf{B}} \subseteq K_r$ of elements of $Q_{\hat{S}}$, and, for each β in $\mathbf{B}$, sequences of vectors $\{\hat{x}_n^\beta\}_{n=1}^\infty$ and $\{\hat{y}_n^\beta\}_{n=1}^\infty$ satisfying the following:

1. $\overline{\mathrm{aco}}^r\{[L_\beta]:\beta\in\mathbf{B}\}\supseteq \mathrm{Ball}^r(K_r)$,
2. $\|\hat{x}_n^\beta\|, \|\hat{y}_n^\beta\|\leqslant 1, \quad \beta\in\mathbf{B}, n\in\mathbf{N}$,
3. $[\hat{x}_n^\beta\otimes\hat{y}_n^\beta]\in K_r, \quad \beta\in\mathbf{B}, n\in\mathbf{N}$,
4. $\|[\hat{x}_n^\beta\otimes\hat{y}_n^\beta]-[L_\beta]\|_r\to 0, \quad \beta\in\mathbf{B}$,
5. For any α and β in $\mathbf{B}$ and m in $\mathbf{N}$ we have $[\hat{x}_n^\beta\otimes\hat{y}_m^\alpha]\in K_r$ and $[\hat{x}_m^\alpha\otimes\hat{y}_n^\beta]\in K_r$ for all but finitely many n in $\mathbf{N}$, and
6. For any α and β in $\mathbf{B}$ and m in $\mathbf{N}$

$$\|[\hat{x}_n^\beta\otimes\hat{y}_m^\alpha]\|_r+\|[\hat{x}_m^\alpha\otimes\hat{y}_n^\beta]\|_r\to 0, \quad (n\to\infty).$$

With $\{[L_\beta]\}_{\beta\in\mathbf{B}}\subseteq K_r$, $\{\hat{x}_n^\beta\}_{n=1}^\infty$, and $\{\hat{y}_n^\beta\}_{n=1}^\infty$ fixed as above, denote by $\mathcal{F}_x$ (resp. $\mathcal{F}_y$) the algebraic span of $\bigcup_{\beta\in\mathbf{B}}\bigcup_{n\in\mathbf{N}}\{x_n^\beta\}$ (resp. $\bigcup_{\beta\in\mathbf{B}}\bigcup_{n\in\mathbf{N}}\{y_n^\beta\}$); then 5 and 6 of the definition imply

$$\|[\hat{x}_n^\beta\otimes\hat{y}]\|_r+\|[\hat{x}\otimes\hat{y}_n^\beta]\|_r\to 0, \quad \beta\in\mathbf{B}, x\in\mathcal{F}_x, y\in\mathcal{F}_y, \quad (n\to\infty).$$

The property $X_{0,1}^r$ is a modification of the property $X_{0,1}$ of [3]. The upcoming result about equation solving in our situation are the direct analogues of Lemmas 2.9, 3.3, and Theorem 3.6 of that work, and so the proofs are merely indicated here. (In fact, minor modifications of these results hold as well in the case analogous to property $X_{\theta,\gamma}^r$ $(0\leqslant\theta\leqslant 1)$ of [3]).

LEMMA 2.3. *Suppose S is an absolutely continuous contraction such that $\mathcal{A}_S$ has property $X_{0,1}^r$ for some $r>0$. Suppose $[L]\in K_r$, $\varepsilon>0$ and vectors $x_0,x_1,\ldots,x_t$ from $\mathcal{F}_x$ and $y_0,y_1,\ldots,y_t$ from $\mathcal{F}_y$ are given. Then there exist vectors $\hat{x}$ and $\hat{y}$ in $\mathcal{F}_x$ and $\mathcal{F}_y$ respectively satisfying the following:*
1. $\|[L]-[\hat{x}\otimes\hat{y}]\|_r\leqslant\varepsilon$,
2. $\|\hat{x}-\hat{x}_0\|^2, \|\hat{y}-\hat{y}_0\|^2\leqslant\|[L]-[\hat{x}_0\otimes\hat{y}_0]\|_r+\varepsilon$,
3. $\|\hat{x}\|^2\leqslant\|\hat{x}_0\|^2+\|[L]-[\hat{x}_0\otimes\hat{y}_0]\|_r+\varepsilon$,
4. $\|\hat{y}\|^2\leqslant\|\hat{y}_0\|^2+\|[L]-[\hat{x}_0\otimes\hat{y}_0]\|_r+\varepsilon$, and
5. $\|[(\hat{x}-\hat{x}_0)\otimes\hat{y}_j]\|_r, \|[\hat{x}_j\otimes(\hat{y}-\hat{y}_0)]\|_r<\varepsilon, \quad 1\leqslant j\leqslant t$.

Proof (sketch). If $[L]=[\hat{x}_0\otimes\hat{y}_0]$ we are done with $\hat{x}=\hat{x}_0$ and $\hat{y}=\hat{y}_0$. So suppose $\|[L]-[\hat{x}_0\otimes\hat{y}_0]\|_r=\eta>0$. Since $\overline{\mathrm{aco}}^r\{[L_\beta]:\beta\in\mathbf{B}\}$ contains $\mathrm{Ball}^r(K_r)$ there are elements $[L_1],[L_2],\ldots,[L_m]$ in $\{[L_\beta]:\beta\in\mathbf{B}\}$ and scalars $\{\alpha_i\}_{i=1}^m$ satisfying

$$\left\|[L]-[\hat{x}_0\otimes\hat{y}_0]-\sum_{i=1}^m\alpha_i[L_i]\right\|_r\leqslant\frac{\varepsilon}{3\eta}\cdot\eta=\frac{\varepsilon}{3} \tag{10}$$

