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FURUTA'S INEQUALITY AND ITS APPLICATION TO THE RELATIVE OPERATOR ENTROPY

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Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

A capital letter means a bounded linear operator on a Hilbert space. An operator T is said to be positive if $(Tx,x) \geqslant 0$ for all x in a Hilbert space. It is well known that $A \geqslant B \geqslant 0$ does not always ensure $A^2 \geqslant B^2$ in general and also that "$\geqslant$" is only a partial order on the set of positive operators, not a total ordering. We recall the following famous inequality: if $A \geqslant B \geqslant 0$, then $A^\alpha \geqslant B^\alpha$ for each $\alpha \in [0,1]$. This inequality is the Löwner-Heinz theorem established firstly in [12] and nice operator algebraic proof was given in [14].

In [7], as an extension of this Löwner-Heinz theorem, we established the Furuta's inequality as follows; if $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,

$$(B^r A^p B^r)^{\frac{1}{q}} \geqslant B^{\frac{p+2r}{q}}$$

and

$$A^{\frac{p+2r}{q}} \geqslant (A^r B^p A^r)^{\frac{1}{q}}$$

hold for each p and q such that $p \geqslant 0$, $q \geqslant 1$ and $(1+2r)q \geqslant p+2r$. Alternative proofs of this inequality are given in [4], [8], [9] and [11]. Recently in [2], the relative operator entropy $S(A|B)$ for positive invertible operators A and B is defined by

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

In this paper, first of all, we shall show that the Furuta's inequality can be applied to estimate the value of this relative operator entropy $S(A|B)$. We cite an example as follows. Let A, B and C be positive invertible operators. Then $\log C \geqslant \log A \geqslant \log B$ holds if and only if

$$S(A^{-2r}|C^p) \geqslant S(A^{-2r}|A^p) \geqslant S(A^{-2r}|B^p)$$

for all $p \geqslant 0$ and all $r \geqslant 0$.

As an immediate consequence of this result, we shall show that $\log C \geqslant \log A^{-1} \geqslant \log B$ ensures $S(A|C) \geqslant -2A\log A \geqslant S(A|B)$ for positive invertible operators A, B and C.

Secondary we shall show somewhat precise upper bound and lower bound of the relative operators entropy.

Finally we shall show an elementary proof of the following result. Let A and B be selfadjoint operators. Then $A \geqslant B$ holds if and only if

$$F_e(p,r) = e^{-rB}(e^{rB}e^{pA}e^{rB})^{\frac{t+2r}{p+2r}}e^{-rB}$$

is an increasing function of both p and r for $p \geqslant t \geqslant 0$ and for a fixed $t \geqslant 0$ and $r \geqslant 0$. This result is an extension of Ando's one [1].

1. THE RELATIVE OPERATOR ENTROPY

In this section we shall show that the Furuta's inequality can be applied to estimate the value of the relative operator entropy. Also we shal show somewhat precise upper bound and lower bound of the relative operator entropy.

As an extension of the operator entropy considered by Nakamura and Umegaki [13] and the relative operator entropy considered by Umegaki [15], recently in [2], the relative operator entropy $S(A|B)$ is defined by

$$A(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$

for positive invertible operators A and B. We remark that $S(A|I) = -A\log A$ is the usual operator entropy.

THEOREM 1. *Let A, B and C be positive invertible operators. Then the following assertions are mutually equivalent.*

(i) $\log C \geqslant \log A \geqslant \log B$,

(ii) $(A^{\frac{p}{2}}C^sA^{\frac{p}{2}})^{\frac{p}{p+s}} \geqslant A^p \geqslant (A^{\frac{p}{2}}B^sA^{\frac{p}{2}})^{\frac{p}{p+s}}$ *for all* $p \geqslant 0$ *and all* $s \geqslant 0$,

(iii) $\log(A^rC^pA^r) \geqslant \log A^{p+2r} \geqslant \log(A^rB^pA^r)$ *for all* $p \geqslant 0$ *and all* $r \geqslant 0$,

(iv) $S(A^{-2r}|C^p) \geqslant S(A^{-2r}|A^p) \geqslant S(A^{-2r}|B^p)$ *for all* $p \geqslant 0$ *and all* $r \geqslant 0$.

COROLLARY 1. *Let A, B and C be positive invertible operators. If* $\log C \geqslant$ $\geqslant \log A^{-1} \geqslant \log B$ *then* $S(A|C) \geqslant -2A\log A \geqslant S(A|B)$.

THEOREM 2. *If A and B are positive invertible operators, then for any positive number x_0*

$$(\log x_0 - 1)A + \frac{B}{x_0} \geqslant S(A|B) \geqslant (1-\log x_0)A - \frac{AB^{-1}A}{x_0}.$$

COROLLARY 2 [3]. *If A and B are positive invertible operators, then*

$$B - A \geqslant S(A|B) \geqslant A - AB^{-1}A.$$

COROLLARY 3 [2]. *If A, B and C are positive invertible operators, then*

$$S(A|B) = 0$$

holds if and only if $A = B$.

In order to give proofs to the results in this section, first of all we cite the following Furuta's inequality in [7] which is an extension of the Löwner-Heinz inequality.

THEOREM A (Furuta's inequality). *Let A and B be positive operators acting on a Hilbert space. If $A \geqslant B \geqslant 0$, then*

$$(B^r A^p B^r)^{\frac{1+2r}{p+2r}} \geqslant B^{1+2r} \tag{1}$$

and

$$A^{1+2r} \geqslant (A^r B^p A^r)^{\frac{1+2r}{p+2r}} \tag{2}$$

hold for all $p \geqslant 1$ and $r \geqslant 0$.

In order to give a proof of Theorem 1, we cite the following Lemmas.

LEMMA 1. *The following* (i), (ii) *and* (iii) *are equivalent.*

(i) $\log A \geqslant \log B$

(ii) $A^p \geqslant (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ *for all $p \geqslant 0$*

(iii) $A^p \geqslant (A^{\frac{p}{2}} B^s A^{\frac{p}{2}})^{\frac{p}{s+p}}$ *for all $p \geqslant 0$ and all $s \geqslant 0$.*

LEMMA 2. *Let A and B be invertible positive operators. For any real number r,*

$$(BAB)^r = BA^{\frac{1}{2}} \left(A^{\frac{1}{2}} B^2 A^{\frac{1}{2}}\right)^{r-1} A^{\frac{1}{2}} B.$$

Proof of Lemma 2. We cite the following proof [10] for the sake of convenience. Let $BA^{\frac{1}{2}} = UH$ be the polar decomposition of the invertible operator $BA^{\frac{1}{2}}$ where U is unitary and $H = \left|BA^{\frac{1}{2}}\right|$.

$$(BAB)^r = (UH^2U^*)^r = UH^{2r}U^* = BA^{\frac{1}{2}} H^{-1} H^{2r} H^{-1} A^{\frac{1}{2}} B =$$
$$= BA^{\frac{1}{2}} (H^2)^{r-1} A^{\frac{1}{2}} B = BA^{\frac{1}{2}} \left(A^{\frac{1}{2}} B^2 A^{\frac{1}{2}}\right)^{r-1} A^{\frac{1}{2}} B,$$

so the proof is complete. ∎

Proof of Lemma 1. First of all, by Lemma 2 we recall the following two equivalent assertions in (*) for any $p \geqslant 0$ and any $s \geqslant 0$:

$$(*) \qquad A^p \geqslant \left(A^{\frac{p}{2}} B^s A^{\frac{p}{2}}\right)^{\frac{p}{s+p}} \text{ if and only if } (B^{\frac{s}{2}} A^p B^{\frac{s}{2}})^{\frac{s}{s+p}} \geqslant B^s.$$

(i)↔(ii) is shown in [1]. (iii)→(ii) is easy by putting $s = p$ in (iii). We show (ii)→(iii). Assume (ii); $A^s \geqslant (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{1}{2}}$ for all $s \geqslant 0$. Then by (2) of Theorem A, we have the following inequality:

$$(3) \qquad A^{s(1+2t)} \geqslant \{A^{st}(A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{m}{2}} A^{st}\}^{\frac{1+2t}{m+2t}}$$

for $m \geqslant 1$ and $t \geqslant 0$. Put $m = 2$ in (3), we have

$$(4) \qquad A^{s(1+2t)} \geqslant \{A^{s(t+\frac{1}{2})} B^s A^{s(t+\frac{1}{2})}\}^{\frac{1+2t}{2+2t}} \text{ for } t \geqslant 0.$$

Put $p = s(1+2t)$ in (4), then $\dfrac{1+2t}{2+2t} = \dfrac{p}{s+p}$, so we have

$$(5) \qquad A^p \geqslant (A^{\frac{p}{2}} B^s A^{\frac{p}{2}})^{\frac{p}{s+p}}$$

for all p and s such that $p \geqslant s \geqslant 0$,because $p = s(1+2t) \geqslant s$.