and

$$\sum_{i=1}^n|\alpha_i|\leqslant\eta. \tag{11}$$

Further, one may choose an integer n_0 so large that

$$\|[L_i] - [\hat{x}^i_n \otimes \hat{y}^i_n]\|_r \leqslant \frac{\varepsilon}{3\eta}, \quad n \geqslant n_0,\ 1 \leqslant i \leqslant m, \tag{12}$$

where the $\{\hat{x}^i_n\}_{n=1}^{\infty}$ and $\{\hat{y}^i_n\}_{n=1}^{\infty}$ are the vectors associated with $[L_i]$ by Definition 2.2. Then for any m-tuple $\nu = (n_1, \ldots, n_m)$ with each $n_i > n_0$ we have

$$\left\|[L] - [\hat{x}_0 \otimes \hat{y}_0] - \sum_{i=1}^{m} \alpha_i \left[\hat{x}^i_n \otimes \hat{y}^i_{n_i}\right]\right\|_r \leqslant \frac{2\varepsilon}{3}. \tag{13}$$

Choose β_i so that $\beta_i^2 = \alpha_i$ $(1 \leqslant i \leqslant m)$. For any ν as above, set

$$\begin{aligned} \hat{x}_\nu &= \sum_{i=1}^{m} \beta_i \hat{x}^i_{n_i} \quad \text{and} \\ \hat{y}_\nu &= \sum_{i=1}^{m} \overline{\beta}_i \hat{y}^i_{n_i}. \end{aligned} \tag{14}$$

It is by now a standard argument (see, e.g., [3, proof of Lemma 2.9]) that an m--tuple $\nu_0 = (n^0_1, n^0_2, \ldots, n^0_m)$ may be chosen (by choosing the n^0_i succesively and making repeated use of (6) of Definition 2.2) so as to ensure that the conditions 1–5 of the Lemma hold with $\hat{x} = \hat{x}_{\nu_0}$ and $\hat{y} = \hat{y}_{\nu_0}$. ■

The next lemma extends the approximation process to systems of simultaneous equation of finite size.

LEMMA 2.4. *Suppose S is an absolutely continuous contraction such that $\mathcal{A}_S$ has property $X^r_{0,1}$ for some $r > 0$. Suppose also that* N *in* $\mathbb{N}$, $\{\varepsilon_{i,j}\}_{i,j=1}^{\mathrm{N}}$ *a set of positive numbers,* $\{[L_{i,j}]\}_{i,j=1}^{\mathrm{N}} \subseteq K_r$, *and sequences* $\{\hat{x}_i\}_{i=1}^{\mathrm{N}}$ *and* $\{\hat{y}_j\}_{j=1}^{\mathrm{N}}$ *contained in* $\mathcal{F}$ *are given such that* $\|[L_{i,j}] - [\hat{x}_i \otimes \hat{y}_j]\|_r < \varepsilon_{i,j}$, *for* $1 \leqslant i, j \leqslant \mathrm{N}$. *Suppose finally that* $\varepsilon > 0$ *and* i_0 *and* j_0 *with* $1 \leqslant i_0, j_0 \leqslant \mathrm{N}$ *are given. Then there exist sequences* $\{\hat{x}'_i\}_{i=1}^{\mathrm{N}}$ *and* $\{\hat{y}'_j\}_{j=1}^{\mathrm{N}}$ *from* $\mathcal{F}$ *satisfying*

1. $\hat{x}'_i = \hat{x}_i, \quad 1 \leqslant i \leqslant \mathrm{N}$ *and* $i \neq i_0$,
2. $\hat{y}'_j = \hat{y}_j, \quad 1 \leqslant j \leqslant \mathrm{N}$ *and* $j \neq j_0$,
3. $\|[L_{i_0 j_0}] - [\hat{x}'_{i_0} \otimes \hat{y}'_{j_0}]\|_r < \varepsilon$,
4. $\|[L_{ij}] - [\hat{x}'_i \otimes \hat{y}'_j]\|_r < \varepsilon_{ij}, \quad 1 \leqslant i, j \leqslant \mathrm{N}$ *and* $(i_0, j_0) \neq (i, j)$,
5. $\|\hat{x}'_{i_0} - \hat{x}_{i_0}\|^2, \|\hat{y}'_{j_0} - \hat{y}_{j_0}\|^2 \leqslant \|[L_{i_0 j_0}] - [\hat{x}_{i_0} \otimes \hat{y}_{j_0}]\|_r + \varepsilon$,
6. $\|\hat{x}'_{i_0}\|^2 \leqslant \|\hat{x}_{i_0}\|^2 + \|[L_{i_0 j_0}] - [\hat{x}_{i_0} \otimes \hat{y}_{j_0}]\|_r + \varepsilon$, *and*
7. $\|\hat{y}'_{j_0}\|^2 \leqslant \|\hat{y}_{j_0}\|^2 + \|[L_{i_0 j_0}] - [\hat{x}_{i_0} \otimes \hat{y}_{j_0}]\|_r + \varepsilon$.

Proof (sketch). Let $\delta > 0$ be chosen so small that $\delta < \varepsilon$ and $\delta < \min_{1 \leqslant i,j \leqslant \mathrm{N}} (\varepsilon_{ij} - \|[L_{ij}] - [\hat{x}_i \otimes \hat{y}_j]\|_r)$. Apply Lemma 2.3 to the coset $[L_{i_0 j_0}]$, with $\hat{x}_0 = \hat{x}_{i_0}$, $\hat{y}_0 = \hat{y}_{j_0}$, the ε of that lemma set to δ, and with $\{\hat{x}_i\}_{i=1}^{\mathbf{N}} \cup \{\hat{y}_j\}_{j=1}^{\mathbf{N}}$ forming the collection $\{\hat{z}_j\}_{j=1}^{t}$ of that lemma. The resulting $\hat{x}$ and $\hat{y}$ form a satisfactory pair $\hat{x}'_{i_0}$ and $\hat{y}'_{j_0}$, and one sets $\hat{x}'_i = \hat{x}_i$ $(i \neq i_0)$ and $\hat{y}'_j = \hat{y}_j$ $(j \neq j_0)$. ∎

The following may be obtained by an iterative use of Lemma 2.4 as applied to larger and larger sub-blocks of an $\aleph_0 \times \aleph_0$ system of simultaneous equations. For the details in the proof of a similar result, see [3, Theorem 3.6]. The subsequent theorem follows via a scaling argument as in [16].

THEOREM 2.5. *Suppose S is an absolutely continuous contraction such that $\mathcal{A}_S$ has property $X_{0,1}^r$ for some $r > 0$. Suppose $\{\varepsilon_i\}_{i=1}^{\infty}$ is a sequence of positive numbers and $\{[L_{ij}]\}_{i,j=1}^{\infty}$ is an array in $K_r \subseteq Q_{\hat{S}}$ satisfying $\sum_{k=1}^{\infty} |||[L_{ik}]|||_r < \infty$ for $i \in \mathbf{N}$, and $\sum_{k=1}^{\infty} |||[L_{kj}]|||_r < \infty$ for $j \in \mathbf{N}$. Then there exist sequences $\{\hat{x}_i\}_{i=1}^{\infty}$ and $\{\hat{y}_j\}_{j=1}^{\infty}$ such that*

$$[L_{ij}] = [\hat{x}_i \otimes \hat{y}_j], \quad i, j \in \mathbf{N},$$

$$\|\hat{x}_i\|^2 \leqslant \sum_{j=1}^{\infty} |||[L_{ij}]|||_r + \varepsilon_i, \quad i \in \mathbf{N}, \text{ and}$$

$$\|\hat{y}_j\|^2 \leqslant \sum_{j=1}^{\infty} |||[L_{ij}]|||_r + \varepsilon_j, \quad j \in \mathbf{N}.$$

THEOREM 2.6. *Let S be as above and $\{[L_{ij}]\}_{i,j=1}^{\infty}$ be an array in $K_r \subseteq Q_{\hat{S}}$. Then there exist sequences $\{\hat{x}_i\}_{i=1}^{\infty}$ and $\{\hat{y}_j\}_{j=1}^{\infty}$ satisfying*

$$[L_{ij}] = [\hat{x}_i \otimes \hat{y}_j], \quad i, j \in \mathbf{N}.$$

The following theorem gives a sample of the dilation results obtainable by modification of the arguments in [2, Theorem 4.1 and Proposition 4.2] and [3, Corollary 5.5]. Recall that T is a dilation of T' if T' is the compression of T to some semi-invariant subspace (see, for example, [3] for full definitions).

THEOREM 2.7. *Suppose S is an absolutely continuous contraction such that $\mathcal{A}_S$ has property $X_{0,1}^r$ for some $r > 0$. Then*

1. *if $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of (not necessarily distinct) points of $r\mathbf{D}$ then S dilates, up to unitary equivalence, a normal operator A whose matrix is* $\mathrm{Diag}(\lambda_k)$, *and*
2. *S dilates, up to unitary equivalence, $0_{\mathcal{K}}$ where $\mathcal{K}$ has dimension $\aleph_0$.*

For consequences of these dilation theorems (*mutatis mutandis*) involving $\mathrm{Lat}(T)$ see [3, Proposition 4.17] and for some involving the solution of equations "in $L^1(\mathbf{T})$" see [12, Theorem 1].

In the applications to follow we employ an auxiliary norm to overestimate $\|[L]\|_r$. Denote by $\mathcal{L}$ the subset of L^1/H_0^1 consisting of those $[f]$ whose associated sequence of negative Fourier coefficients is absolutely summable. Define a norm by $\|[f]\|_\ell \equiv \equiv \|(f_0, f_{-1}, \ldots)\|_{\ell^1}$ for $[f]$ in $\mathcal{L}$. A computation using the duality of L^1/H_0^1 and H^∞ shows $\|[f]\|_\ell \geqslant \|[f]\|_{L^1/H_0^1}$ for $[f]$ in $\mathcal{L}$.

Denote by $K_{r,\ell}$ the set of those $[f]$ in L^1/H_0^1 for which $D_{1/r}[f] \in \mathcal{L}$. One may define a norm on $K_{r,\ell}$ by $\|[f]\|_{r,\ell} \equiv \|D_{1/r}[f]\|_\ell$. It is easy to show that

$$\|[f]\|_{r,\ell} \geqslant \|[f]\|_r \geqslant \|[f]\|_{L^1/H_0^1}, \quad [f] \in K_{r,\ell}. \tag{15}$$

Finally, import these definitions as before to the predual Q_T of some T in $\mathbf{A}$, producing a norm $\|[L]\|_{r,\ell}$ on Q_T and a subset $K_{r,\ell}$ of Q_T in the obvious way.

3. SHIFTS WHOSE ADJOINTS HAVE RICH POINT SPECTRUM

Before applying the technique of Section 2 weighted shifts let us recall some relevant results and definitions from [3]. Recall that an operator T is *power bounded* if $\{\|T^n\|\}_{n=1}^\infty$ is a bounded sequence. The following is straightforward from [3, Theorem 10.5] and the fact that a power bounded unilateral weighted shift is similar to a contraction (see, e.g., [17]). Denote by U the unweighted unilateral shift.