On the other hand (ii) is equivalent to the following (6) by (*),

$$(6) \qquad (B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} \geqslant B^p$$

for all $p \geqslant 0$. Then applying (1) of Theorem 1 to (6), we have the following (7)

$$(7) \qquad \{B^{pu}(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{m}{2}} B^{pu}\}^{\frac{1+2u}{m+2u}} \geqslant B^{p(1+2u)}$$

for $m \geqslant 1$ and $u \geqslant 0$. Put $m = 2$ in (7). Then we have

$$(8) \qquad \{B^{p(u+\frac{1}{2})} A^p B^{p(u+\frac{1}{2})}\}^{\frac{1+2u}{2+2u}} \geqslant B^{p(1+2u)} \text{ for } u \geqslant 0.$$

Put $s = p(1+2u)$ in (8), then $\dfrac{1+2u}{2+2u} = \dfrac{s}{p+s}$, so we have

$$(9) \qquad (B^{\frac{s}{2}} A^p B^{\frac{s}{2}})^{\frac{s}{s+p}} \geqslant B^s$$

for all p and s such that $s \geqslant p \geqslant 0$, because $s = p(1+2u) \geqslant p$. (9) is equivalent to the following (10) by (*),

$$(10) \qquad A^p \geqslant (A^{\frac{p}{2}} B^s A^{\frac{p}{2}})^{\frac{p}{s+p}}$$

for all p and s such that $s \geqslant p \geqslant 0$.

Whence the proof of (iii) is complete by (5) and (10). ■

Proof of Theorem 1. (i)→(ii). The hypothesis (i) $\log C \geqslant \log A$, that is, $\log A^{-1} \geqslant \log C^{-1}$ which is equivalent to

$$A^{-p} \geqslant (A^{-\frac{p}{2}}C^{-s}A^{-\frac{p}{2}})^{\frac{p}{s+p}}$$

by (i) and (iii) in Lemma 1. Taking inverses ensures

$$(A^{\frac{p}{2}}C^{s}A^{\frac{p}{2}})^{\frac{p}{s+p}} \geqslant A^{p}$$

and the rest of (ii) is already obtained by (i) and (iii) in Lemma 1. For the proof from (ii) to (i) we have only to trace the reverse implication in the proof from (i)→(ii).

(ii)→(iii). (ii) ensures the following inequality since $\log t$ is an operator monotone function

$$p\log(A^{\frac{p}{2}}C^{s}A^{\frac{p}{2}}) \geqslant (p+s)p\log A \geqslant p\log(A^{\frac{p}{2}}B^{s}A^{\frac{p}{2}})$$

for all $p \geqslant 0$ and all $s \geqslant 0$, and we interchange $\frac{p}{2}$ with r and also we interchange s with p, then

$$\log(A^{r}C^{p}A^{r}) \geqslant \log(A^{p+2r}) \geqslant \log(A^{r}B^{p}A^{r})$$

for all $p \geqslant 0$ and all $r \geqslant 0$.

(iii)↔(iv). (iii) is equivalent to the following inequality

$$A^{-r}\log(A^{r}C^{p}A^{r})A^{-r} \geqslant A^{-r}\log(A^{p+2r})A^{-r} \geqslant A^{-r}\log(A^{r}B^{p}A^{r})A^{-r}$$

for all $p \geqslant 0$ and all $r \geqslant 0$, that is,

$$S(S^{-2r}|C^{p}) \geqslant S(A^{-2r}|A^{p}) \geqslant S(A^{-2r}|B^{p})$$

for all $p \geqslant 0$ and all $r \geqslant 0$.