THEOREM 3.1. *Let S be a power bounded injective unilateral weighted shift of multiplicity one with $r(S) = 1$. Then either*

S similar to U, $\mathcal{A}_S$ has property $(\mathbf{A}_1)$ but not property $(\mathbf{A}_2)$ or $(\mathbf{A}_{\aleph_0})$,

or

$S \in C_{00}$ and $\mathbf{A}_S$ has property $(\mathbf{A}_{\aleph_0})$.

If in addition $\|S\| = 1$ then $S \in \mathbf{A}$ and the properties $(\mathbf{A}_n)$ may be replaced by the classes $\mathbf{A}_n$ throughout.

With this background, turn to the case that $S \in \mathcal{L}(\mathcal{H})$ is a contractive, injective unilateral shift of multiplicity one with positive weights, and continue to follow the notational convention of Section 2. Regard S as shifting the orthonormal basis $e_0, e_1, \ldots$ of $\mathcal{H}$. We suppose further in this section that $r_2 > 0$ so $r_2\mathbf{D} \subseteq \sigma_p(S^*)$, and consider almost exclusively the case $r_2 = r$.

The cosets needed for Definition 2.2 and the technique of Section 2 will be the $[C_\lambda]$ with λ in $r_2\mathbf{D}$. For such a λ consider the sequence of vectors $\{f_\lambda^n\}_{n=1}^\infty \subseteq \mathcal{H}$, where f_λ^n is indicated below:

$$f_\lambda^n = \left(0, \ldots, 0, \overset{n^{\text{th}}}{1}, \frac{\lambda}{w_n}, \frac{\lambda^2}{w_n w_{n+1}}, \ldots\right). \tag{16}$$

Set

$$e_\lambda^n = \frac{f_\lambda^n}{\|f_\lambda^n\|}. \tag{17}$$

It is easy to check via (2) that

$$[\hat{e}_\lambda^n \otimes \hat{e}_\lambda^n] = [C_\lambda]_{\hat{S}}. \tag{18}$$

The following lemma summarizes some "vanishing conditions" satisfied by the vectors $\hat{e}_\lambda^n$ if S/r is in the class C_{00}. Note that in light of Theorem 3.1 some such condition is required to arrive at the situation of Definition 2.2.

LEMMA 3.2. *Suppose S is a contractive injective unilateral shift of multiplicity one with positive weights $\{w_0, w_1, \ldots\}$. Suppose further that $r_2(S) = r(S) = r > 0$ and that S/r is in the class C_{00}. Let λ be arbitrary in $r\mathbf{D}$. Then for any μ in $r\mathbf{D}$, m in $\mathbf{N}$, and v in $\mathcal{H}$ of finite support,*

1. $[\hat{e}_\mu^m \otimes \hat{e}_\lambda^n]_{\hat{S}}$ *and* $[\hat{e}_\lambda^n \otimes \hat{e}_\mu^m]_{\hat{S}}$ *are in* $K_{r,\ell}$, $n > m$,
2. $\left\|[\hat{e}_\mu^m \otimes \hat{e}_\lambda^n]\right\|_{r,\ell} + \left\|[\hat{e}_\lambda^n \otimes \hat{e}_\mu^m]\right\|_{r,\ell} \to 0, \quad (n \to \infty)$,
3. $[\hat{v} \otimes \hat{e}_\lambda^n]_{\hat{S}}$ *and* $[\hat{e}_\lambda^n \otimes \hat{v}]_{\hat{S}}$ *are in* $K_{r,\ell}$, *and*
4. $\|[\hat{v} \otimes \hat{e}_\lambda^n]\|_{r,\ell} + \|[\hat{e}_\lambda^n \otimes \hat{v}]\|_{r,\ell} \to 0, \quad (n \to \infty)$.

Proof. Examine first the sequence of negative Fourier coefficients $(c_{-j})_{j=0}^\infty$ arising from $[f_\mu^m \otimes f_\lambda^n]_{\hat{S}}$ under the assumption that $n > m$. Recall that these are obtained as in (3) by

$$c_{-j} = \left(\hat{S}^j \hat{f}_\mu^m, \hat{f}_\lambda^n\right)_{\mathcal{H}\oplus\mathcal{K}} = (S^j f_\mu^m, f_\lambda^n)_{\mathcal{H}}, \quad j \geqslant 0, \tag{19}$$

where we use the notational convention. A computation yields

$$(c_0, c_{-1}, \ldots) = N_1(\mu, \lambda, m, n) + N_2(\mu, \lambda, m, n) \tag{20}$$

where

$$\begin{aligned} N_1(\mu, \lambda, m, n) &= w_m \cdots w_{n-1} \cdot \Big(\overbrace{0, 0, \ldots, 0}^{n-m}, 1, \overline{\lambda}, \overline{\lambda}^2, \ldots\Big) + \\ &+ \frac{\mu}{w_m} \cdot w_{m+1} \cdots w_{n-1} \cdot \Big(\overbrace{0, 0, \ldots,}^{n-m-1} 1, \overline{\lambda}, \overline{\lambda}^2, \ldots\Big) + \ldots\ldots + \\ &+ \frac{\mu_{n-m-1}}{w_m \cdots w_{n-2}} \cdot w_{n-1} \cdot \left(0, 1, \overline{\lambda}, \overline{\lambda}^2, \ldots\right) \end{aligned} \tag{21}$$

and

$$N_2 = \frac{\mu^{n-m}}{w_m \cdots w_{n-1}} \cdot \left(1, \overline{\lambda}, \overline{\lambda}^2, \dots\right) + \frac{\mu^{n-m+1}}{w_m \cdots w_n} \cdot \left(\frac{\overline{\lambda}}{w_n}, \frac{\overline{\lambda}^2}{w_n}, \dots\right) + \cdots \tag{22}$$
$$+ \frac{\mu^{n-m+j}}{w_m \cdots w_{n+j-1}} \cdot \left(\frac{\overline{\lambda}^j}{w_n \cdots w_{n+j-1}}, \frac{\overline{\lambda}^{j+1}}{w_n \cdots w_{n+j-1}}, \dots\right) + \cdots .$$

(Each row of the above pair of expressions arises from the negative Fourier coefficients from products with a single non-zero entry of f_λ^n.) Denote by $D_{1/r}N_1$ and $D_{1/r}N_2$ the sequences obtained from $N_1(\mu, \lambda, m, n)$ and $N_2(\mu, \lambda, m, n)$ by the usual termwise product with $\{1/r^j\}_{j=0}^{\infty}$. We turn next to showing that $D_{1/r}N_1$ and $D_{1/r}N_2$ are absolutely summable and estimating their ℓ^1 norms. First consider $D_{1/r}N_2$. A computation gives

$$\left\|D_{1/r}N_2\right\|_{\ell^1} \leqslant \frac{|\mu|^{n-m}}{w_m \cdots w_{n-1}} \cdot \sum_{j=0}^{\infty} \left(\frac{|\lambda|}{r}\right)^j \cdot \left|\left(\hat{f}^n_{|\mu|}, \hat{f}^n_{|\mu|}\right)_{\mathcal{H}\oplus\mathcal{K}}\right| \tag{23}$$
$$\leqslant \frac{|\mu|^{n-m}}{w_m \cdots w_{n-1}} \cdot \frac{1}{1 - |\lambda|/r} \cdot \left\|\hat{f}^n_{|\mu|}\right\| \cdot \left\|\hat{f}^n_{|\lambda|}\right\|.$$

With similar computation for $D_{1/r}N_1$, we arrive at

$$\left\|\left[\hat{e}^m_\mu \otimes \hat{e}^n_\lambda\right]\right\|_{r,\ell} \leqslant \frac{w_m \cdots w_{n-1}}{r^{n-m}} \cdot \frac{1}{1 - |\lambda|/r} \cdot \frac{\left\|\hat{f}^m_{\sqrt{r|u|}}\right\|^2}{\left\|\hat{f}^m_\mu\right\| \cdot \left\|\hat{f}^n_\lambda\right\|} + \tag{24}$$
$$+ \frac{|\mu|^{n-m}}{w_m \cdots w_{n-1}} \cdot \frac{1}{1 - |\lambda|/r} \cdot \frac{\left\|\hat{f}^n_{|\mu|}\right\| \cdot \left\|\hat{f}^n_{|\lambda|}\right\|}{\left\|\hat{f}^m_\mu\right\| \cdot \left\|\hat{f}^n_\lambda\right\|}.$$

(This shows half of what is needed for assertion 1 of the lemma.)

A computation using $\left(\frac{|\mu|^{n-m}}{w_m \cdots w_{n-1}}\right)^2 \cdot \frac{\left\|\hat{f}^n_{|\mu|}\right\|^2}{\left\|\hat{f}^m_{|\mu|}\right\|^2} \to 0$ as $(n \to \infty)$ for the second term on the right hand side of (24) and $S/r \in C_{00}$ for the first term yields $\left\|\left[\hat{e}^m_\mu \otimes \hat{e}^n_\lambda\right]\right\|_{r,\ell} \to 0$ as $(n \to \infty)$ which is half of what is needed for assertion 2 of the lemma.

The computations to establish the results for $\left[\hat{e}^n_\lambda \otimes \hat{e}^m_\mu\right]$ are similar and therefore omitted, but yield the other half of what is needed for assertions 1 and 2 of the lemma. Finally, assertions 3 and 4 follow easily. ■

Let $\mathcal{F}'$ denote the set of all $v = w + \sum_{i=1}^{m} \alpha_i e^m_{\lambda_i}$ where w is of finite support in $\mathcal{H}$ and $\{\lambda_i\}_{i=1}^m \subseteq r\mathbf{D}$. (Thus $\mathcal{F}'$ adjoins the vectors of finite support to the set $\mathcal{F}_x \cup \mathcal{F}_y$

of Section 2.) The first part of the corollary below is immediate from Lemma 3.2 and the second then follows from (15).

COROLLARY 3.3. *Let S be as in Lemma 3.2. Then for any λ in $r\mathbb{D}$ and v in $\mathcal{F}'$, $[\hat{v} \otimes \hat{e}_\lambda^n]$ and $[\hat{e}_\lambda^n \otimes \hat{v}]$ are in $K_{r,\ell}$ and*

$$\|[\hat{v} \otimes \hat{e}_\lambda^n]\|_{r,\ell} + \|[\hat{e}_\lambda^n \otimes \hat{v}]\|_{r,\ell} \to 0 \quad (n \to \infty).$$

Then also $[\hat{v} \otimes \hat{e}_\lambda^n]$ and $[\hat{e}_\lambda^n \otimes \hat{v}]$ are in K_r and

$$\|[\hat{v} \otimes \hat{e}_\lambda^n]\|_r + \|[\hat{e}_\lambda^n \otimes \hat{v}]\|_r \to 0 \quad (n \to \infty).$$

The next theorem follows from the preceding corollary, Lemma 3.2, (17) and the trivial observation that $r\mathbb{D}$ is dominating for itself.