(iv)→(i). Put $r=0$ and $p=1$ in (iv), then

$$\log C \geqslant \log A \geqslant \log B$$

so the proof of Theorem 1 is complete. ■

Proof of Corollary 1. In Theorem 1 we interchange A with A^{-1} and also we put $p = 2r = 1$, so we have the desired result. ■

Proof of Theorem 2. First of all, we cite the following obvious inequality for any positive real numbers x and x_0

$$\log x_0 - 1 + \frac{x}{x_0} \geqslant \log x \geqslant 1 - \log x_0 - \frac{1}{x_0 x}. \tag{11}$$

We can interchange x with positive operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in (11), then

$$A^{\frac{1}{2}}\left\{\log x_0 - 1 + \frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{x_0}\right\}A^{\frac{1}{2}} \geqslant S(A|B) \geqslant$$
$$\geqslant A^{\frac{1}{2}}\left\{1 - \log x_0 - \frac{A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}}{x_0}\right\}A^{\frac{1}{2}}$$

that is,

$$(\log x_0 - 1)A + \frac{B}{x_0} \geqslant S(A|B) \geqslant (1 - \log x_0)A - \frac{AB^{-1}A}{x_0}$$

so the proof is complete. ∎

Proof of Corollary 2. Put $x_0 = 1$ in Theorem 2. ∎

Proof of Corollary 3. Put $S(A|B) = 0$ in Corollary 2, then

$$B - A \geqslant 0 \geqslant A - AB^{-1}A$$

that is, $B \geqslant A$ and $AB^{-1}A \geqslant A$, namely $B^{-1} \geqslant A^{-1}$, so that $A \geqslant B$ and we have $A = B$ and the reverse inplication is obvious. ∎

2. ELEMENTARY PROOF OF EXTENDED RESULTS OF ANDO'S ONE

In [1], Ando established the following fine result.

THEOREM B. *Let A and B be selfadjoint operators. The following assertions are mutually equivalent.*

(i) $A \geqslant B$.

(ii) $e^{-\frac{r}{2}A}\left(e^{\frac{r}{2}A}e^{rB}e^{\frac{r}{2}A}\right)^{\frac{1}{2}}e^{-\frac{r}{2}A} \leqslant 1$ *for all* $r \geqslant 0$.

(iii) $e^{-\frac{r}{2}A}\left(e^{\frac{r}{2}A}e^{rB}e^{\frac{r}{2}A}\right)^{\frac{1}{2}}e^{-\frac{r}{2}A}$ *is a decreasing function of* $r \geqslant 0$.

Related to Theorem B, we shall state the following results.

THEOREM 3. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(i) $\log A \geqslant \log B$.

(ii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $F_t(p) = (B^rA^pB^r)^{\frac{t+2r}{p+2r}}$ *is an increasing function of* p *for* $p \geqslant t \geqslant 0$.

(iii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $G_t(p) = (A^rB^pA^r)^{\frac{t+2r}{p+2r}}$ *is a decreasing function of* p *for* $p \geqslant t \geqslant 0$.

THEOREM 4. *Let A and B be positive invertible operators. Then the following assertions are mutually equivalent.*

(i) $\log A \geqslant \log B$.

(ii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $F(p,r) = B^{-r}(B^r A^p B^r)^{\frac{t+2r}{p+2r}} B^{-r}$ *is an increasing function of both* p *and* r *for* $p \geqslant t \geqslant 0$.

(iii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $G(p,r) = A^{-r}(A^r B^p A^r)^{\frac{t+2r}{p+2r}} A^{-r}$ *is a decreasing function of both* p *and* r *for* $p \geqslant t \geqslant 0$.

THEOREM 5. *Let* A *and* B *be selfadjoint operators. The following assertions are mutually equivalent.*

(i) $A \geqslant B$.

(ii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $F_{\mathrm{e}}(p,r) = \mathrm{e}^{-rB}\left(\mathrm{e}^{rB}\mathrm{e}^{pA}\mathrm{e}^{rB}\right)^{\frac{t+2r}{p+2r}} \mathrm{e}^{-rB}$ *is an increasing function of both* p *and* r *for* $p \geqslant t \geqslant 0$.

(iii) *For a fixed* $t \geqslant 0$ *and* $r \geqslant 0$, $G_{\mathrm{e}}(p,r) = \mathrm{e}^{-rA}\left(\mathrm{e}^{rA}\mathrm{e}^{pB}\mathrm{e}^{rA}\right)^{\frac{t+2r}{p+2r}} \mathrm{e}^{-rA}$ *is a decreasing function of both* p *and* r *for* $p \geqslant t \geqslant 0$.