THEOREM 3.4. *Suppose S is a contractive, injective unilateral shift of multiplicity one. Suppose further that $r_2 = r > 0$ and that S/r is in the class C_{00}. Then $\mathcal{A}_S$ has property $X_{0,1}^r$ and therefore the dilation properties of Theorem 2.7.*

Let us observe in passing that the proof of the key "vanishing lemma" (Lemma 3.2) may be modified to yield a partial result essentially (usefully?) analogous to the property $\mathcal{E}_{0,1}^r$ of [11] or [10] even with no C_{00} condition.

4. SHIFTS WITH RICH LEFT ESSENTIAL SPECTRUM

Now switch attention to contractive injective unilateral weighted shifts S with the radius $r_1 = r_1(S) < r = r(S)$, and continue the application of the machinery of Section 2 (and to abide by the notational convention of that section). For such shifts the left essential spectrum $\sigma_{\mathrm{le}}(S)$ satisfies $\sigma_{\mathrm{le}}(S) = \{\lambda : r_1 \leqslant \lambda \leqslant r\}$ (see [15]) and thus $\sigma_{\mathrm{le}}(S) \cap r\mathbb{D}$ is dominating for $r\mathbb{D}$. For convenience, denote the annulus $\{z : r_1 < |z| < r\}$ by $A(r_1, r)$. The collection $\{[C_\lambda]_{\hat{S}} : \lambda \in A(r_1, r)\}$ will turn out in some cases to be a satisfactory collection to show $\mathcal{A}_S$ has property $X_{0,1}^r$.

It is well known (see, e.g., [15]) that $r_1(S) = \liminf_n \inf_k |w_{k+1} \cdots w_{k+n}|^{\frac{1}{n}}$ and $r(S) = \lim_n \sup_k |w_{k+1} \cdots w_{k+n}|^{\frac{1}{n}}$. They may be used as in [15] to construct a pair of vectors associated with some λ in $A(r_1, r)$. Let $\varepsilon > 0$ be given, and choose a and b so that $r_1 < a < |\lambda| < b < r$. Choose l so large that $\left(\frac{|\lambda|}{b}\right)^l < \varepsilon$ and k so that $|w_{k+1} \cdots w_{k+l}|^{\frac{1}{l}} > b$. Choose p so large that $\left(\frac{a}{|\lambda|}\right)^p < \varepsilon$ and m so large that $m > l + k$ and also

$|w_{m+1}\cdots w_{m+p}|^{\frac{1}{p}} < a$. (Call some (finite) sequence of indices $(k+1,\ldots,m+p)$ satisfying these conditions a (λ,ε)-Ridge block.) Define the vector $g_\lambda = g_\lambda(k,l,m,n)$ by

$$g_\lambda \equiv \left(0,0,\ldots,\overset{k+1^{st}}{1},\frac{w_{k+1}}{\lambda},\frac{w_{k+1}w_{k+2}}{\lambda^2},\ldots,\frac{w_{k+1}\cdots w_{m+p}}{\lambda^{m+p-k}},0,\ldots\right). \tag{25}$$

A computation in [15] shows $\dfrac{\|Sg_\lambda - \lambda g_\lambda\|}{\|g_\lambda\|} < \varepsilon$. Then defining $c_\lambda = c_\lambda(k,l,m,p)$ by normalizing,

$$c_\lambda \equiv \frac{g_\lambda}{\|g_\lambda\|}, \tag{26}$$

we have

$$\|\hat{S}\hat{c}_\lambda - \lambda\hat{c}_\lambda\| < \varepsilon. \tag{27}$$

A computation of the (finitely non-zero) negative Fourier coefficients of $[\hat{c}_\lambda \otimes \hat{c}_\lambda]_{\hat{S}}$ (where we use the notational convention of Section 2) yields a sequence $(d_0, d_{-1},\ldots)$ satisfying

$$|d_{-j}| \leqslant |\lambda^j|, \quad 0 \leqslant j, \tag{28}$$

independent of k,l,m and p. Observe for later use that the negative Fourier coefficients of $[C_\lambda]_{\hat{S}}$ satisfy a similar inequality.

For λ in $A(r_1,r)$ we next construct the needed *sequences* of vectors for $[C_\lambda]_{\hat{S}}$. It is noted in [15] that for any such $\lambda, \varepsilon > 0$, and M there exists a (λ,ε)-Ridge block with lowest index $k+1$ larger than M. Choose a positive sequence $\{\varepsilon_n\}_{n=1}^\infty$ converging to 0, and choose then a sequence of (λ,ε_n)-Ridge blocks $R_n(k_n,l_n,m_n,p_n)$ satisfying $k_{n+1}+1 > p_n$. Denote by c_λ^n the unit vector associated with R_n as in (25) and (26). It follows easily from (28) that

$$[\hat{c}_\lambda^n \otimes \hat{c}_\lambda^n]_{\hat{S}} \in K_{r,l}. \tag{29}$$

In fact, it will be convenient to have the following uniform bound:

$$\|\,[\hat{c}_\lambda^n \otimes \hat{c}_\lambda^n]\,\|_{r,l} \leqslant \frac{1}{1-|\lambda|/r}, \quad n \in \mathbf{N}. \tag{30}$$

It is clear from (3) and (26) that the 0^{th} Fourier coefficient of any $[\hat{c}_\lambda^n \otimes \hat{c}_\lambda^n]$ is 1, as is the 0^{th} Fourier coefficient of $[C_\lambda]_{\hat{S}}$. From $\varepsilon_n \to 0$ and (27) as applied to the $\hat{c}_\lambda^n$ it is easy to show that the -1^{st} Fourier coefficient of $[\hat{c}_\lambda^n \otimes \hat{c}_\lambda^n]$ approaches λ, the -1^{st} Fourier coefficient of $[C_\lambda]_{\hat{S}}$, as $n \to \infty$. The argument extends as well to

Fourier coefficients of higher negative index, and this, (30) and a similar observation for $[C_\lambda]_{\hat{S}}$, and the definition of $\|\cdot\|_{r,l}$ yield the following lemma.