The following result is obtained in [10].

THEOREM C. *If* $A \geqslant B \geqslant 0$, *then for fixed* $t \geqslant 0$ *and* $r \geqslant 0$,

(i) $F_t(p) = (B^r A^p B^r)^{\frac{t+2r}{p+2r}}$ *is an increasing function of* p *for* $p \geqslant t \geqslant 0$.

(ii) $G_t(p) = (A^r B^p A^r)^{\frac{t+2r}{p+2r}}$ *is a decreasing function of* p *for* $p \geqslant t \geqslant 0$.

In order to give proofs of results in this section, we cite the following Lemma 3.

LEMMA 3. *Let* A *and* B *be positive invertible operators. If* $\log A \geqslant \log B$, *then for each* $r \geqslant 0$,

(i) $B^r A^{p+s} B^r \geqslant (B^r A^p B^r)^{\frac{p+s+2r}{p+2r}}$

(ii) $(A^r B^p A^r)^{\frac{p+s+2r}{p+2r}} \geqslant A^r B^{p+s} A^r$ *hold for each* p *and* s *such that* $p \geqslant s \geqslant 0$.

Proof of Lemma 3. First of all, we cite (12) by (i) and (iii) of Lemma 1.

$$A^p \geqslant \left(A^{\frac{p}{2}} B^{2r} A^{\frac{p}{2}}\right)^{\frac{p}{2r+p}} \tag{12}$$

for $p \geqslant 0$ and $r \geqslant 0$.

The Löwner-Heinz theorem ensures the following (13) by (12)

$$A^s \geqslant \left(A^{\frac{p}{2}} B^{2r} A^{\frac{p}{2}}\right)^{\frac{s}{p+2r}} \tag{13}$$

for $p \geqslant s \geqslant 0$ and $r \geqslant 0$.

$$\begin{aligned}(B^r A^p B^r)^{\frac{p+s+2r}{p+2r}} &= B^r A^{\frac{p}{2}} \left(A^{\frac{p}{2}} B^{2r} A^{\frac{p}{2}}\right)^{\frac{s}{p+2r}} A^{\frac{p}{2}} B^r \qquad \text{by Lemma 2}\\ &\leqslant B^r A^{\frac{p}{2}} A^s A^{\frac{p}{2}} B^r \qquad \text{by (13)}\\ &= B^r A^{p+s} B^r\end{aligned}$$

so we have (i) in Lemma 3. By the hypothesis we have $\log B^{-1} \geqslant \log A^{-1}$. Then by (i) in Lemma 3,

$$(A^{-r}B^{-p}A^{-r})^{\frac{p+s+2r}{p+2r}} \leqslant A^{-r}B^{-(p+s)}A^{-r}$$

for each $r \geqslant 0$ and for each p and s such that $p \geqslant s \geqslant 0$. Taking inverses gives (ii) in Lemma 3. ■

Proof of Theorem 3.

(i)→(iii). In Lemma 3, $\dfrac{t+2r}{p+s+2r} \in [0,1]$ by the hypothesis $p \geqslant t \geqslant 0$, by (i) in Lemma 3 we have the following inequality by the Löwner-Heinz theorem

$$F_t(p+s) = (B^rA^{p+s}B^r)^{\frac{t+2r}{p+s+2r}} \geqslant (B^rA^pB^r)^{\frac{t+2r}{p+2r}} = F_t(p)$$

which is the desired result (ii).

(i)→(iii). This proof is easily shown by the same way as one in (i)→(ii) by using (ii) in Lemma 3.

(iii)→(i). Put $t=0$ in (iii), for any sequence $\{p_k\}$ of increasing real numbers such that $p_n \geqslant p_{n-1} \geqslant \cdots \geqslant p_2 \geqslant p_1 = 0$, then

$$A^{2r} \geqslant (A^rB^{p_2}A^r)^{\frac{2r}{p_2+2r}} \geqslant \cdots \geqslant (A^rB^{p_n}A^r)^{\frac{2r}{p_n+2r}}$$

so we have $A^{2r} \geqslant (A^rB^pA^r)^{\frac{2r}{p+2r}}$ for $p \geqslant 0$ and $r \geqslant 0$, which means that (iii) in Lemma 1 holds, so we have (i) by Lemma 1.

(ii)→(i). This proof is also easily shown by the same way as one in (iii)→(i). ■

Proof of Theorem 4.