LEMMA 4.1. *Let S be a contractive injective unilateral weighted shift with positive weights and $r_1 < r$. Let λ be in the annulus $A(r_1, r)$, and let $\{\hat{c}_\lambda^n\}_{n=1}^\infty$ be a sequence of unit vectors constructed for λ as above. Then*

$$\| [\hat{c}_\lambda^n \otimes \hat{c}_\lambda^n] - [C_\lambda] \|_{r,l} \to 0 \quad (n \to \infty).$$

One may begin the effort to use Definition 2.2 by choosing, for each λ in $A(r_1, r)$, a sequence of unit vectors $\{\hat{c}_\lambda^n\}_{n=1}^\infty$ as above. Since for each λ and μ in $A(r_1, r)$ and n and m in $\mathbb{N}$ the vectors c_λ^n and c_μ^m are of finite support the next lemma is immediate from S a shift and (3).

LEMMA 4.2. *Let S be a contractive injective unilateral weighted shift with positive weights and $r_1 < r$. Then we have*

$$\| [\hat{c}_\lambda^n \otimes \hat{c}_\mu^m] \|_{\hat{S}} \in K_{r,l}, \quad \lambda, \mu \in A(r_1, r), \quad n, m \in \mathbb{N}, \tag{31}$$

and, for any λ and μ in $A(r_1, r)$ and m in $\mathbb{N}$,

$$\| [\hat{c}_\lambda^n \otimes \hat{c}_\mu^m] \|_{r,l} \to 0, \quad n \to \infty. \tag{32}$$

The other vanishing condition is more troublesome. The next lemma is the best we have been able to achieve and includes only partial results even with a C_{00} hypothesis. (Contrast the case of T in $\mathbb{A}$ for which it is automatic for such sequences associated with $\sigma_{le}(T)$; see also Li [13] for a discussion of similar difficulties in another setting.) For any λ in $A(r_1, r)$, $\varepsilon > 0$, and k in $\mathbb{N}$ denote by $L_k(\lambda, \varepsilon)$ the length of the shortest (λ, ε)-Ridge block with lowest index greater than k.

LEMMA 4.3. *Let S be a contractive injective unilateral weighted shift with positive weights and $r_1 < r$. Suppose there exists a set $\Lambda \subseteq A(r_1, r)$, dominating for $r\mathbb{D}$, such that one of the following conditions holds:*

1. *For all λ in Λ and $\varepsilon > 0$,* $\dfrac{w_0 \cdots w_k}{r^k} \cdot \sqrt{L_k(\lambda, \varepsilon)} \to 0, \quad k \to \infty,$
2. *$S/r \in C_{00}$, and for any λ in Λ and $\varepsilon > 0$ there is a sequence of (disjoint) (λ, ε)-Ridge blocks of uniformly bounded length,*
3. *$S/r \in C_{00}$, and for any λ in Λ we have the collection of all terms*

$$\frac{1 + \sum_{j=1}^{n} \left(\frac{w_{k+1} \cdots w_{k+j}}{r^j} \right)^2 \cdot \left(\frac{r}{|\lambda|} \right)^j}{\left(1 + \sum_{j=1}^{n} \left(\frac{w_{k+1} \cdots w_{k+j}}{r^j} \right)^2 \cdot \left(\frac{r}{|\lambda|} \right)^{2j} \right)^{1/2}}$$

uniformly bounded in k and n together,

4. *For all λ in Λ we have*

$$\frac{1+\sum_{j=1}^{n}\left(\frac{w_1\cdots w_{k+j}}{r^j}\right)^2\cdot\left(\frac{r}{|\lambda|}\right)^j}{\left(1+\sum_{j=1}^{n}\left(\frac{w_1\cdots w_{k+j}}{r^j}\right)^2\cdot\left(\frac{r}{|\lambda|}\right)^{2j}\right)^{1/2}}\to 0,\quad n\to\infty.$$

Then there exists $\Lambda'\subseteq\Lambda$ dominating for $r\mathbf{D}$ and sequences $\{\hat{c}_\lambda^n\}_{n=1}^\infty$ of unit vectors as usual for $\{[C_\lambda]_{\mathcal{S}}:\lambda\in\Lambda'\}$ satisfying, for any v in $\mathcal{H}$ of finite support and λ in Λ',

$$\text{(33)}\qquad \|\,[\hat{v}\otimes\hat{c}_\lambda^n]\,\|_{r,l}\to 0,\quad n\to\infty.$$

In particular, for any μ and λ in Λ' and m in $\mathbf{N}$,

$$\text{(34)}\qquad \|\,[\hat{c}_\mu^m\otimes\hat{c}_\lambda^n]\,\|_{r,l}\to 0,\quad n\to\infty.$$

Proof. The basic estimate to show each of these conditions sufficient is as follows. Suppose λ in $A(r_1,r)$ and a (λ,ε)-Ridge block $(k+1,\dots,m+p)$ with c_λ the associated unit vector are given. Recall that e_j $(j=0,1,\dots)$ denotes the standard j^{th} basis vector for $\mathcal{H}$. Suppose $M<k$ is given. One may compute that, for all $j\leqslant M$,

$$\|[\hat{e}_j\otimes\hat{c}_\lambda]\|\leqslant\frac{w_j\cdots w_{M-1}}{r^{M-j}}\cdot\frac{w_M\cdots w_k}{r^{k-M+1}}\cdot\frac{1+\sum_{j=1}^{m+p-k}\left(\frac{w_{k+1}\cdots w_{k+j}}{r^j}\right)^2\cdot\left(\frac{r}{|\lambda|}\right)^j}{\left(1+\sum_{j=1}^{m+p-k}\left(\frac{w_{k+1}\cdots w_{k+j}}{r^j}\right)^2\cdot\left(\frac{r}{|\lambda|}\right)^{2j}\right)^{1/2}}.$$

Observe that the first term on the right hand side is, for a fixed M, uniformly bounded for all $j\leqslant M$. Further computations using any one of the hypotheses yield (33). ■

The following theorem then obtains from Lemmas 4.1, 4.2, and 4.3, (15), and Definition 2.2.

THEOREM 4.4. *Let S be a contractive injective unilateral weighted shift with $r_1<r$. Suppose there exists a set $\Lambda\subseteq A(r_1,r)$, dominating for $r\mathbf{D}$, for which one of the conditions of Lemma 4.3 holds. Then $\mathcal{A}_S$ has property $X_{0,1}^r$ and therefore the dilation properties of Theorem 2.7.*

It is easy to construct shifts S for which S/r is not power bounded and to which Condition 1 of the lemma applies; essentially, what is needed is that the weights vary rapidly from blocks of large weights to blocks of small weights. To construct a shift to which 4 applies, the following numerical lemma is useful.

LEMMA 4.5. *For any* $R > 1$,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \frac{R^n}{n}}{\sqrt{\sum_{j=1}^{n} \frac{R^{2n}}{n}}} = 0.$$

Proof. Estimate each sum by the sum taken from $n/2$ to n and compute. ■

The shift whose weights are $1, \sqrt{1/2}, \sqrt{2/3}, \ldots$ then satisfies Condition 4 with $r = 1$. It is easy then to intersperse these weights with blocks of large and blocks of small weights to produce a shift T with $r = 1$ which is not power bounded and has $r < 1$; let $S = T/\|T\|$.

The construction via Ridge blocks of sequences of vectors for the cosets $[C_\lambda]$ may work even in the absence of a C_{00} assumption. Consider a shift whose weights are as indicated:

$$\Bigg(1, \frac{1}{4}, \overbrace{\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}}^{n_1}, \overbrace{\frac{\sqrt{2}}{2}, \ldots, \frac{\sqrt{2}}{2}}^{4}, \frac{1}{4}, \frac{1}{4}, \overbrace{\frac{\sqrt[n_1]{2}}{2}, \ldots, \frac{\sqrt[n_1]{2}}{2}}^{n_1^2},$$

$$\overbrace{\frac{1}{4}, \frac{1}{4}, \ldots, \frac{1}{4}}^{n_2}, \overbrace{\frac{\sqrt[3]{2}}{2}, \ldots, \frac{\sqrt[3]{2}}{2}}^{9}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \overbrace{\frac{\sqrt[n_2]{2}}{2}, \ldots, \frac{\sqrt[n_2]{2}}{2}}^{n_2^2}, \ldots\Bigg).$$

We leave to the reader to check that this is a contractive shift S with spectral radius $r = 1/2$ and $r_1 = 1/4$ and that S/r is not power bounded. Further, the numbers $n_1, n_2, \ldots$ may be chosen so as to ensure (33).

We know of no shift, in the class C_{00} or not, for which this construction using points of the left essential spectrum can not provide suitable sequences of vectors.

5. REMARKS AND QUESTIONS

It is appropriate to remark at the outset that this work shares some of the goals of the work of W.-S. Li [13] on polynomially bounded operators and uses a complimentary if distinct approach: to generalize the situation for an operator T in $\mathbb{A}$ we give up $r_{\text{spec}}(T) = 1$ while she gives up $\|T\| = 1$. While these studies were pursued independently we wish to thank Li for early access to her results and some enlightening discussions on that approach.

The property $X^r_{0,1}$ yields subspaces invariant (not merely semi-invariant) for an operator. The following definition generalizes that used in [10], [14], and [7] from $r = 1$ to $0 < r \leqslant 1$. Let $\vee$ denote closed linear span.

Definition 5.1. Let T be a contraction and $\mathcal{M}$ be an element of $\mathrm{Lat}(T)$, the lattice of subspaces invariant for T. We say $\mathcal{M}$ is an *r-analytic invariant subspace* for T if there exists a non-zero conjugate analyitic function $e\colon \lambda \to e_\lambda$ from $r\mathbb{D}$ into $\mathcal{M}$ such that

$$(T|\mathcal{M} - \lambda)^* e_\lambda = 0, \quad \lambda \in r\mathbb{D}. \tag{35}$$

If in addition

$$\bigvee_{\lambda \in r\mathbb{D}} e_\lambda = \mathcal{M} \tag{36}$$

we say $\mathcal{M}$ is a full r-analytic invariant subspace for T.

We may obtain the following much as in [10], [14], and [7].

Proposition 5.2. *Let $S \in \mathcal{L}(\mathcal{H})$ be an absolutely continuous contraction such that $\mathcal{A}_S$ has property $X^r_{0,1}$ for some $r > 0$. Then S has a cyclic r-analytic invariant subspace.*

With more work one may obtain a cyclic full r-analytic invariant subspace (as in [10, Proposition 5.3]), and if S is one of the shifts studied then we may obtain a dense set of vectors generating cyclic full r-analytic invariant subspaces. These results are relegated to the remarks because there appear to be impediments to their hoped-for culmination in the reflexivity of $\mathcal{A}_S$ as in [10]. The reflexivity result [17, Proposition 37] appears encouraging nevertheless.

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