(i)→(ii). First of all, we show the following inequality

$$(14) \qquad \left(A^{\frac{p}{2}}B^{2r}A^{\frac{p}{2}}\right)^{\frac{t-p}{2r+p}} \geqslant \left(A^{\frac{p}{2}}B^{2s}A^{\frac{p}{2}}\right)^{\frac{t-p}{2s+p}}$$

for $r \geqslant s \geqslant 0$. By (iii) of Theorem 3, we have

$$(15) \qquad \left(A^{\frac{p}{2}}B^{2r}A^{\frac{p}{2}}\right)^{\frac{t_1+p}{2r+p}} \leqslant \left(A^{\frac{p}{2}}B^{2s}A^{\frac{p}{2}}\right)^{\frac{t_1+p}{2s+p}}$$

for $2r \geqslant 2s \geqslant t_1 \geqslant 0$.

Put $\alpha = \dfrac{p-t}{p+t_1} \in [0,1]$ since $p \geqslant t \geqslant 0$ and $t_1 \geqslant 0$, by the Löwner-Heinz Theorem, taking α as exponents of both sides of (15) and moreover taking inverses of these both sides, we have (14).

$$\begin{aligned}
F(p,r) &= B^{-r}\left(B^rA^pB^r\right)^{\frac{t+2r}{p+2r}}B^{-r} = A^{\frac{p}{2}}\left(A^{\frac{p}{2}}B^{2r}A^{\frac{p}{2}}\right)^{\frac{t-p}{p+2r}}A^{\frac{p}{2}} && \text{by Lemma 2}\\
&\geqslant A^{\frac{p}{2}}\left(A^{\frac{p}{2}}B^{2s}A^{\frac{p}{2}}\right)^{\frac{t-p}{p+2s}}A^{\frac{p}{2}} \text{ for } r \geqslant s \geqslant 0 && \text{by (14)}\\
&= B^{-s}\left(B^sA^pB^s\right)^{\frac{t+2s}{p+2s}}B^{-s} && \text{by Lemma 2}\\
&= F(p,s) \text{ for } r \geqslant s \geqslant 0 \cdot
\end{aligned}$$

so we have (ii) since $F(p,r)$ is an increasing function of p by Theorem 3.

(i)→(iii). Also by the same way as one (i)→(ii), we have (iii) from (i) using (ii) of Theorem 3.

(iii)→(i). Put $t = 0$ in (iiii), for any sequence $\{p_k\}$ of increasing real numbers such that $p_n \geqslant p_{n-1} \geqslant \cdots \geqslant p_2 \geqslant p_1 \geqslant 0$, then we have

$$A^{-r}\left(A^r \cdot A^r\right)A^{-r} \geqslant A^{-r}\left(A^r B^{p_2} A^r\right)^{\frac{2r}{p_2+2r}} A^{-r} \geqslant \cdots \geqslant$$
$$\geqslant A^{-r}\left(A^r B^{p_n} A^r\right)^{\frac{2r}{p_n+2r}} A^{-r}$$

so we have the following inequality

$$A^{2r} \geqslant \left(A^r B^p A^r\right)^{\frac{2r}{p+2r}} \text{ for } p \geqslant 0 \text{ and } r \geqslant 0. \tag{16}$$

(16) means that (iii) in Lemma 1 holds, so by Lemma 1 we have $\log A \geqslant \log B$ which is desired (i).

(ii)→(i). This proof is easily shown by the same way as one in (iii)→(i). ∎

Proof of Theorem 5. We have only to replace A by e^A and also B by e^B in Theorem 4. ∎

Proof of Theorem B. In (iii) of Theorem 5, we put $t = 0$ and $2r = p$ and also we recall that $A \geqslant B$ holds if and only if $\left(e^{\frac{r}{2}A} e^{rB} e^{\frac{r}{2}A}\right)^{\frac{1}{2}} \leqslant e^{rA}$ hols by Lemma 1, so that Theorem 5 easily implies Theorem B. ∎

Proof of theorem C. As $\log t$ is an operator monotone function, so the hypothesis $A \geqslant B \geqslant 0$ ensures $\log A \geqslant \log B$ and Theorem 3 implies Theorem C. ∎

Mean theoretic proofs of the results in this section will appear in [6] which is an extension of [5].

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